

Lipschitz conformal immersions from degenerating Riemann surfaces with L^2 - bounded second fundamental forms.

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Abstract : *We give an asymptotic lower bound for the Willmore energy of weak immersions with degenerating conformal class. This lower bound is used in several other works. It is for instance one of the ingredients used in [Ri3] for providing an alternative proof of the one by L. Simon of the existence of a smooth torus minimizing the Willmore energy. The main result of the present paper has been independently obtained in [KuLi].*

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I Introduction

In [Ri3] the author introduced a suitable framework for proceeding to the calculus of variations of Willmore Lagrangian of a surface. To that aim he introduced the space of Lipschitz immersions with L^2 -bounded second fundamental forms.

Let g_0 be a reference smooth metric on Σ . One defines the Sobolev spaces $W^{k,p}(\Sigma, \mathbb{R}^m)$ of measurable maps from Σ into \mathbb{R}^m in the following way

$$W^{k,p}(\Sigma, \mathbb{R}^m) = \left\{ f \text{ meas. } \Sigma \rightarrow \mathbb{R}^m \text{ s.t. } \sum_{l=0}^k \int_{\Sigma} |\nabla^l f|_{g_0}^p \, dvol_{g_0} < +\infty \right\}$$

Since Σ is assumed to be compact it is not difficult to see that this space is independent of the choice we have made of g_0 .

First we need to have a weak first fundamental form that is we need $\vec{\Phi}^* g_{\mathbb{R}^m}$ to define an L^∞ metric with a bounded inverse. The last requirement is satisfied if we assume that $\vec{\Phi}$ is in $W^{1,\infty}(\Sigma)$ and if $d\vec{\Phi}$ has maximal rank 2 at every point with some uniform quantitative control of "how far" $d\vec{\Phi}$ is from being degenerate : there exists $c_0 > 0$ s.t.

$$|d\vec{\Phi} \wedge d\vec{\Phi}|_{g_0} \geq c_0 > 0 \quad . \quad (I.1)$$

where $d\vec{\Phi} \wedge d\vec{\Phi}$ is a 2-form on Σ taking values into 2-vectors from \mathbb{R}^m and given in local coordinates by $2 \partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi} \, dx \wedge dy$. The condition (I.1) is again independent of the choice of the metric g_0 . For a Lipschitz immersion satisfying (I.1) we can define the Gauss map as being the following measurable map in $L^\infty(\Sigma)$ taking values in the Grassmanian of oriented $m - 2$ -planes in \mathbb{R}^m .

$$\vec{n}_{\vec{\Phi}} := \star \frac{\partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi}}{|\partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi}|} \quad .$$

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We then introduce the space \mathcal{E}_Σ of Lipschitz immersions¹ of Σ with bounded second fundamental form as follows :

$$\mathcal{E}_\Sigma := \left\{ \begin{array}{l} \vec{\Phi} \in W^{1,\infty}(\Sigma, \mathbb{R}^m) \quad \text{s.t. } \vec{\Phi} \text{ satisfies (I.1) for some } c_0 \\ \text{and} \quad \int_\Sigma |d\vec{n}|_g^2 \, dvol_g < +\infty \end{array} \right\} .$$

Where $g := \vec{\Phi}^* g_{\mathbb{R}^m}$ is the pull back by $\vec{\Phi}$ of the flat canonical metric $g_{\mathbb{R}^m}$ of \mathbb{R}^m and $dvol_g$ is the volume form associated to g .

It is proved in [Ri1] that any Lipschitz immersion $\vec{\Phi}$ in \mathcal{E}_Σ defines a smooth conformal structure on Σ . A conformal structure c being given on the two manifold Σ we define \mathcal{E}_Σ^c to be the subspace of \mathcal{E} made of weak immersions which are conformal with respect to c . The conformal class to which c belongs to will be denoted $[c]$.

Let $\vec{\Phi}$ be in \mathcal{E}_Σ . Denote by $\pi_{\vec{n}_{\vec{\Phi}}}$ the orthonormal projections of vectors in \mathbb{R}^m onto the $m - 2$ -plane given by $\vec{n}_{\vec{\Phi}}$. With these notations the *second fundamental form*

$$\forall X, Y \in T_p \Sigma \quad \vec{\mathbb{I}}_p(X, Y) := \pi_{\vec{n}_{\vec{\Phi}}} d^2 \vec{\Phi}(X, Y)$$

² The *mean curvature vector* of the immersion at p is given by

$$\vec{H} := \frac{1}{2} \text{tr}_g(\vec{\mathbb{I}}) = \frac{1}{2} \left[\vec{\mathbb{I}}(\varepsilon_1, \varepsilon_1) + \vec{\mathbb{I}}(\varepsilon_2, \varepsilon_2) \right] ,$$

where $(\varepsilon_1, \varepsilon_2)$ is an orthonormal basis of $T_p \Sigma$ for the metric $g_{\vec{\Phi}}$.

In the present paper we are mainly interested with the Lagrangian given by the L^2 norm of the second fundamental form :

$$E(\vec{\Phi}) := \int_\Sigma |\vec{\mathbb{I}}|_g^2 \, dvol_g ,$$

An elementary computation gives

$$E(\vec{\Phi}) := \int_\Sigma |\vec{\mathbb{I}}|_g^2 \, dvol_g = \int_\Sigma |d\vec{n}_{\vec{\Phi}}|_g^2 \, dvol_g .$$

This energy E can be hence seen as being the *Dirichlet Energy* of the Gauss map $\vec{n}_{\vec{\Phi}}$ with respect to the induced metric $g_{\vec{\Phi}}$. The Gauss Bonnet theorem implies that

$$E(\vec{\Phi}) := \int_\Sigma |\vec{\mathbb{I}}|_g^2 \, dvol_g = 4 \int_\Sigma |\vec{H}|^2 \, dvol_g - 4\pi \chi(\Sigma) , \quad (\text{I.2})$$

where $\chi(\Sigma)$ is the *Euler characteristic* of the surface Σ . The energy

$$W(\vec{\Phi}) := \int_\Sigma |\vec{H}|^2 \, dvol_g ,$$

is the so called *Willmore energy*

In [Ri3] we studied the existence minimizers of W in \mathcal{E}_Σ with or without constraints on the conformal class c realized by $\vec{\Phi}$. We provided in particular a new flexible approach for proving L.Simon's result of the existence of a minimizer for Σ being the torus. Following [Si] we denote

$$\beta_g^m := \inf \left\{ W(\vec{\Phi}) \ ; \ \vec{\Phi} \text{ is an immersion of the genus } g \text{ closed surface} \right\} .$$

¹We will also simply call these immersions : *weak immersions*.

²In order to define $d^2 \vec{\Phi}(X, Y)$ one has to extend locally the vector X or Y by a vector-field but it is not difficult to check that $\pi_{\vec{n}_{\vec{\Phi}}} d^2 \vec{\Phi}(X, Y)$ is independent of this extension.

and

$$\omega_g^m := \min \left\{ 4\pi + \sum_{i=1}^p (\beta_{g_i}^m - 4\pi) \quad ; \quad g = g_1 + \cdots + g_p \quad , \quad 1 \leq g_i < g \right\} .$$

In [BK] it is proved that for $g \geq 2$

$$\beta_g^m < \omega_g^m . \quad (I.3)$$

In [Ri3] we gave the existence of a smooth minimizer of the Willmore energy in \mathcal{E}_Σ for an arbitrary closed 2-manifold Σ provided the conformal class defined by some minimizing sequence is not degenerating. This fact is then guaranteed by the following result which has been proved first for $m = 3, 4$ in [KS1] has also been independently proved in [KuLi] for general m .

Theorem I.1 *Let (Σ, c_k) be a sequence of closed riemann surface of genus g but with degenerating conformal class $[c_k]$ diverging to the boundary of the Moduli Space of Σ . Let $\vec{\Phi}_k$ be a sequence of conformal immersions in \mathcal{E}_Σ then*

$$\liminf_{k \rightarrow +\infty} \int_{\Sigma} |\vec{H}_{\vec{\Phi}_k}|^2 \, dvol_{\vec{\Phi}_k^* g_{\mathbb{R}^m}} \geq \min\{8\pi, \omega_g^m\} . \quad (I.4)$$

□

This result plays a crucial role in [?] for proving the compactness of Willmore surfaces below $\min\{8\pi, \omega_g^m\}$ modulo the action of conformal diffeomorphisms.

One of the main tool we are using for proving (I.4) is the following consequence of Simon's monotonicity formula with boundary which was probably known by the experts in the field but for which we still give a proof below.

Lemma I.1 *Let Σ be a compact surface with boundary. Let $\vec{\Phi}$ be a weak immersion in \mathcal{E}_Σ then the following inequality holds*

$$4\pi \leq \int_M |\vec{H}|^2 \, dvol_g + 2 \frac{\mathcal{H}^1(\partial M)}{d(\partial M, M)} , \quad (I.5)$$

where $\mathcal{H}^1(\partial M)$ is the 1-dimensional Hausdorff measure of the boundary of the immersion ∂M and d is the usual distance³ between two sets. □

Observe that the inequality is optimal since it is an equality for M being the flat 2 dimensional disc.

II The normalization procedure.

The aim of this section is to prove the "normalization" result proposition II.1 . This result asserts that a sequence of conformal immersions $\vec{\Phi}_k$ of \mathcal{E}_Σ realizing a degenerating sequence of conformal classes on Σ being given, either (I.4) holds or, for any connected component σ of the limit $\bar{\Sigma}$ topologically obtained from Σ by removing collapsing geodesics for the constant Gauss curvature associated to $(\Sigma, \vec{\Phi}_k^* g_{\mathbb{R}^m})$, one can find a sequence of Möbius transformations Ξ_k of \mathbb{R}^m such that , away from possibly finitely many blow up points on σ , the sequence of immersion $\Xi_k \circ \vec{\Phi}_k$ converges weakly on σ in $W^{2,2} \cap (W^{1,\infty})^*$ (with respect to some reference constant Gauss curvature metric) to a non constant conformal immersion of the component σ .

³Let A and B be two sets in \mathbb{R}^m we define

$$d(A, B) := \sup_{p \in A} \inf_{q \in B} |p - q| + \sup_{p \in B} \inf_{q \in A} |q - p| .$$

The proof of this normalization result will be a consequence of Deligne-Mumford's description of the loss of compactness of the conformal class for a sequence of riemann surfaces, the three point renormalization lemma A.1, the Simon's monotonicity formula with boundary and Müller-Sverak-Hélein local control of conformal factors.

First we recall Deligne-Mumford's description of the loss of compactness of the conformal class for a sequence of riemann surfaces with a fixed topology (see for instance [Zh] and proposition 5.1 of [Hum]). We restrict below in the presentation of the proof to the more complex case where $g(\Sigma) > 1$, the case of tori is following the same lines and is easier to present.

Let (Σ, c_k) be a sequence of closed riemann surface of fixed topology $g(\Sigma) > 1$. Denote by h_k the hyperbolic metric associated to c_k . Let $\vec{\Phi}_k$ a sequence of immersions in \mathcal{E}_Σ with uniformly bounded energy. Then, modulo extraction of a subsequence, there exists

- i) an integer $n \in \mathbb{N}^*$
- ii) a sequence $\mathcal{L}_k = l_k^i$; $i = 1 \dots n$ of finitely many pairwise disjoint simple closed geodesics of (Σ, h_k) with length converging to zero.
- iii) a closed riemann surface $(\bar{\Sigma}, \bar{c})$.
- iv) a complete hyperbolic riemann surface $(\tilde{\Sigma}, \tilde{h})$ with $2n$ cusps $\{q^i\}_{i=1 \dots n} = \{(q_1^i, q_2^i)\}_{i=1 \dots n}$ - or punctures - and no boundary such that $\tilde{\Sigma}$ has been obtained topologically after removing the geodesics \mathcal{L}_k and after closing each component of the boundary of the open surface $\Sigma \setminus \mathcal{L}_k$ by adding a puncture q_i^i at each of these components. Moreover $\bar{\Sigma}$ is topologically equal to $\tilde{\Sigma}$ and the complex structure defined by \tilde{h} on $\tilde{\Sigma} \setminus \{q_i\}$ extends uniquely to \bar{c} (i.e. there exists a conformal diffeomorphism from $(\tilde{\Sigma} \setminus \mathcal{L}_k, \tilde{h})$ into $(\bar{\Sigma} \setminus \{q^i\}_{i=1 \dots n}, \bar{c})$).

