

A PROOF OF THE MULTIPLICITY ONE CONJECTURE FOR MIN-MAX MINIMAL SURFACES IN ARBITRARY CODIMENSION

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ABSTRACT. Given any admissible k -dimensional family of immersions of a given closed oriented surface into an arbitrary closed Riemannian manifold, we prove that the corresponding min-max width for the area is achieved by a smooth (possibly branched) immersed minimal surface with multiplicity one and Morse index bounded by k .

1. INTRODUCTION

Recently, a new theory for the construction of branched immersed minimal surfaces of arbitrary topology, in an assigned closed Riemannian manifold \mathcal{M}^m , was proposed in [13]. This method is based on a penalization of the area functional by means of the second fundamental form A of the immersion.

Namely, for a fixed parameter $\sigma > 0$, one first finds an immersion $\Phi : \Sigma \rightarrow \mathcal{M}^m$ which is critical for the perturbed area functional

$$(1.1) \quad A^\sigma(\Phi) := \int_{\Sigma} d \operatorname{vol}_{g_\Phi} + \sigma^2 \int_{\Sigma} (1 + |A|_{g_\Phi}^2)^2 d \operatorname{vol}_{g_\Phi},$$

where Σ is a fixed closed oriented surface and g_Φ is the metric induced by Φ , with volume form $\operatorname{vol}_{g_\Phi}$. This functional A^σ enjoys a sort of Palais–Smale condition up to diffeomorphisms.

We should mention that the idea of considering perturbed functionals goes back to the paper [17] by Sacks–Uhlenbeck, where a perturbation of the Dirichlet energy is used to build minimal immersed spheres. However, in order to find minimal immersed surfaces with higher genus, one should give up working with the Dirichlet energy and use a more tensorial functional like (1.1): among closed orientable surfaces, only the sphere has a unique conformal structure (up to diffeomorphisms) and, as a consequence, a harmonic map (i.e. a critical point for the Dirichlet energy) $\Phi : \Sigma \rightarrow \mathcal{M}^m$ could fail to be conformal and minimal if Σ has positive genus. In principle, one can overcome this issue by introducing the conformal structure as an additional parameter in the variational problem: this program was carried out by Zhou in [21].

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Considering any sequence $\sigma_j \downarrow 0$, one gets a sequence $\Phi_j : \Sigma_j \rightarrow M$ of conformal immersions (with area bounded above and below), where Σ_j denotes Σ endowed with the induced conformal structure. Assuming for simplicity that we are dealing with a constant conformal structure (in general one gets a limiting Riemann surface in the Deligne–Mumford compactification), the sequence Φ_j is then bounded in $W^{1,2}$ and we can consider its weak limit Φ_∞ , up to subsequences. A priori it is not clear whether the strong $W^{1,2}$ -convergence holds, even away from a finite bubbling set. However, in [13] the second author shows that, if the sequence σ_j is carefully chosen so as to satisfy a certain *entropy condition*, then the surfaces $\Phi_j(\Sigma_j)$ converge to a *parametrized stationary varifold* (a notion introduced in [13, 12] and recalled in Section 4 below) which we call $(\Sigma_\infty, \Theta_\infty, N_\infty)$ in the present paper. The limiting multiplicity N_∞ a priori could be bigger than one.

A consequence of the main regularity result contained in [12] is that the multiplicity N_∞ is locally constant. This result, which is optimal for the class of parametrized stationary varifolds, leaves nonetheless open the question whether one can have $N_\infty > 1$ on some connected component of Σ_∞ .

This question should be compared with the *multiplicity one conjecture* by Marques and Neves. In [10], the following upper bound for the Morse index of a minimal hypersurface with locally constant multiplicity is established: if

$$\Sigma = \sum_{j=1}^{\ell} n_j \Sigma_j$$

is a minimal hypersurface with locally constant multiplicity, given by a min-max with k parameters in the context of Almgren–Pitts theory, then

$$\text{index}(\text{supp}(\Sigma)) \leq k, \quad \text{supp}(\Sigma) := \bigsqcup_{j=1}^{\ell} \Sigma_j.$$

In other words, this is a bound for the Morse index of the hypersurface obtained by replacing all the multiplicities n_j with 1. In order for this estimate to give more information about Σ , or at least its unstable part, the authors make the following conjecture.

Conjecture 1.1 (Multiplicity one conjecture). *For generic metrics on M^{n+1} , with $3 \leq n+1 \leq 7$, two-sided unstable components of closed minimal hypersurfaces obtained by min-max methods must have multiplicity one.*

It is natural to demand for extra information for one-sided stable components with unstable double cover, as well. Marques and Neves were able to prove this conjecture for one-parameter sweepouts, leaving the general case open. For metrics with positive Ricci curvature, related results were already obtained by Marques and Neves in [9] and later by Zhou in [20].