$(\tilde{\Sigma}, \tilde{h})$ is called the *nodal surface* of the converging sequence and $(\bar{\Sigma}, \bar{c})$ is its renormalization. These objects are related with another by the mean of a sequence of continuous maps ϕ_k from $(\tilde{\Sigma} \setminus \cup_i q^i, \tilde{h})$ into $(\Sigma \setminus \mathcal{L}_k, h_k)$ such that

- [1] ϕ_k realizes a C_{loc}^∞ diffeomorphism from $(\tilde{\Sigma} \setminus \cup_i q^i, \tilde{h})$ into $(\Sigma \setminus \mathcal{L}_k, h_k)$,
- [2] $\tilde{h}_k := \phi_k^* h_k$ converges in C_{loc}^∞ topology on $\tilde{\Sigma} \setminus \cup_i q^i$ to \tilde{h} .

Now, under the notations above, the aim of this section is to prove the following proposition.

Proposition II.1 *Let (Σ, c_k) be a sequence of closed riemann surface of fixed topology $g(\Sigma) > 1$ with degenerating conformal class c_k . And let $\vec{\Phi}_k$ be a sequence of weak immersions of Σ with L^2 -bounded second fundamental form (i.e. $\vec{\Phi}_k \in \mathcal{E}_\Sigma$). Then either*

$$\liminf_{k \rightarrow +\infty} \int_{\Sigma} |\vec{H}_{\vec{\Phi}_k}|^2 dvol_{\vec{\Phi}_k^* g_{\mathbb{R}^m}} \geq 8\pi \quad . \quad (\text{II.6})$$

or, if this is not true, then for any connected component σ of $\tilde{\Sigma}$ and there exists a subsequence still denoted $\vec{\Phi}_k$ and a subsequence of Moebius transformations Ξ_k of \mathbb{R}^m such that

$$\Xi_k \circ \vec{\Phi}_k(\Sigma_k) \subset B_R(0) \quad (\text{II.7})$$

moreover

$$\mathcal{H}^2(\Xi_k \circ \vec{\Phi}_k(\Sigma_k)) \leq C R^2 \sup_k W(\vec{\Phi}_k) \quad , \quad (\text{II.8})$$

where R and C only depend on m , and there exists at most finitely many points $\{a_1, \dots, a_N\}$ of σ such that, if we denote $\vec{\xi}_k := \Xi_k \circ \vec{\Phi}_k \circ \phi_k$, then

$$\vec{\xi}_k \rightharpoonup \vec{\xi}_\infty \quad \text{weakly in } W_{loc}^{2,2} \left(\sigma \cap \tilde{\Sigma} \setminus \{a_1, \dots, a_N\} \cup \{q^1, \dots, q^n\}, \tilde{h} \right) \quad (\text{II.9})$$

and moreover, for any compact $K \subset \sigma \cap \tilde{\Sigma} \setminus \{a_1 \cdots a_N\} \cup \{q^1, \dots, q_n\}$ there exists $C_K > 0$ such that

$$\sup_{k \in \mathbb{N}} \|\log |d\vec{\xi}_k|_{\phi_k^* h_k}\|_{L^\infty(K)} \leq C_K < +\infty \quad . \quad (\text{II.10})$$

and the limit $\vec{\xi}_\infty$ is an element of $\mathcal{E}_\sigma^{\bar{c}}$: it realizes a non constant weak conformal immersion of $(\sigma \cap \bar{\Sigma}, \bar{c})$ into $B_R(0)$. \square

Proof of proposition II.1.

We work of course under the assumption that

$$\int_{\Sigma} |\vec{H}_{\vec{\Phi}_k}|^2 d\text{vol}_{\vec{\Phi}_k^* g_{\mathbb{R}^m}} < 8\pi \quad (\text{II.11})$$

Let P_1, P_2 and P_3 be three distinct points in a given component $\tilde{\sigma}$ of $\tilde{\Sigma} \setminus \{q^1, \dots, q^n\}$. We apply the 3-points renormalization lemma A.1 to $\vec{\Phi}_k$ on Σ for the three points $\phi_k(P_1), \phi_k(P_2)$ and $\phi_k(P_3)$ and we obtain the existence of a Möbius transformation Ξ_k such that

$$\Xi_k \circ \vec{\Phi}_k(\Sigma_k) \subset B_R(0) \quad (\text{II.12})$$

moreover

$$\forall i \neq j \quad |\Xi_k \circ \vec{\Phi}_k \circ \phi_k(P_i) - \Xi_k \circ \vec{\Phi}_k \circ \phi_k(P_j)| \geq r > 0 \quad , \quad (\text{II.13})$$

and

$$\mathcal{H}^2(\Xi_k \circ \vec{\Phi}_k(\Sigma_k)) \leq C R^2 \sup_k W(\vec{\Phi}_k) \quad , \quad (\text{II.14})$$

where R, r and C are positive constants depending only on m since (II.11) holds.

We denote $\vec{\xi}_k$ the conformal immersion from $(\tilde{\Sigma}, \tilde{h}_k = \phi_k^* h_k)$ given by $\vec{\xi}_k := \Xi_k \circ \vec{\Phi}_k \circ \phi_k$. Let $\delta > 0$ be a small positive constant and denote by $\tilde{\Sigma}_\delta := \tilde{\Sigma} \setminus \cup_{j=1}^n B_\delta^{\tilde{h}}(q^j)$ where $B_\delta^{\tilde{h}}(q^j)$ is the geodesic ball in $(\tilde{\Sigma}, \tilde{h})$ of radius δ and center q^j . On $\tilde{\Sigma}_\delta$ there exists an L^∞ bounded smooth function l such that $\tilde{h} = e^l \bar{h}$.

To each $p \in \tilde{\Sigma}_\delta$ we assign $\rho_p > 0$ such that

$$\int_{B_{\rho_p}^{\tilde{h}}(p)} |d\vec{n}_{\vec{\xi}_k}|_{\tilde{h}_k}^2 d\text{vol}_{\tilde{h}_k} = \int_{B_{\rho_p}^{\bar{h}}(p)} |d\vec{n}_{\vec{\xi}_k}|_{g_k}^2 d\text{vol}_{g_k} = 8\pi/3 \quad ,$$

where $B_{\rho_p}^{\tilde{h}}(p)$ is the geodesic ball in $(\tilde{\Sigma}, \tilde{h})$ of center p and radius ρ_p and $g_k := \vec{\xi}_k^* g_{\mathbb{R}^m}$.

We extract a Besicovitch covering : each point in $\tilde{\Sigma}_\delta$ is covered by at most N of such balls where N only depends on $(\tilde{\Sigma}_\delta, \tilde{h})$. Since $\int_{\tilde{\Sigma}} |d\vec{n}|_{g_k}^2 d\text{vol}_{g_k}$ is uniformly bounded independent of k the number of balls in the extracted Besicovitch covering has to be uniformly bounded independently of k .

Let $(B_{\rho_k^i}(p_k^i))_{i \in I}$ be this finite covering (we shall omit now the superscript \tilde{h}). We can extract a subsequence such that I is independent of k , such that each p_k^i converges to a limit p_∞^i and each ρ_k^i converges to a limit ρ_∞^i . Let

$$I_0 := \{i \in I \quad \text{s. t.} \quad \rho_\infty^i = 0\} \quad .$$

Let $I_1 := I \setminus I_0$. It is clear that the union of the closures of the balls $\cup_{i \in I_1} \overline{B_{\rho_\infty^i}(p_\infty^i)}$ covers $\tilde{\Sigma}_\delta$. Because of the strict convexity of the balls (the metric \tilde{h} has constant curvature either 0 or -1) the points in $\tilde{\Sigma}_\delta$ which are not contained in the union of the open balls $\cup_{i \in I_1} B_{\rho_\infty^i}(p_\infty^i)$ cannot accumulate and therefore are isolated and hence finite. Denote

$$\{a_1 \cdots a_N\} := \tilde{\Sigma}_\delta \setminus \cup_{i \in I_1} B_{\rho_\infty^i}(p_\infty^i) \quad . \quad (\text{II.15})$$

We prove now that the following claim holds.

Claim 1. *On the connected component $\tilde{\sigma}$ of $\tilde{\Sigma}$ either there exists a subsequence k' such that for any compact $K \subset \tilde{\sigma} \cap \tilde{\Sigma}_\delta \setminus \{a_1 \cdots a_N\}$*

$$\sup_{k' \in \mathbb{N}} \|\log |d\vec{\xi}_{k'}|_{\tilde{h}_{k'}}\|_{L^\infty(K)} < +\infty \quad . \quad (\text{II.16})$$

or for any compact $K \subset \tilde{\sigma} \cap \tilde{\Sigma}_\delta \setminus \{a_1 \cdots a_N\}$

$$\log |d\vec{\xi}_k|_{\tilde{h}_k} \rightarrow -\infty \quad \text{uniformly on } K \quad (\text{II.17})$$

Proof of claim 1.

Let $i \in I_1$. For such an i we consider on the ball $B_{\rho_k^i}(p_k^i)$ conformal coordinates $(x_k^i, y_k^i) \in D^2$ for the metric \tilde{h}_k which converge in C^∞ -norm to conformal coordinates (x^∞, y^∞) for \tilde{h} (this is clearly possible since the sequence of metrics \tilde{h}_k converge to \tilde{h} in C^∞ norm). Denote by f_k^i the inverse of (x_k^i, y_k^i) (i.e. f_k^i is then a conformal diffeomorphism from D^2 into $(B_{\rho_k^i}(p_k^i), \tilde{h}_k)$). Due to conformal invariance one has

$$\int_{D^2} |\nabla \vec{n}_{\tilde{\xi}_k \circ f_k^i}|^2 dx dy = \int_{B_{\rho_k^i}^{\tilde{h}}(p_k^i)} |d\vec{n}|_{\tilde{h}_k}^2 d\text{vol}_{\tilde{h}_k} = \int_{B_{\rho_k^i}^{\tilde{h}}(p_k^i)} |d\vec{n}|_{g_k}^2 d\text{vol}_{g_k} = 8\pi/3$$

and therefore $\vec{\xi}_k \circ f_k^i$ satisfies all the assumptions of lemma A.2. We then apply lemma A.2 and since f_k^i converges in C^∞ -norm to some limiting conformal coordinates from D^2 into $B_{\rho_\infty^i}^{\tilde{h}}(p_\infty^i)$ we deduce the following alternative :

either there exists a subsequence k' such that for any compact $K \subset B_{\rho_\infty^i}^{\tilde{h}}(p_\infty^i)$

$$\limsup_{k' \in \mathbb{N}} \|\log |d\vec{\xi}_{k'}|_{\tilde{h}_{k'}}\|_{L^\infty(K)} < +\infty \quad . \quad (\text{II.18})$$

or for any compact $K \subset B_{\rho_\infty^i}^{\tilde{h}}(p_\infty^i)$

$$\log |d\vec{\xi}_k|_{\tilde{h}_k} \rightarrow -\infty \quad \text{uniformly on } K \quad (\text{II.19})$$

This alternative holds on any ball $B_{\rho_\infty^i}^{\tilde{h}}(p_\infty^i)$, it is then clear that, since none of this ball is "separated" from the others in a given connected component of $\tilde{\Sigma}$, the claim 1 is proved.

We are now going to prove the following claim :

Claim 2. *If the second alternative of claim 1, (II.17), holds on $\tilde{\sigma}$ then (II.6) holds :*

$$\liminf_{k \rightarrow +\infty} \int_{\Sigma} |\vec{H}_{\vec{\Phi}_k}|^2 d\text{vol}_{\vec{\Phi}_k^* g_{\mathbb{R}^m}} \geq 8\pi \quad .$$

Proof of claim 2.

Assuming the second alternative (II.17) of claim 1 implies that, modulo extraction of a subsequence, the complement of $\{a_1, \cdots, a_N\}$ in $\sigma \cap \tilde{\Sigma}_\delta$ is collapsing to a point $Q \in B_R(0)$.

Because of (II.13), among the three points P_1, P_2 and P_3 we need at least two P_j to coincide with two distinct blowing points a^i , say $P_1 = a^1$ and $P_2 = a^2$, and such that

$$\text{for } i = 1, 2 \quad \liminf_{k \rightarrow +\infty} |\vec{\xi}_k(P_i) - Q| \geq r/2 > 0 \quad (\text{II.20})$$

Let $s > 0$ such that the different balls $B_s^{\tilde{h}_k}(a_i)$ included in $\tilde{\Sigma}_\delta$ are disjoint. Consider $M_k^i := \tilde{\xi}_k(B_s^{\tilde{h}_k}(a^i))$ and take conformal coordinates $(x_k^i, y_k^i) \in D^2$ for the metric \tilde{h}_k which converge in C^∞ -norm to conformal coordinates (x^∞, y^∞) for \tilde{h} (this is clearly possible since the sequence of metrics \tilde{h}_k converge to \tilde{h} in C^∞ norm). Denote by f_k^i the inverse of (x_k^i, y_k^i) (i.e. f_k^i is then a conformal diffeomorphism from D^2 into $(B_s^{\tilde{h}_k}(a^i), \tilde{h}_k)$) We have in these coordinates that

$$\mathcal{H}^1(\partial M_k^i) = \int_{\partial D^2} e^{\lambda_k^i} d\theta \quad .$$

where $\lambda_k^i = \log |\partial_t heta \tilde{\xi}_k \circ f_k^i|$ combining the fact that s is fixed, and the fact that \tilde{h}_k converges in any C^l norm to a limiting metric \tilde{h} we have that ∂M_k^i stays contained in a compact subset K of $\tilde{\Sigma}_\delta \setminus \{a^1 \cdots a^N\}$. Since moreover f_k^i and it's inverse converge strongly in any C^l norm to a limiting conformal parametrization of $B_s^{\tilde{h}}(a^i)$ and since we are assuming that the second alternative (II.17) of claim 1 holds we deduce that

$$\lambda_k^i \longrightarrow -\infty \quad \text{uniformly on } \partial D^2 \quad .$$

Thus we have that

$$\lim_{k \rightarrow +\infty} \mathcal{H}^1(\partial M_k^i) = 0 \quad (\text{II.21})$$

Consider now the monotonicity formula (A.13) for M_k^1 (resp. M_k^2) and $\bar{x}^0 = \tilde{\xi}_k(P_1)$ (resp. $\bar{x}^0 = \tilde{\xi}_k(P_2)$). For a fixed k we make $t \rightarrow 0$. Since

$$\frac{1}{t^2} \int_{M_k^i \cap B_t(\bar{x}^0)} \langle \bar{x} - \bar{x}^0, \vec{H}_{\tilde{\xi}_k} \rangle dvol_{g_k} \leq (t^{-2} Area(M_k^i \cap B_t(\bar{x}^0)))^{1/2} W(\tilde{\xi}_k(\tilde{\Sigma}) \cap B_t(\bar{x}^0))$$

and since $t^{-2} Area(M_k^i \cap B_t(\bar{x}^0))$ is uniformly bounded (from the classical monotonicity formula without boundary) and since clearly $W(\tilde{\xi}_k(\tilde{\Sigma}) \cap B_t(\bar{x}^0)) \rightarrow 0$ we deduce that for a fixed k

$$\lim_{t \rightarrow +\infty} \frac{1}{t^2} \int_{M_k^i \cap B_t(\bar{x}^0)} \langle \bar{x} - \bar{x}^0, \vec{H}_{\tilde{\xi}_k} \rangle dvol_{g_k} = 0 \quad . \quad (\text{II.22})$$

Since $\bar{x}^0 = \tilde{\xi}_k(P_1)$ belongs to the interior of M_k^i , using again the classical monotonicity formula without boundary for $\tilde{\Phi}_k(\Sigma)$ we have that

$$\liminf_{t \rightarrow 0} t^{-2} Area(M_k^i \cap B_t(\bar{x}^0)) \geq \pi \quad (\text{II.23})$$

It is clear that

$$\lim_{t \rightarrow 0} -\frac{1}{2} \int_{\partial M_k^i \cap B_T(\bar{x}^0)} \frac{1}{\rho_t^2} \langle \bar{x} - \bar{x}^0, \vec{\nu} \rangle dl_{\partial M_k^i} = -\frac{1}{2} \int_{\partial M_k^i \cap B_T(\bar{x}^0)} \frac{1}{|\bar{x} - \bar{x}^0|^2} \langle \bar{x} - \bar{x}^0, \vec{\nu} \rangle dl_{\partial M_k^i} \quad (\text{II.24})$$

Hence by combining (II.22)...(II.24) and making $T \rightarrow +\infty$ in (A.13) gives

$$\frac{1}{4} \int_{M_k^i} |\vec{H}|^2 dvol_g \geq \pi - \frac{1}{2} \int_{\partial M_k^i} \frac{1}{|\bar{x} - \bar{x}^0|^2} \langle \bar{x} - \bar{x}^0, \vec{\nu} \rangle dl_{\partial M_k^i} \quad . \quad (\text{II.25})$$

Since for $i = 1, 2$ one has (II.20) and since the boundary of M_k^i is converging to Q , for k large enough one has

$$\text{for } i = 1, 2 \quad \forall \bar{x} \in \partial M_k^i \quad r/4 \leq |\bar{x} - \bar{x}^0| \leq 2R$$

Inserting this information in (II.25) gives

$$\text{for } i = 1, 2 \quad \frac{1}{4} \int_{M_k^i} |\vec{H}|^2 d\text{vol}_g \geq \pi - \frac{8R}{r^2} \mathcal{H}^1(\partial M_k^i) \quad (\text{II.26})$$

Since (II.21) holds and since M_k^1 and M_k^2 are disjoint we have proved (II.6) and claim 2 is proved.

Assuming now that (II.6) would not hold then there exists a subsequence k' such that for the chosen component σ of $\tilde{\Sigma}$ one has (II.10) : for any compact K in $\sigma \setminus \{a^1 \cdots a^N\} \cup \{q^1, \dots, q^n\}$ one has

$$\limsup_{k' \in \mathbb{N}} \|\log |d\vec{\xi}_{k'}|_{\tilde{h}_{k'}}\|_{L^\infty(K)} < +\infty \quad .$$

Since \tilde{h}_k strongly converges to \tilde{h} and since \tilde{h} and \bar{h} are comparable norms away from the punctures $\{q^1, \dots, q^n\}$, one has that for any compact K in $\sigma \setminus \{a^1 \cdots a^N\} \cup \{q^1, \dots, q^n\}$

$$\limsup_{k' \in \mathbb{N}} \|\log |d\vec{\xi}_{k'}|_{\bar{h}}\|_{L^\infty(K)} < +\infty \quad . \quad (\text{II.27})$$

Since the area of $\vec{\xi}_{k'}(\tilde{\Sigma})$ is uniformly bounded (see (II.14)), the $W^{1,2}$ norm of $\vec{\xi}_k$ is uniformly bounded with respect to the norm $g_k = \vec{\xi}_k^* g_{\mathbb{R}^m}$ as well as with respect to the norm \tilde{h}_k since these two metrics are conformally equivalent and one has

$$\int_{\tilde{\Sigma}} |d\vec{\xi}_{k'}|_{\tilde{h}_{k'}}^2 d\text{vol}_{\tilde{h}_{k'}} \leq CR^2 \sup_k W(\vec{\Phi}_k) \quad . \quad (\text{II.28})$$

Since \tilde{h}_k converges strongly to \tilde{h} away from the punctures one has that for any compact $K \subset \sigma \setminus \{q^1, \dots, q^n\}$

$$\left| \int_K |d\vec{\xi}_k|_{\tilde{h}_k}^2 - \int_K |d\vec{\xi}_k|_{\tilde{h}}^2 \right| \longrightarrow 0$$

And since \tilde{h} and \bar{h} are conformally equivalent one has

$$\left| \int_K |d\vec{\xi}_k|_{\tilde{h}_k}^2 - \int_K |d\vec{\xi}_k|_{\bar{h}}^2 \right| \longrightarrow 0 \quad (\text{II.29})$$

Combining (II.28) and (II.29) gives then that, modulo extraction of a subsequence $\vec{\xi}_k$ converges weakly in $W_{loc}^{1,2}(\sigma \setminus \{q^1, \dots, q^n\})$ to a limit $\vec{\xi}_\infty$ moreover

$$\int_{\tilde{\Sigma}} |d\vec{\xi}_\infty|_{\bar{h}}^2 d\text{vol}_{\bar{h}} \leq CR^2 \sup_k W(\vec{\Phi}_k) \quad . \quad (\text{II.30})$$

This last inequality implies that $\vec{\xi}_\infty \in W^{1,2}(\sigma, \bar{h})$. Since (II.27) holds, using very classical arguments of functional analysis - see for instance [Ri1] beginning of section VI.7.1 - one obtains the following fact

$$\vec{\xi}_k \rightharpoonup \vec{\xi}_\infty \quad \text{weakly in } W_{loc}^{2,2} \left(\sigma \setminus \{a^1, \dots, a^N\} \cup \{q^1, \dots, q^n\}, \tilde{h} \right) \quad (\text{II.31})$$

Moreover $\vec{\xi}_\infty$ is a weak immersion away from $\{a^1, \dots, a^N\} \cup \{q^1, \dots, q^n\}$ satisfying

$$\int_\sigma |d\vec{n}_{\vec{\xi}_\infty}|_{\bar{h}}^2 d\text{vol}_{\bar{h}} \leq \liminf_{k' \rightarrow +\infty} \int_\sigma |d\vec{n}_{\vec{\xi}_{k'}}|_{\tilde{h}_{k'}}^2 d\text{vol}_{\tilde{h}_{k'}} < +\infty \quad (\text{II.32})$$

Using now Lemma A.5 of [Ri3] permits to extend $\vec{\xi}_\infty$ as a conformal, possibly branched, immersion of σ . If there is one branched point, then, arguing as in [Ri3] end of section 3 we would obtain that

$$8\pi \leq W(\vec{\xi}_\infty(\sigma)) \leq \liminf_{k' \rightarrow +\infty} W(\vec{\xi}_{k'}(\sigma)) \leq \liminf_{k' \rightarrow +\infty} W(\vec{\Xi}_{k'} \circ \vec{\Phi}_{k'}(\Sigma)) = \liminf_{k' \rightarrow +\infty} W(\vec{\Phi}_{k'}(\Sigma))$$

which contradicts our assumption that (II.21) does not hold. Hence $\vec{\xi}_\infty$ extends to a weak conformal immersion with L^2 -bounded second fundamental form all over σ . \square

III Proof of theorem I.1.