Further results, such as the two-sidedness of Σ when the metric has positive Ricci curvature, were obtained by Ketover, Marques and Neves in [7], using the catenoid estimate.

We also mention that Ketover, Liokumovich and Song in [6, 19] started to settle the generic, one-parameter case for the simpler and more effective Simon–Smith variant of Almgren–Pitts theory, specially designed for 3-manifolds.

Very recently, in [1], Chodosh and Mantoulidis established the conjecture for bumpy metrics in 3-manifolds, i.e. when $n = 2$, in the setting of Allen–Cahn level set approach. Also, half a year after this paper was written, Zhou announced a proof for bumpy metrics in all dimensions $3 \leq n + 1 \leq 7$, in the context of Almgren–Pitts theory (see [22]).

The importance of this conjecture in relation to the Morse index of Σ is twofold. First of all, there is no satisfactory definition for the Morse index of an embedded minimal hypersurface with multiplicity bigger than one: such Σ could be thought as the limiting object of many qualitatively different sequences, e.g. the elements of the sequence could realize different covering spaces of the limit, or more pathologically they could be minimal and have many catenoidal necks (hence Σ would be the limit of a sequence of highly unstable hypersurfaces).

Also, if one is able to establish a lower bound on the Morse index such as

$$k \leq \text{index}(\text{supp}(\Sigma)) + \text{nullity}(\text{supp}(\Sigma)),$$

then the multiplicity one conjecture gives infinitely many *geometrically distinct* minimal hypersurfaces, provided there exists at least one for every value of k . This was precisely the strategy used in [1] to prove Yau’s conjecture for generic metrics: in [1] the authors obtained the multiplicity one result and the equality $\text{index}(\Sigma) = k$ (the nullity vanishing automatically for bumpy metrics). This was later extended to higher ambient dimension (but in codimension one) in [22].

In this work we establish the natural counterpart of this conjecture in our setting, namely for minimal surfaces produced by the viscous relaxation method.

Theorem 1.2. *We have $N_\infty \equiv 1$.*

We stress that this result holds in *arbitrary codimension* and *without any genericity assumption*. This should be seen as a multiplicity one statement from the perspective of the parametrization *domain*, in that localization in the domain (away from branch points) gives a genuine embedded minimal surface, but a priori it does not exclude multiple covers of the image surface $\Theta_\infty(\Sigma_\infty)$ globally. It seems to be optimal for a min-max approach involving parametrizations, rather than e.g. approaches involving level sets of functions, and it is sufficient to obtain an upper bound on the Morse index. This bound, detailed in [15], relies on having a branched immersion at our disposal, for which a good definition of Morse index is available.

We remark that, in view of earlier work in [14], Theorem 1.2 would imply by itself the main result of [12], for parametrized stationary varifolds arising as a limit of stationary points for the relaxed functionals. However, the proof of Theorem 1.2 relies substantially on the regularity result obtained in [12], needed in several compactness arguments.

The main idea is to define a sort of *macroscopic multiplicity*, on balls $B_\ell^q(p)$ in an ambient Euclidean space $\mathbb{R}^q \supseteq \mathcal{M}^m$, before passing to the limit (i.e. looking at the immersed surfaces Φ_j rather than their limit). This macroscopic multiplicity is roughly the closest integer to the average of a *projected multiplicity*, issued by the map $\Pi \circ \Phi_j|_{B_r^2(z)}$, where $B_r^2(z)$ is a small domain ball and Π a 2-plane close to the image of $\Phi_j|_{\partial B_r^2(z)}$. Then we will use a continuity argument to show that this number stays constant as we pass from scale 1 to scale $\sqrt{\sigma_j}$. At the latter scale we have a very clear understanding of the behaviour of Φ_j and in particular we are able to say that here the macroscopic multiplicity equals 1. Thus the same holds at the original scale and this is sufficient to get $N_\infty \equiv 1$.

Most of the work is contained in Section 6. A more detailed discussion of the strategy, together with an informal explanation of the technical statements contained in Section 6, is deferred to the beginning of that section.

On the other hand, Section 5 contains two auxiliary facts about harmonic maps, the second of which is rather technical in nature, while Section 7 is more standard and uses a simple covering argument, together with some ideas similar to those in Section 6.