Assume that the assumptions theorem I.1 are fulfilled and that (II.21) does not hold.

Let $(\sigma^j)_{j=1\dots M}$ be the connected components of $\tilde{\Sigma}$.

Consider the sequence $\vec{\xi}_k := \Xi_k^1 \circ \vec{\Phi}_k \circ \phi_k$ given by proposition II.1 for the component σ^1 . Let $\{a_1, \dots, a_N\}$ be the possible blow-up points for of $\vec{\xi}_k$ on σ^1 and let $(q^j)_{j \in J^1}$ be the punctures of (σ^1, \tilde{h}) and denote for any $\delta > 0$

$$\sigma_\delta^1 = \sigma^1 \setminus \cup_{i=1}^N B_\delta^{\tilde{h}}(a_i) \cup_{j \in J} B_\delta^{\tilde{h}}(q^j) \quad .$$

Because of the convergences (II.9) and (II.10) we have

$$\lim_{\delta \rightarrow 0} \liminf_{k \rightarrow +\infty} W(\vec{\xi}_k(\sigma_\delta^1)) \geq W(\vec{\xi}_\infty(\sigma^1)) \quad .$$

Observe first that

$$\forall j \neq j' \in J \quad \vec{\xi}_\infty(q^j) \neq \vec{\xi}_\infty(q^{j'}) \quad . \quad (\text{III.1})$$

Indeed if it would not be the case, using Li Yau inequality we would obtain

$$8\pi \leq W(\vec{\xi}_\infty(\sigma^1)) \leq \liminf_{k' \rightarrow +\infty} W(\vec{\xi}_{k'}(\sigma^1)) \leq \liminf_{k' \rightarrow +\infty} W(\vec{\Xi}_{k'} \circ \vec{\Phi}_{k'}(\Sigma)) = \liminf_{k' \rightarrow +\infty} W(\vec{\Phi}_{k'}(\Sigma))$$

Claim 1. *Assume there is a node q^{j_1} for $j_1 \in J^1$ such that $\tilde{\Sigma} \cup \cup_{j \neq j_1} q^j$ is not disconnected then we claim that (II.21) holds which contradict our assumption.*

Proof of claim 1. Because of (III.1) the image of the complement of σ^1 by $\vec{\xi}_k$ is connecting at the limit two distinct points of \mathbb{R}^m images of two punctures of σ^1 by $\vec{\xi}_\infty$.

For any $\delta > 0$ small enough, we cut the surface Σ in two separated surfaces with boundaries : in one hand we consider

$$\Sigma_\delta^+ := \phi_k(\sigma \setminus \cup_{j \in J^1} B_\delta^{\tilde{h}}(q^j)) \subset \Sigma \quad ,$$

and in the other hand we consider

$$\Sigma_\delta^- := \Sigma \setminus \Sigma_\delta^+ \quad .$$

We split the immersions $\Xi_k^1 \circ \vec{\Phi}_k$ according to this decomposition for some δ to be fixed later on. We observe that, since $\vec{\xi}_k$ converges weakly in $(W_{loc}^{1,\infty}(\sigma^1 \setminus [\{a_1 \dots a_N\} \cup \{q^j\}_{j \in J^1}]))^*$ we have that $\vec{\xi}_k$ converges strongly in $C_{loc}(\sigma^1 \setminus [\{a_1 \dots a_N\} \cup \{q^j\}_{j \in J^1}])$. Hence for any $\delta > 0$

$$\forall j \in J^1 \quad \lim_{k \rightarrow +\infty} \sup_{2^{-1}\delta < t < \delta} d(\vec{\xi}_k(\partial B_t^{\tilde{h}}(q^j)), \vec{\xi}_k(\partial B_t^{\tilde{h}}(q^j))) = 0 \quad (\text{III.2})$$

where d is the usual distance between two sets in \mathbb{R}^m defined in the introduction.

Since \tilde{h}_k is converging in C^∞ norm to the metric \tilde{h} in $B_\delta^{\tilde{h}}(q^j) \setminus B_{\delta/2}^{\tilde{h}}(q^j)$ which is itself conformally equivalent to the smooth constant scalar curvature metric \bar{h} in $B_\delta^{\tilde{h}}(q^j)$ We can construct conformal coordinates (x_k^j, y_k^j) for the metric \tilde{h}_k in $B_\delta^{\tilde{h}}(q^j) \setminus B_{\delta/2}^{\tilde{h}}(q^j)$ converging in C^∞ norm to some limiting (x_∞^j, y_∞^j) conformal coordinates for the metric \tilde{h} (and hence also for the metric \bar{h}). Denote f_k^j the inverse of (x_k^j, y_k^j) and f_∞^j it's limit. We can moreover choose (x_k^j, y_k^j) in such a way that (x_∞^j, y_∞^j) goes from $B_\delta^{\tilde{h}}(q^j) \setminus B_{\delta/2}^{\tilde{h}}(q^j)$ into some annulus in the plane $D_\delta^2 \setminus D_{c\delta}^2$ ($c > 0$ since \bar{h} is smooth around q^j and conformally equivalent to $(f_\infty^j)^{-1}(dx^2 + dy^2)$).

Denote $\lambda_k := \log |\partial_x(\vec{\xi}_k \circ f_k^j)| = \log |\partial_y(\vec{\xi}_k \circ f_k^j)|$. The uniform bound on the area (II.8) gives that

$$\int_{D_\delta^2 \setminus D_{c\delta}^2} e^{2\lambda_k} \leq C R^2 \quad . \quad (\text{III.3})$$

Since the convergence of $\vec{\xi}_k$ holds true also weakly in $W_{loc}^{2,2}(\sigma^1 \setminus \{a_1 \cdots a_N\} \cup \cup_{j \in J} \{q^j\}, \bar{h})$ and since (II.10) holds too, one has

$$\lim_{k \rightarrow +\infty} \int_{D_\delta^2 \setminus D_{c\delta}^2} |\lambda_k - \lambda_\infty|^2 = 0 \quad . \quad (\text{III.4})$$

The combination of (III.3) and (III.4) give that

$$\lim_{k \rightarrow +\infty} \int_{D_\delta^2 \setminus D_{c\delta}^2} |e^{\lambda_k} - e^{\lambda_\infty}| = 0 \quad . \quad (\text{III.5})$$

Hence there exists $\alpha_k \in (c, 1)$ such that

$$\lim_{k \rightarrow +\infty} \int_{\partial D_{\alpha_k \delta}^2} |e^{\lambda_k} - e^{\lambda_\infty}| = 0 \quad . \quad (\text{III.6})$$

Since $\vec{\xi}_\infty \in \mathcal{E}_\Sigma$, $\vec{\xi}_\infty \circ f_\infty^j \in W^{1,\infty}(D^2)$ and $e^{\lambda_\infty} \in L^\infty(D^2)$. Thus

$$\int_{D_{\alpha_k \delta}^2} e^{\lambda_\infty} = O(\delta) \quad (\text{III.7})$$

Combining (III.6) and (III.7) we obtain that there exists $C > 0$ such that for any $j \in J$ and for any $\delta > 0$ there exists $k_\delta \in \mathbb{N}$ verifying

$$\forall k \geq k_\delta \quad \exists \alpha_k^j \in (1/2, 1) \quad \text{s.t.} \quad \mathcal{H}^1(\vec{\xi}_k(\partial B_{\alpha_k^j \delta}^{\bar{h}})) \leq C \delta \quad (\text{III.8})$$

Since there is a node q^{j_1} for $j_1 \in J^1$ such that $\tilde{\Sigma} \cup \cup_{j \neq j_1} q^j$ is not disconnected and since (III.1) holds, there must exist a sequence of point $p_k \in \Sigma_{\delta/2}^-$ such that

$$\liminf_{k \rightarrow +\infty} \inf_{j \in J^1} |p_k - q^j| > 0$$

Cutting now Σ into at least two disjoint connected components by removing the curves $\partial B_{\alpha_k^j \delta}^{\bar{h}}$. Applying twice the monotonicity formula with boundary lemma A.3 with $T \rightarrow +\infty$, $t \rightarrow 0$ and with respect respectively to an arbitrary fixed point $p \in \Sigma_{\delta/2}^+$ for the immersion $\vec{\xi}_k$ restricted to $\Sigma_{\delta/2}^+$ and with respect to p_k for the immersion $\vec{\xi}_k$ restricted to $\Sigma_{\delta/2}^-$ one obtains

$$\liminf_{k \rightarrow +\infty} W(\vec{\Phi}_k) = \liminf_{k \rightarrow +\infty} W(\Xi_k^1 \circ \vec{\Phi}_k) \geq \liminf_{k \rightarrow +\infty} W(\vec{\xi}_k(\Sigma_\delta^+)) + W(\vec{\xi}_k(\Sigma_\delta^-)) \geq 4\pi + 4\pi - C \delta = 8\pi - C \delta \quad .$$

Since this holds for any $\delta > 0$ we have then (II.21) which contradicts our assumption and we have proved claim 1. \square

Denote by $g(\sigma^i)$ the genus of each component σ^i of $\tilde{\Sigma}$, if each q^j for $j = 1 \cdots n$ is disconnecting $\tilde{\Sigma}$ the Deligne-Mumford's description of the loss of compactness in the Moduli space gives

$$g(\Sigma) = \sum_{j=1}^M g(\sigma^j) \quad . \quad (\text{III.9})$$

Starting now from this normalized sequence $\Xi_k^1 \circ \Phi_k$ for the first component σ^1 , because of the convergences (II.9) and (II.10) we have

$$\lim_{\delta \rightarrow 0} \liminf_{k \rightarrow +\infty} W(\vec{\xi}_k(\sigma_\delta^1)) \geq W(\vec{\xi}_\infty(\sigma^1)) \geq \beta_{g(\sigma^1)} \quad .$$

For each of the other components σ^i , $i \neq 1$, working under the condition that (II.21) does not hold, proposition II.1 gives for each i the existence of a subsequence - still denoted ξ_k - and the existence of a sequence of Moebius transformations such that $\vec{\Xi}_k^i \circ \vec{\Xi}_k^1 \circ \vec{\Phi}_k$ converges in the sense (II.9) and (II.10). Taking subsequences of subsequences we can assume that the subsequence we are working with is common to all the σ^i . As for σ^1 we have for any i

$$\lim_{\delta \rightarrow 0} \liminf_{k \rightarrow +\infty} W(\vec{\Xi}_k^i \circ \vec{\xi}_k(\sigma_\delta^i)) \geq W(\vec{\xi}_i^\infty) \geq \beta_{g(\sigma^i)} \quad .$$

where $\vec{\xi}_i^\infty$ is the weak local limit of $\vec{\Xi}_k^i \circ \vec{\xi}_k$ on σ^i minus the blow up points and the punctures. Using now lemma A.4 we deduce that for any $i \neq 1$

$$\lim_{\delta \rightarrow 0} \liminf_{k \rightarrow +\infty} W(\vec{\xi}_k(\sigma_\delta^i)) \geq W(\vec{\xi}_i^\infty) - 4\pi \geq \beta_{g(\sigma^i)} - 4\pi \quad .$$

summing over i gives that

$$\begin{aligned} \liminf_{k \rightarrow +\infty} W(\vec{\Phi}_k) &= \liminf_{k \rightarrow +\infty} W(\vec{\Xi}_k^1 \circ \vec{\Phi}_k) \\ &\geq \lim_{\delta \rightarrow 0} \liminf_{k \rightarrow +\infty} \sum_{i=1}^M W(\vec{\xi}_k(\sigma_\delta^i)) \geq \beta_{g(\sigma^1)} + \sum_{i=1}^M (\beta_{g(\sigma^i)} - 4\pi) \quad . \end{aligned}$$

This concludes the proof of theorem I.1. □

A Appendix

Lemma A.1 [3-points normalization lemma] *For any $\Lambda > 0$ there exists $R, r > 0$ such that for any closed two dimensional manifold Σ , for any choice of 3 distinct points P_1, P_2 and P_3 in Σ and for any embedding $\vec{\Phi}$ of Σ into \mathbb{R}^m satisfying*

$$\int_{\Sigma} |d\vec{n}_{\vec{\Phi}}|_g^2 dvol_g < \Lambda \quad , \quad (\text{A.1})$$

where $g := \vec{\Phi}^* g_{\mathbb{R}^m}$, then there exists a Moebius transformation Ξ of \mathbb{R}^m such that

$$\Xi \circ \vec{\Phi}(\Sigma) \subset B_R(0) \quad \text{and} \quad \forall i \neq j \quad |\Xi \circ \vec{\Phi}(P_i) - \Xi \circ \vec{\Phi}(P_j)| \geq r \quad . \quad (\text{A.2})$$

Moreover the following control of the total area of $\Xi \circ \vec{\Phi}(\Sigma)$ holds

$$\mathcal{H}^2(\Xi \circ \vec{\Phi}(\Sigma)) \leq C R^2 \Lambda \quad , \quad (\text{A.3})$$

where $C > 0$ is a universal constant. □

Proof of lemma A.1. We apply a translation and a dilation in such a way that $P_1 = 0$ and

$$|\vec{\Phi}(P_1) - \vec{\Phi}(P_2)| = \min_{i \neq j} |\vec{\Phi}(P_i) - \vec{\Phi}(P_j)| = 1 \quad .$$

We keep denoting $\vec{\Phi}$ the resulting embedding - observe that due to it's conformal invariance the Willmore energy has not been modified. From lemma A.3 there exists a universal constant $C > 0$ such that for any $x_0 \in \mathbb{R}^m$ and $0 < \sigma < \rho < +\infty$

$$\sigma^{-2} \mathcal{H}^2(\vec{\Phi}(\Sigma) \cap B_\sigma(x_0)) \leq C \left[\rho^{-2} \mathcal{H}^2(\vec{\Phi}(\Sigma) \cap B_\rho(x_0)) + \int_{\vec{\Phi}^{-1}(B_\rho(x_0))} |\vec{H}|^2 dvol_g \right] . \quad (\text{A.4})$$

We claim that there exists ρ_0 depending only on $\Lambda > W(\vec{\Phi})$ and $x_1 \in B_1(0)$ such that $\vec{\Phi}(\Sigma) \cap B_{\rho_0}(x_1) = \emptyset$. For $y \in \vec{\Phi}(\Sigma)$ one has

$$\lim_{\sigma \rightarrow 0} \sigma^{-2} \mathcal{H}^2(\vec{\Phi}(\Sigma) \cap B_\sigma(y)) = \pi \quad . \quad (\text{A.5})$$

For $0 < \rho < 1/2$ we consider a regular covering of $B_1(0)$ by balls $B_\rho(z_l)$ in such a way that any point in $B_1(0)$ is contained in at most $C(m)$ balls of the form $B_{2\rho}(z_l)$. The number of l such that $\int_{\vec{\Phi}^{-1}(B_{2\rho}(z_l))} |\vec{H}|^2 d\text{vol}_g > C^{-1} \pi/2$ is bounded by $2 \Lambda C C(m)$. For an l such that $\int_{\vec{\Phi}^{-1}(B_{2\rho}(z_l))} |\vec{H}|^2 d\text{vol}_g < C^{-1} \pi/2$ and such that there exists $y \in B_\rho(z_l) \cap \Sigma \neq \emptyset$, combining (A.4) and (A.5) one obtains that

$$(2\rho)^{-2} \mathcal{H}^2(\vec{\Phi}(\Sigma) \cap B_{2\rho}(z_l)) > C^{-1} \pi/2 \quad .$$

the number of such l is then bounded by ρ^{-2} times a number depending only on m and Λ - where we are using again (A.4) but for $x_0 = 0$, $\sigma = 1$ and $\rho \rightarrow +\infty$. The total number of ball $B_\rho(z_l)$ is proportional to ρ^{-m} . Since $m > 2$, for $\rho = \rho_0$ chosen small enough, depending only on m and Λ we deduce the claim.

Let x_1 and ρ_0 given by the claim we choose Ξ to be the inversion with respect to x_1 : $\Xi(x) := (x - x_1)/|x - x_1|^2$. We have then

$$\Xi(\vec{\Phi}(\Sigma)) \subset B_{1/\rho_0}(0) \quad , \quad (\text{A.6})$$

moreover, since none of the P_i is in $B_{\rho_0}(x_1)$, we have that $\forall i = 1, 2, 3 \Xi(\vec{\Phi}(P_i)) \in B_{1/\rho_0}(0)$. We have also that $|\vec{\Phi}(P_2) - x_1| + |\vec{\Phi}(P_1) - x_1| < 3$ hence $\Xi(\vec{\Phi}(P_1))$ and $\Xi(\vec{\Phi}(P_2))$ are contained in $\mathbb{R}^m \setminus B_{1/3}(0)$ thus

$$|\Xi(\vec{\Phi}(P_1)) - \Xi(\vec{\Phi}(P_2))| \|\nabla \Xi^{-1}\|_{L^\infty(B_{1/\rho_0}(0) \setminus B_{1/3}(0))} \geq |\vec{\Phi}(P_1) - \vec{\Phi}(P_2)| = 1 \quad ,$$

which implies that

$$|\Xi(\vec{\Phi}(P_1)) - \Xi(\vec{\Phi}(P_2))| \geq 9 \quad . \quad (\text{A.7})$$

Either $\vec{\Phi}(P_3) \in B_{10}(0)$ or $\vec{\Phi}(P_3) \in \mathbb{R}^m \setminus B_{10}(0)$. In the first case one has that all the $\Xi(\vec{\Phi}(P_i))$ are included in $B_{1/\rho_0}(0) \setminus B_{1/11}(0)$ hence we have

$$\begin{aligned} \forall i \neq j \quad & |\Xi(\vec{\Phi}(P_i)) - \Xi(\vec{\Phi}(P_j))| \|\nabla \Xi^{-1}\|_{L^\infty(B_{1/\rho_0}(0) \setminus B_{1/11}(0))} \\ & \geq |\vec{\Phi}(P_i) - \vec{\Phi}(P_j)| \geq 1 \end{aligned}$$

which implies that

$$\forall i \neq j \quad |\Xi(\vec{\Phi}(P_i)) - \Xi(\vec{\Phi}(P_j))| \geq 11^2 \quad . \quad (\text{A.8})$$

This implies (A.2) in this case. In the case when $\vec{\Phi}(P_3) \in \mathbb{R}^m \setminus B_{10}(0)$ we deduce that $\Xi(\vec{\Phi}(P_3)) \in B_{1/9}(0)$ and since $\Xi(\vec{\Phi}(P_1))$ and $\Xi(\vec{\Phi}(P_2))$ are contained in $\mathbb{R}^m \setminus B_{1/3}(0)$, we obtain that

$$\forall i = 1, 2 \quad |\Xi(\vec{\Phi}(P_i)) - \Xi(\vec{\Phi}(P_3))| \geq 2/9 \quad . \quad (\text{A.9})$$

This lower bound combined with (A.7) gives (A.2) in this case too.

Regarding the proof of estimate (A.3), we first observe that inequality (A.4) (which holds also for $\vec{\Phi}$ replaced by $\Xi \circ \vec{\Phi}$) implies that , for any $\rho \geq \rho_0^{-1}$

$$\mathcal{H}^2(\Xi \circ \vec{\Phi}(\Sigma)) \leq C \rho_0^{-2} \rho^{-2} \mathcal{H}^2(\Xi \circ \vec{\Phi}(\Sigma)) + \rho_0^{-2} \Lambda \quad .$$

Letting ρ converge to $+\infty$ yields the desired estimate (A.3). Hence lemma A.1 is proved. \square

The following lemma is more or less implicitly contained in [MS] and [Hel]. We prove it however for the convenience of the reader.

Lemma A.2 Let $\vec{\Phi}_k$ be a sequence of conformal immersions of the disc D^2 into \mathbb{R}^m such that

$$\sup_{k \in \mathbb{N}} \int_{D^2} |\nabla \vec{n}_{\vec{\Phi}_k}|^2 dx dy \leq 8\pi/3 \quad , \quad (\text{A.10})$$

and

$$\limsup_{k \rightarrow +\infty} \int_{D^2} e^{2\lambda_k} dx dy < +\infty, \quad (\text{A.11})$$

where $e^{\lambda_k} = |\partial_x \vec{\Phi}_k| = |\partial_y \vec{\Phi}_k|$.

Then the following alternative holds : either

$$\forall \omega \subset\subset D^2 \quad \lim_{k \rightarrow +\infty} \lambda_k = -\infty \quad \text{unif. on } \omega$$

or there exists a subsequence k' such that

$$\forall \omega \subset\subset D^2 \quad \limsup_{k' \rightarrow +\infty} \|\lambda_{k'}\|_{L^\infty(\omega)} < +\infty$$

□

Proof of lemma A.2. Since (A.10) holds, lemma 5.1.4 of [Hel] gives the existence of a moving frame $(\vec{e}_1, \vec{e}_2) \in (W^{1,2}(D^2, S^{m-1}))^2$ such that

$$\int_{D^2} |\nabla \vec{e}_1|^2 + |\nabla \vec{e}_2|^2 \leq C \int_{D^2} |\nabla \vec{n}_{\vec{\Phi}_k}|^2 dx dy \leq 8\pi/3 \quad ,$$

where $C > 0$ only depends on m and

$$\star \vec{n}_{\vec{\Phi}} = \vec{e}_1 \wedge \vec{e}_2 \quad .$$

Moreover λ_k satisfies

$$\Delta \lambda_k = (\nabla^\perp \vec{e}_1, \nabla \vec{e}_2) \quad .$$

Let μ_k be the solution of

$$\begin{cases} \Delta \mu_k = (\nabla^\perp \vec{e}_1, \nabla \vec{e}_2) & \text{in } D^2 \\ \mu_k = 0 & \text{on } \partial D^2 \end{cases}$$

Wente theorem (see theorems 3.1.2 and 3.1.9 in [Hel]) asserts that

$$\|\mu_k\|_{L^\infty(D^2)} \leq (2\pi)^{-1} \|\nabla \vec{e}_1\|_{L^2} \|\nabla \vec{e}_2\|_{L^2} \leq \pi^{-1} C \int_{D^2} |\nabla \vec{n}_{\vec{\Phi}_k}|^2 dx dy \quad (\text{A.12})$$

Hence μ_k is uniformly bounded in L^∞ norm on D^2 .

The function $\nu_k := \lambda_k - \mu_k$ is harmonic on D^2 and satisfies

$$\limsup_{k \rightarrow +\infty} \int_{D^2} e^{2\nu_k} dx dy < +\infty$$

Let ω be an open set strictly included in D^2 (i.e. $\bar{\omega} \subset D^2$) and let U be an open set strictly included in D^2 and such that ω is itself strictly included in U : $\omega \subset\subset U \subset\subset D^2$. Since $e^{2\nu_k}$ is subharmonic ($\Delta e^{2\nu_k} \geq 0$) and positive, from Harnack inequality there exists a constant C depending on U such that

$$\nu_k^+ = \sup_U \nu_k \leq \log \left[C \int_{D^2} e^{2\nu_k} \right] \quad .$$

So ν_k^+ is uniformly bounded from above. If

$$\nu_k^+ \longrightarrow -\infty$$

then the first alternative in the lemma holds for this special ω . Assuming in the contrary that

$$\limsup \nu_k^+ > -\infty \quad ,$$

then there exists a subsequence k' such that

$$\nu_{k'}^+ \longrightarrow \nu_\infty^+ \in \mathbb{R} \quad .$$

Consider now the sequence of positive harmonic functions $\nu_{k'}^+ - \nu_{k'}$ on U , Harnack inequality again gives the existence of a constant $C > 0$ independent of k' such that

$$\sup_\omega \nu_{k'}^+ - \nu_{k'} \leq C \inf_\omega \nu_{k'}^+ - \nu_{k'} \quad .$$

It is clear also that the supremum of $\nu_{k'}$ is bounded from above on ω therefore

$$\limsup_{k' \rightarrow +\infty} \inf_\omega \nu_{k'}^+ - \nu_{k'} < +\infty \quad .$$

Combining the two last inequalities gives that

$$\liminf \inf_\omega \nu_{k'} > +\infty \quad .$$

Thus the second alternative of the lemma holds for this special ω .

So we have proved that for any strict sub-domain of D^2 one of the two alternative always hold. Now it is clear that if for some sub-domain of D^2 the first alternative hold then the second cannot hold for another sub-domain and vice versa. Thus the lemma A.2 is proved. \square

The following lemma is the extension of Simon's monotonicity formula⁴ in the presence of a boundary. This lemma might have been already published somewhere but the author could not find it anywhere and is making the computation related to it available to the reader.

Lemma A.3 [Monotonicity formula with boundary] *Let Σ be a smooth compact surface with boundary and let $\vec{\Phi}$ be an element of \mathcal{E}_Σ , a Lipschitz immersion of the surface Σ into \mathbb{R}^m with L^2 bounded second fundamental form. Denote by $M := \vec{\Phi}(\Sigma)$ the immersed surface. Then for any point $\vec{x}^0 \in \mathbb{R}^m$ and any choice of two radii $0 < t < T < +\infty$ the following identity holds*

$$\begin{aligned} & T^{-2} \text{Area}(M \cap B_T(\vec{x}^0)) - t^{-2} \text{Area}(M \cap B_t(\vec{x}^0)) \\ &= \int_{M \cap B_T(\vec{x}^0) \setminus B_t(\vec{x}^0)} \left| \frac{(\vec{x} - \vec{x}^0)^\perp}{|\vec{x} - \vec{x}^0|^2} + \frac{\vec{H}}{2} \right|^2 d\text{vol}_g - \frac{1}{4} \int_{M \cap B_T(\vec{x}^0) \setminus B_t(\vec{x}^0)} |\vec{H}|^2 d\text{vol}_g \\ & - \frac{1}{T^2} \int_{M \cap B_T(\vec{x}^0)} \langle \vec{x} - \vec{x}^0, \vec{H} \rangle d\text{vol}_g + \frac{1}{t^2} \int_{M \cap B_t(\vec{x}^0)} \langle \vec{x} - \vec{x}^0, \vec{H} \rangle d\text{vol}_g \\ & + \frac{1}{2} \int_{\partial M \cap B_T(\vec{x}^0)} \left(\frac{1}{T^2} - \frac{1}{\rho_t^2} \right) \langle \vec{x} - \vec{x}^0, \vec{\nu} \rangle dl_{\partial M} \end{aligned} \tag{A.13}$$

where $(\vec{x} - \vec{x}^0)^\perp$ is the orthogonal projection of the vector $\vec{x} - \vec{x}^0$ onto $(T_{\vec{x}}M)^\perp$ the normal plane to the surface at \vec{x} and $\rho_t := \max\{|\vec{x} - \vec{x}^0|, t\}$. \square

⁴(A.13) is exactly the formula as it is written in [KS], formula (A.3) page 353, except that the boundary term was not considered in this work.

Proof of lemma A.2. To simplify the presentation we give the argument for $\Sigma = D^2$. We follow step by step the computations in [Sim] pages 82-84.

Let \vec{X} be a smooth vector-field in \mathbb{R}^m . We define the divergence of \vec{X} along $M := \vec{\Phi}(D^2)$ to be the following quantity

$$div_M \vec{X} := \sum_{k=1}^2 \left\langle d\vec{X} \cdot \vec{e}_k, \vec{e}_k \right\rangle . \quad (\text{A.14})$$

where (\vec{e}_1, \vec{e}_2) is an arbitrary local orthonormal frame on M . Decomposing \vec{X} along M in the sum of it's tangential $\vec{X}^T := \sum_{k=1}^2 \langle \vec{X}, \vec{e}_k \rangle \vec{e}_k$ and it's vertical part $\vec{X}^\perp = \vec{X} - \vec{X}^T$ gives⁵

$$\begin{aligned} div_M \vec{X} &= div_M \vec{X}^T + div_M \vec{X}^\perp = \sum_{k=1}^2 \left\langle d\vec{X}^T \cdot \vec{e}_k, \vec{e}_k \right\rangle - \left\langle \vec{X}^\perp, \sum_{k=1}^2 d\vec{e}_k \cdot \vec{e}_k \right\rangle \\ &= \sum_{k=1}^2 \left\langle d\vec{X}^T \cdot \vec{e}_k, \vec{e}_k \right\rangle - 2 \langle \vec{X}, \vec{H} \rangle . \end{aligned} \quad (\text{A.15})$$

We can assume that $\vec{\Phi}$ is conformal since such a conformal reparametrization always exists (see [Ril]). Denote $e^\lambda := |\partial_{x_1} \vec{\Phi}| = |\partial_{x_2} \vec{\Phi}|$ and take $\vec{e}_k := e^{-\lambda} \partial_{x_k} \vec{\Phi}$. Write $\vec{X}^T := \sum_{i=1}^2 X_i \partial_{x_i} \vec{\Phi}$. We have

$$div_M \vec{X}^T = \sum_{k=1}^2 \left\langle d\vec{X} \cdot \vec{e}_k, \vec{e}_k \right\rangle = \sum_{i=1}^2 \frac{\partial X_i}{\partial x_i} + \sum_{i=1}^2 X_i e^{-2\lambda} \partial_{x_i} e^{2\lambda} . \quad (\text{A.16})$$

Hence

$$\begin{aligned} \int_M div_M \vec{X}^T dvol_g &= \int_{D^2} \sum_{i=1}^2 \frac{\partial}{\partial x_i} (e^{2\lambda} X_i) dx_1 dx_2 = \int_0^{2\pi} e^{2\lambda} \sum_{i=1}^2 X_i x_i d\theta \\ &= \int_{\partial M} \langle \vec{X}^T, \vec{\nu} \rangle dl_{\partial M} = \int_{\partial M} \langle \vec{X}, \vec{\nu} \rangle dl_{\partial M} \end{aligned} \quad (\text{A.17})$$

where $\vec{\nu}$ is the unit limiting tangent vector to M on ∂M orthogonal to it and oriented in the outward direction : $\vec{\nu} := e^{-\lambda} \partial_r \vec{\Phi}$. Combining (A.15) and (A.17) gives then

$$\int_M div_M \vec{X} dvol_g = \int_{\partial M} \langle \vec{X}, \vec{\nu} \rangle dl_{\partial M} - 2 \int_M \langle \vec{X}, \vec{H} \rangle dvol_g . \quad (\text{A.18})$$

As in [Sim] we choose $\vec{X} := \gamma(\rho)(\vec{x} - \vec{x}^0)$ where \vec{x}^0 is an arbitrary point in \mathbb{R}^m and $\rho = |\vec{x} - \vec{x}^0|$. We have for this choice of \vec{X}

$$div_M \vec{X} = 2\gamma + \dot{\gamma} \sum_{k=1}^2 dr \cdot \vec{e}_k \langle \vec{x} - \vec{x}^0, \vec{e}_k \rangle = 2\gamma + \rho \dot{\gamma} \left[1 - \frac{|(\vec{x} - \vec{x}^0)^\perp|^2}{|\vec{x} - \vec{x}^0|^2} \right] .$$

We choose now $\gamma(\rho)$ to be a function depending on a parameter $s > 0$: $\gamma_s(\rho) = \varphi(\rho/s)$ where later on φ will be chosen to be closer and closer to the characteristic function of the unit interval $[0, 1]$. (A.18) becomes then

$$\begin{aligned} &2 \int_M \varphi\left(\frac{\rho}{s}\right) dvol_g + \int_M \dot{\varphi}\left(\frac{\rho}{s}\right) \frac{\rho}{s} \left[1 - \frac{|(\vec{x} - \vec{x}^0)^\perp|^2}{|\vec{x} - \vec{x}^0|^2} \right] dvol_g \\ &= \int_{\partial M} \varphi\left(\frac{\rho}{s}\right) \langle \vec{x} - \vec{x}^0, \vec{\nu} \rangle dl_{\partial M} - 2 \int_M \varphi\left(\frac{\rho}{s}\right) \langle \vec{x} - \vec{x}^0, \vec{H} \rangle dvol_g . \end{aligned} \quad (\text{A.19})$$

⁵In comparison with formula in [Sim] observe that our definition of the mean curvature vector differs by a factor 2.

Observe that

$$\frac{d}{ds} \left[\varphi \left(\frac{\rho}{s} \right) \frac{1}{s^2} \right] = -\frac{1}{s^3} \left[2\varphi \left(\frac{\rho}{s} \right) + \dot{\varphi} \left(\frac{\rho}{s} \right) \frac{\rho}{s} \right]$$

Hence we deduce that

$$\begin{aligned} -\frac{d}{ds} \left[\frac{1}{s^2} \int_M \varphi \left(\frac{\rho}{s} \right) dvol_g \right] &= -\frac{1}{s^2} \frac{d}{ds} \left[\int_M \varphi \left(\frac{\rho}{s} \right) \frac{|(\vec{x} - \vec{x}^0)^\perp|^2}{|\vec{x} - \vec{x}^0|^2} dvol_g \right] \\ &+ \frac{1}{s^3} \int_{\partial M} \varphi \left(\frac{\rho}{s} \right) \langle \vec{x} - \vec{x}^0, \vec{\nu} \rangle dl_{\partial M} - \frac{2}{s^3} \int_M \varphi \left(\frac{\rho}{s} \right) \langle \vec{x} - \vec{x}^0, \vec{H} \rangle dvol_g \end{aligned} \quad (\text{A.20})$$

Taking φ closer and closer to the characteristic function of the unit interval $[0, 1]$, (A.20) implies at the limit the following formula⁶

$$\begin{aligned} \frac{d}{ds} \left[\frac{1}{s^2} \int_{M \cap B_s(\vec{x}^0)} dvol_g \right] &= \frac{d}{ds} \left[\int_{M \cap B_s(\vec{x}^0)} \frac{|(\vec{x} - \vec{x}^0)^\perp|^2}{|\vec{x} - \vec{x}^0|^4} dvol_g \right] \\ &- \frac{1}{s^3} \int_{\partial M \cap B_s(\vec{x}^0)} \langle \vec{x} - \vec{x}^0, \vec{\nu} \rangle dl_{\partial M} + \frac{2}{s^3} \int_{M \cap B_s(\vec{x}^0)} \langle \vec{x} - \vec{x}^0, \vec{H} \rangle dvol_g \end{aligned} \quad (\text{A.21})$$

Integrating this formula between $0 < t < T < +\infty$ gives

$$\begin{aligned} &T^{-2} \text{Area}(M \cap B_T(\vec{x}^0)) - t^{-2} \text{Area}(M \cap B_t(\vec{x}^0)) \\ &= \int_{M \cap B_T(\vec{x}^0) \setminus B_t(\vec{x}^0)} \frac{|(\vec{x} - \vec{x}^0)^\perp|^2}{|\vec{x} - \vec{x}^0|^4} dvol_g \\ &+ \frac{1}{2} \int_{\partial M \cap B_T(\vec{x}^0)} \left(\frac{1}{T^2} - \frac{1}{t^2} \right) \langle \vec{x} - \vec{x}^0, \vec{\nu} \rangle dl_{\partial M} \\ &+ \int_{M \cap B_T(\vec{x}^0)} \left(\frac{1}{\rho_t^2} - \frac{1}{T^2} \right) \langle \vec{x} - \vec{x}^0, \vec{H} \rangle dvol_g \end{aligned} \quad (\text{A.22})$$

where $\rho_t(\vec{x}) := \max \{ \rho(\vec{x}) = |\vec{x} - \vec{x}^0|, t \}$. We write

$$\begin{aligned} \int_{M \cap B_T(\vec{x}^0) \setminus B_t(\vec{x}^0)} \frac{|(\vec{x} - \vec{x}^0)^\perp|^2}{|\vec{x} - \vec{x}^0|^4} dvol_g &= \int_{M \cap B_T(\vec{x}^0) \setminus B_t(\vec{x}^0)} \left| \frac{(\vec{x} - \vec{x}^0)}{\rho^2} + \frac{\vec{H}}{2} \right|^2 dvol_g \\ &- \frac{1}{4} \int_{M \cap B_T(\vec{x}^0) \setminus B_t(\vec{x}^0)} |\vec{H}|^2 - \int_{M \cap B_T(\vec{x}^0) \setminus B_t(\vec{x}^0)} \frac{\langle \vec{x} - \vec{x}^0, \vec{H} \rangle}{\rho^2} dvol_g \end{aligned} \quad (\text{A.23})$$

Combining (A.22) and (A.23) gives (A.13) and lemma A.3 is proved. \square

The following lemma is more or less contained in previous works on the subject but, for the convenience of the reader.

Lemma A.4 *Let (Σ, c) be a connected closed riemann surface. Let $a_1 \cdots a_N$ be a finite family of points in the surface. Let \bar{h} be some smooth conformal metric on Σ and denote $\Sigma_\delta := \Sigma \setminus \cup_{i=1}^N B_\delta^{\bar{h}}(a_i)$ where $B_\delta^{\bar{h}}(a_i)$ is the geodesic ball of center a_i and radius δ for the metric \bar{h} . Assume there exists a weak conformal lipshitz immersion $\vec{\xi}_\infty$ of Σ into \mathbb{R}^m with L^2 -bounded second fundamental form (i.e. $\vec{\xi}_\infty$ is an element*

⁶(A.21) generalizes in dimension 2 the formula 17.3 page 84 of [Sim].

of \mathcal{E}_Σ^c) and assume there exists a sequence of weak conformal lipshitz immersion $\vec{\xi}_k$ of Σ_δ into \mathbb{R}^m (for any $\delta > 0$ provided k is large enough) such that

$$\forall \delta > 0 \quad \vec{\xi}_k \longrightarrow \vec{\xi}_\infty \quad \text{weakly in } W^{2,2} \cap (W^{1,\infty})^*(\Sigma_\delta) \quad ,$$

and

$$\forall \delta > 0 \quad \limsup_{k \rightarrow +\infty} \|\log |d\vec{\xi}_k|_{\bar{h}}\|_{L^\infty(\Sigma_\delta)}$$

where the different norms $W^{2,2}$, $W^{1,\infty}$ and L^∞ are taken with respect to the \bar{h} metric. We assume $\vec{\xi}_\infty(a_i) \neq \vec{\xi}_\infty(a_j)$ for $i \neq j$. Let x_k be a sequence of points such that $x_k \notin \vec{\xi}_k(\Sigma_\delta)$ for any $\delta > 0$ for k large enough and converging to $\vec{\xi}_\infty(a_1)$. Denote by I_{x_k} the inversion with respect to the point x_k : $I_{x_k}(x) := x_k + (x - x_k)/|x - x_k|^2$. Then the following inequality holds

$$\lim_{\delta \rightarrow 0} \liminf_{k \rightarrow +\infty} W(I_{x_k} \circ \vec{\xi}_k(\Sigma_\delta)) \geq W(\vec{\xi}_\infty) - 4\pi \quad . \quad (\text{A.24})$$

Assume now that the sequence of points either diverges to ∞ or converges to a point $x_\infty \notin \vec{\xi}_\infty(\Sigma)$ then

$$\lim_{\delta \rightarrow 0} \liminf_{k \rightarrow +\infty} W(I_{x_k} \circ \vec{\xi}_k(\Sigma_\delta)) \geq W(\vec{\xi}_\infty) \quad . \quad (\text{A.25})$$

□

Proof of lemma A.4. Denote $g_k := \vec{\xi}_k^* g_{\mathbb{R}^m}$ the metric on Σ_δ induced by the immersion $\vec{\xi}_k$. Since I_{x_k} is a conformal diffeomorphism from $\mathbb{R}^m \cup \{\infty\}$ into itself, the induced metric $\tilde{g}_k := (I_{x_k} \circ \vec{\xi}_k)^* g_{\mathbb{R}^m} = \vec{\xi}_k^*(I_{x_k}^* g_{\mathbb{R}^m})$ is conformally equivalent to g_k . One has $I_{x_k}^* g_{\mathbb{R}^m} = \frac{1}{|x - x_k|^4} g_{\mathbb{R}^m}$ thus we obtain

$$\tilde{g}_k = \frac{1}{|\vec{\xi}_k - x_k|^4} g_k \quad .$$

We denote K_{g_k} and $K_{\tilde{g}_k}$ the Gauss curvatures with respect to g_k resp. \tilde{g}_k . A classical computation in the differential geometry of surfaces gives

$$K_{g_k} - \frac{1}{|\vec{\xi}_k - x_k|^4} K_{\tilde{g}_k} = -\Delta_{g_k} \log |\vec{\xi}_k - x_k|^2 \quad .$$

The integration of this identity over Σ_δ with respect to the g_k volume form gives

$$\begin{aligned} \int_{\Sigma_\delta} K_{g_k} \, dvol_{g_k} - \int_{\Sigma_\delta} K_{\tilde{g}_k} \, dvol_{\tilde{g}_k} &= - \int_{\Sigma_\delta} \Delta_{g_k} \log |\vec{\xi}_k - x_k|^2 \, dvol_{g_k} \\ &= \int_{\partial \Sigma_\delta} d \log |\vec{\xi}_k - x_k|^2 \cdot \nu_k \, dl_{g_k} \quad , \end{aligned} \quad (\text{A.26})$$

where ν_k is the unit outward normal in Σ to $\partial \Sigma_\delta$ with respect to the g_k metric and dl_{g_k} is the length form with respect to g_k along $\partial \Sigma_\delta$. Let $A_{g_k}^0$, resp. $A_{\tilde{g}_k}^0$, denote the trace free second fundamental form of $\vec{\xi}_k$, resp. $I_{x_k} \circ \vec{\xi}_k$. We have $|A_{g_k}^0|^2 = 4(|\vec{H}_{\vec{\xi}_k}|^2 - K_{g_k})$ and $|A_{\tilde{g}_k}^0|^2 = 4(|\vec{H}_{I_{x_k} \circ \vec{\xi}_k}|^2 - K_{\tilde{g}_k})$. A classical result in conformal geometry (see [Ri1] theorem VI.1) gives

$$|A_{g_k}^0|^2 \, dvol_{g_k} = |A_{\tilde{g}_k}^0|^2 \, dvol_{\tilde{g}_k} \quad .$$

Hence we have

$$|\vec{H}_{\vec{\xi}_k}|^2 \, dvol_{g_k} - K_{\vec{\xi}_k} \, dvol_{g_k} = |\vec{H}_{I_{x_k} \circ \vec{\xi}_k}|^2 \, dvol_{\tilde{g}_k} - K_{I_{x_k} \circ \vec{\xi}_k} \, dvol_{\tilde{g}_k}$$

Integrating this identity over Σ_δ and combining this with (A.26) gives

$$\int_{\Sigma_\delta} |\vec{H}_{\vec{\xi}_k}|^2 d\text{vol}_{g_k} - \int_{\Sigma_\delta} |\vec{H}_{I_{x_k} \circ \vec{\xi}_k}|^2 d\text{vol}_{\bar{g}_k} = \int_{\partial\Sigma_\delta} d\log |\vec{\xi}_k - x_k|^2 \cdot \nu_k dl_{g_k} \quad (\text{A.27})$$

Lemma A.5 of [Ri3] gives an asymptotic expansion of $\vec{\xi}_\infty$ in a neighborhood of a_1 from which we deduce that

$$\text{dist}(\vec{\xi}_\infty(\partial B_\delta(a_1)), \vec{\xi}_\infty(a_1)) \geq C_\delta > 0$$

for some positive constant C_δ . Since x_k converges to $\vec{\xi}_\infty$ and since $\vec{\xi}_k$ converges in C^0 norm to $\vec{\xi}_\infty$ on Σ_δ , we have that for any $\delta > 0$ there exists $c_\delta > 0$ such that, for k large enough

$$\text{dist}(\vec{\xi}_k(\partial B_\delta(a_1)), x_k) \geq c_\delta > 0 \quad . \quad (\text{A.28})$$

Since $\vec{\xi}_k$ weakly converges in $W^{2,2}$ norm towards $\vec{\xi}_\infty$ on Σ_δ , $d\vec{\xi}_k$ strongly converges towards $d\vec{\xi}_\infty$ in $L^p(\Sigma_\delta)$ for any $p < +\infty$. In $\Sigma_\delta \setminus \Sigma_{2\delta}$ we denote $\vec{\nu}$ be the vector-field equal to the outward unit normal to $\partial\Sigma_s$ for $s \in (\delta, 2\delta)$ for the metric \bar{h} . Since g_k is conformally equivalent to \bar{h} we have that $\nu_k dl_{g_k} = \vec{\nu} dl_{\bar{h}}$ and we have for $s \in (\delta, 2\delta)$

$$\int_{\partial\Sigma_s} d\log |\vec{\xi}_k - x_k|^2 \cdot \nu_k dl_{g_k} = \int_{\partial\Sigma_s} d\log |\vec{\xi}_k - x_k|^2 \cdot \vec{\nu} dl_{\bar{h}} \quad .$$

Using now the strong convergence of $d\vec{\xi}_k$ towards $d\vec{\xi}_\infty$ in $L^p(\Sigma_\delta)$ for $p = 2$ together with (A.28), with the C^0 convergence of $\vec{\xi}_k$ towards $\vec{\xi}_\infty$ and with the help of Fubini theorem, we deduce that for almost every $s \in (\delta, 2\delta)$ there exists a subsequence still denoted $\vec{\xi}_k$ such that

$$\lim_{k \rightarrow +\infty} \int_{\partial\Sigma_s} d\log |\vec{\xi}_k - x_k|^2 \cdot \vec{\nu} dl_{\bar{h}} - \int_{\partial\Sigma_s} d\log |\vec{\xi}_\infty - \vec{\xi}_\infty(a_1)|^2 \cdot \vec{\nu} dl_{\bar{h}} = 0 \quad . \quad (\text{A.29})$$

As before, since $g_\infty := \vec{\xi}_\infty^* g_{\mathbb{R}^m}$ is conformally equivalent to \bar{h} we have

$$\int_{\partial\Sigma_s} d\log |\vec{\xi}_\infty - \vec{\xi}_\infty(a_1)|^2 \cdot \vec{\nu} dl_{\bar{h}} = \int_{\partial\Sigma_{s_k}} d\log |\vec{\xi}_\infty - \vec{\xi}_\infty(a_1)|^2 \cdot \nu_\infty dl_{g_\infty} \quad . \quad (\text{A.30})$$

For all $i \neq 1$ $|\vec{\xi}_\infty - \vec{\xi}_\infty(a_1)| \geq c > 0$ on $\partial B_s(a_i)$ and, since $\vec{\xi}_\infty$ is lipshitz we easily get that

$$\forall i \neq 1 \quad \lim_{s \rightarrow 0} \int_{\partial B_s(a_i)} d\log |\vec{\xi}_\infty - \vec{\xi}_\infty(a_1)|^2 \cdot \nu_\infty dl_{g_\infty} = 0 \quad (\text{A.31})$$

At this stage, if $\vec{\xi}_\infty$ would be a smooth immersion we could easily pass to the limit $s \rightarrow 0$ for $i = 1$ as well and prove that this boundary integral generates a residue equal to 4π . In the case of weak immersion $\vec{\xi}_\infty \in \mathcal{E}_\Sigma$ this passage to the limit does not necessarily holds and we have to pick some well chosen sequence s_i .

We take conformal coordinates $x = (x_1, x_2)$ around a_1 such that $x(a_1) = y(a_1) = 0$. We assume to simplify notations that $\vec{\xi}_\infty(a_1) = 0$. Let f be the inverse of these coordinates. We assume to simplify the presentation that our metric \bar{h} coincides with the flat metric in these coordinates. Let λ_∞ be the conformal factor associated to $f^*g_\infty : e^{2\lambda_\infty} [dx_1^2 + dx_2^2] = f^*g_\infty$. By an abuse of notation we keep denoting $\vec{\xi}_\infty$ the composition of $\vec{\xi}_\infty$ with f . Let

$$\vec{e}_1 := e^{-\lambda_\infty} \partial_{x_1} \vec{\xi}_\infty \quad \text{and} \quad \vec{e}_2 := e^{-\lambda_\infty} \partial_{x_2} \vec{\xi}_\infty$$

. For any radius $\rho > 0$ we denote $\overline{\xi_\infty}^{-\rho} := |\partial B_\rho(0)|^{-1} \int_{\partial B_\rho(0)} \vec{\xi}_\infty$. We have

$$\overline{\xi_\infty}^{-\rho} = \frac{1}{2\pi\rho} \int_{\partial B_\rho(0)} \int_0^\rho \frac{\partial \vec{\xi}_\infty}{\partial r} dr = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\rho \frac{e^{\lambda_\infty}}{|x|} [\cos\theta \vec{e}_1 + \sin\theta \vec{e}_2] r dr d\theta \quad (\text{A.32})$$

For such a $\vec{\xi}_\infty \in \mathcal{E}_\Sigma$ λ is continuous (see [Hel], [Ri1] for instance). Hence we have

$$\int_{B_\rho(0)} \frac{|e^{\lambda_\infty} - e^{\lambda_\infty(0)}|}{|x|} = o(1) \rho \quad . \quad (\text{A.33})$$

For $i = 1, 2$ we denote $\vec{e}_i^\rho := |\partial B_\rho(0)|^{-1} \int_{\partial B_\rho(0)} \vec{e}_i$. Since \vec{e}_i is $W^{1,2}$ we have using Hölder and Sobolev-Poincaré inequalities

$$\int_{B_\rho(0)} \frac{|\vec{e}_i - \vec{e}_i^\rho|}{|x|} \leq C \rho \left[\frac{1}{|B_\rho(0)|} \int_{B_\rho(0)} |\vec{e}_i - \vec{e}_i^\rho|^3 \right]^{1/3} \leq C \rho \left[\int_{B_\rho(0)} |\nabla \vec{e}_i|^2 \right]^{1/2} = \rho o(1) \quad . \quad (\text{A.34})$$

Since

$$\int_0^{2\pi} \cos\theta \vec{e}_1^\rho + \sin\theta \vec{e}_2^\rho = 0 \quad ,$$

combining (A.32), (A.33) and (A.34) gives

$$|\overline{\xi_\infty}^{-\rho}| \leq \rho o(1) \quad . \quad (\text{A.35})$$

Let $\vec{X}_i^\rho := |B_\rho(0)|^{-1} \int_{B_\rho(0)} \partial_{x_i} \vec{\xi}_\infty = |B_\rho(0)|^{-1} \int_{B_\rho(0)} e^{\lambda_\infty} \vec{e}_i$. Using Poincaré inequality we have

$$\frac{1}{|B_\rho(0)|} \int_{B_\rho(0)} |\partial_{x_i} \vec{\xi}_\infty - \vec{X}_i^\rho|^2 \leq C \int_{B_\rho(0)} |\nabla^2 \vec{\xi}_\infty|^2 = o(1) \quad . \quad (\text{A.36})$$

Thus there exists $x \in B_\rho(0)$ such that

$$|\partial_{x_i} \vec{\xi}_\infty - \vec{X}_i^\rho|^2 = o(1)$$

Since $\partial_{x_i} \vec{\xi}_\infty = e^{\lambda_\infty} \vec{e}_i = e^{\lambda_\infty(0)} \vec{e}_i + o(1)$ one has

$$|\vec{X}_i^\rho| = e^{\lambda_\infty} + o(1) \quad . \quad (\text{A.37})$$

We have moreover

$$\begin{aligned} |\vec{X}_1^\rho \cdot \vec{X}_2^\rho| &= \left| \frac{1}{|B_\rho(0)|^2} \int_{B_\rho(0)} \int_{B_\rho(0)} \partial_{x_1} \vec{\xi}_\infty(x) \cdot \partial_{x_2} \vec{\xi}_\infty(y) - \partial_{x_1} \vec{\xi}_\infty(y) \cdot \partial_{x_2} \vec{\xi}_\infty(x) dx_1 dx_2 dy_1 dy_2 \right| \\ &\leq \frac{\|\nabla \vec{\xi}_\infty\|_\infty}{|B_\rho(0)|} \int_{B_\rho(0)} |\nabla \vec{\xi}_\infty - |B_\rho(0)|^{-1} \int_{B_\rho(0)} \nabla \vec{\xi}_\infty| \leq C \int_{B_\rho(0)} |\nabla^2 \vec{\xi}_\infty| = o(1) \quad . \end{aligned} \quad (\text{A.38})$$

On $B_\rho(0)$ we define $\vec{u}_\rho := \vec{\xi}_\infty - x_1 \vec{X}_1^\rho - x_2 \vec{X}_2^\rho$. Using again Poincaré inequality we have

$$\int_{B_\rho(0)} |\nabla u_\rho|^2 = \int_{B_\rho(0)} |\nabla \vec{\xi}_\infty - |B_\rho(0)|^{-1} \int_{B_\rho(0)} \nabla \vec{\xi}_\infty|^2 \leq C \rho^2 \int_{B_\rho(0)} |\nabla^2 \vec{\xi}_\infty|^2 = o(1) \rho^2 \quad . \quad (\text{A.39})$$

Using Hölder inequality, Fubini theorem and the mean value formula we deduce from (A.39) that there exists $s \in (\rho/2, \rho)$ such that

$$\begin{cases} \|u_\rho - \overline{u_\rho}^s\|_{L^\infty(\partial B_s(0))} \leq \int_{\partial B_s(0)} |\nabla u_\rho| \leq o(1) s \quad . \\ \int_{\partial B_s(0)} |\partial_{x_i} \vec{\xi}_\infty - \vec{X}_i^\rho|^2 = o(1) s \quad \text{for } i = 1, 2 \end{cases} \quad (\text{A.40})$$

(A.35) gives that

$$|\overline{u_\rho}^s| = o(1) s$$

Hence from (A.37), (A.38) and (A.40), proceeding to a classical Schmidt orthonormalization of $e^{-\lambda_\infty(0)} (\vec{X}_1^\rho, \vec{X}_1^\rho)$, we deduce that there exists a radius $s \in (\rho/2, \rho)$ and a pair of unit vectors $(\vec{f}_1^\rho, \vec{f}_2^\rho)$ orthogonal with another such that

$$e^{-\lambda_\infty(0)} (\vec{X}_1^\rho, \vec{X}_1^\rho) = (\vec{f}_1^\rho, \vec{f}_2^\rho) + o(1) \quad (\text{A.41})$$

and

$$\|\vec{\xi}_\infty - e^{\lambda_\infty(0)} [x_1 \vec{f}_1^\rho - x_2 \vec{f}_2^\rho]\|_{L^\infty(\partial B_s(0))} = o(1) s \quad (\text{A.42})$$

This implies in particular that

$$\| |\vec{\xi}_\infty| - e^{\lambda_\infty(0)} |x| \|_{L^\infty(\partial B_s(0))} = o(1) s \quad (\text{A.43})$$

Combining (A.43), (A.41) and the second line of (A.40) gives

$$\begin{aligned} \int_{\partial B_s(a_1)} d \log |\vec{\xi}_\infty - \vec{\xi}_\infty(a_1)|^2 \cdot \nu_\infty \, dl_{g_\infty} &= 2s \int_0^\theta \frac{\partial \vec{\xi}_\infty}{\partial r} \cdot \frac{\vec{\xi}_\infty}{|\vec{\xi}_\infty|^2} \, d\theta \\ &= 4\pi + o(1) \quad . \end{aligned} \quad (\text{A.44})$$

Thus combining (A.27), (A.31) and (A.44), for this special choice of $s \in (\delta/2, \delta)$ we have found a subsequence still denoted $\vec{\xi}_k$ such that

$$\liminf_{k \rightarrow +\infty} \int_{\Sigma_s} |\vec{H}_{I_{x_k} \circ \vec{\xi}_k}|^2 \, dvol_{\tilde{g}_k} \geq \liminf_{k \rightarrow +\infty} \int_{\Sigma_s} |\vec{H}_{\vec{\xi}_k}|^2 \, dvol_{g_k} - 4\pi + o_\delta(1) \quad (\text{A.45})$$

Since

$$\liminf_{k \rightarrow +\infty} \int_{\Sigma_s} |\vec{H}_{\vec{\xi}_k}|^2 \, dvol_{g_k} \geq \int_{\Sigma_s} |\vec{H}_{\vec{\xi}_\infty}|^2 \, dvol_{g_\infty} \quad (\text{A.46})$$

we have proved that for all $\varepsilon > 0$ there exists $s > 0$ and a subsequence still denoted $\vec{\xi}_k$ such that

$$\liminf_{k \rightarrow +\infty} W(I_{x_k} \circ \vec{\xi}_k(\Sigma_s)) \geq W(\vec{\xi}_\infty(\Sigma_s)) - 4\pi - \varepsilon \quad (\text{A.47})$$

Using a diagonal argument we can choose a unique subsequence $\vec{\xi}_k$ such that (A.44) holds for $\delta = 2^{-j}$ for each $j \in \mathbb{N}$ and with a well chosen $s_j \in (2^{-j-1}, 2^{-j})$. Since $\liminf_{k \rightarrow +\infty} W(I_{x_k} \circ \vec{\xi}_k(\Sigma_s))$ is a decreasing function of δ we have proved that

$$\lim_{\delta \rightarrow 0} \liminf_{k \rightarrow +\infty} W(I_{x_k} \circ \vec{\xi}_k(\Sigma_\delta)) \geq W(\vec{\xi}_\infty(\Sigma)) - 4\pi - \varepsilon \quad (\text{A.48})$$

Hence we have proved that from any sequence satisfying the assumption of the lemma one can extract a subsequence such that (A.48) holds we have then proved (A.24). The last case when no subsequence of x_k converge to a point of $\vec{\xi}_\infty(\Sigma)$ is covered by the previous analysis since the problem of the limiting point $\vec{\xi}_\infty(a_1)$ as being an accumulation point of x_k does not show up and one has (A.25) in that case. Lemma A.4 is then proved. \square

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