Corollary 1.3. *If there is no bubbling or degeneration of the underlying conformal structure, we have strong $W^{1,2}$ -convergence $\Phi_k \rightarrow \Phi_\infty$. In general we have a bubble tree convergence.*

Theorem 1.2 and Corollary 1.3 pave the way to obtain meaningful Morse index bounds. Indeed, although Theorem 1.2 does not rule out the possibility of having a surface covered multiple times by Φ_∞ , a crucial advantage of having a parametrization at our disposal is that we have a reasonable definition of Morse index and nullity: they are defined with respect to the area functional and variations in $C_c^\infty(\Sigma_\infty \setminus \{z_1, \dots, z_s\})$, the points z_1, \dots, z_s being the branch points of the immersion Φ_∞ .¹

The natural expected inequalities would be

$$\text{index}(\Phi_\infty) \leq k \leq \text{index}(\Phi_\infty) + \text{nullity}(\Phi_\infty).$$

An abstract framework to show upper bounds for the Morse index, dealing with general penalized functionals on Banach manifolds, is developed in [11]. Combining Corollary 1.3 with the general result obtained in [11] and with [15], we reach the following conclusion (we refer the reader to [11] for the notion of *admissible family*).

Corollary 1.4. *Given an admissible family $\mathcal{A} \subseteq \mathcal{P}(\text{Imm}(\Sigma, \mathcal{M}^m))$ of dimension k and calling*

$$W_{\mathcal{A}} := \inf_{A \in \mathcal{A}} \sup_{\Phi \in A} \text{area}(\Phi)$$

¹Although we are dealing with a weakly conformal map Φ_∞ , for which area and energy are the same, it is important to remark that the Morse indices for area and energy, denoted index_A and index_E respectively, should not be expected to agree. The relationship between the two is a subtle problem: in this direction, we mention the inequality $\text{index}_E(\Psi) \leq \text{index}_A(\Psi) \leq \text{index}_E(\Psi) + r$ established in [2], for a branched minimal immersion Ψ , where $r = r(g, b)$ depends on the genus and the number of branch points of Ψ .

the width of \mathcal{A} , there exists a (possibly branched) minimal immersion Φ of a closed surface S into \mathcal{M}^m such that

- (i) $\text{genus}(S) \leq \text{genus}(\Sigma)$,
- (ii) $W_{\mathcal{A}} = \text{area}(\Phi)$,
- (iii) $\text{index}(\Phi) \leq k$.

However, proving the second inequality, namely $k \leq \text{index}(\Phi) + \text{nullity}(\Phi)$, seems to require a finer understanding of the convergence $\Phi_k \rightarrow \Phi_{\infty}$. We hope to be able to deal with this question elsewhere.

Also, it would be interesting to adapt Gromov's notion of volume spectrum (and higher codimension generalizations), used to produce infinitely many minimal hypersurfaces in many settings, to the present situation. To this aim, a natural topological question concerns how much *genus* is needed to realize a nontrivial p -sweepout (in the sense of Gromov–Guth), and how to realize the sweepout within the space of immersions or in an appropriate closure of it.

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3. NOTATION

- We will assume, without loss of generality, that \mathcal{M}^m is isometrically embedded in some Euclidean space \mathbb{R}^q . Given $p \in \mathcal{M}^m$ and $\ell > 0$, we set $\mathcal{M}_{p,\ell}^m := \ell^{-1}(\mathcal{M}^m - p)$.
- In what follows, Π will always denote a 2-plane through the origin, which we identify with the corresponding orthogonal projection $\Pi : \mathbb{R}^q \rightarrow \Pi$. We call Π^{\perp} the orthogonal $(q - 2)$ -subspace, identified with the corresponding orthogonal projection. Given 2-planes Π, Π' , we denote by $\text{dist}(\Pi, \Pi')$ an arbitrary distance on the Grassmannian $\text{Gr}(2, \mathbb{R}^q)$, e.g. the one induced by Plücker's embedding of $\text{Gr}(2, \mathbb{R}^q)$ into the projectivization of $\Lambda_2 \mathbb{R}^q$.

The adjoint maps, which are just the inclusions $\Pi \hookrightarrow \mathbb{R}^q$ and $\Pi^{\perp} \hookrightarrow \mathbb{R}^q$, are denoted Π^* and $(\Pi^{\perp})^*$, so that

$$\text{id}_{\mathbb{R}^q} = \Pi^* \Pi + (\Pi^{\perp})^* \Pi^{\perp}.$$

Also, Π_0 is the canonical 2-plane, so that $\Pi_0 : \mathbb{R}^q \rightarrow \mathbb{R}^2$ is the projection onto the first two coordinates, while $\Pi_0^{\perp} : \mathbb{R}^q \rightarrow \mathbb{R}^{q-2}$ is the projection onto the remaining $q - 2$.

- We call $B_r^2(x)$ the ball of center x and radius r in the plane $\mathbb{C} = \mathbb{R}^2$, while $B_s^q(p)$ will denote the ball of center p and radius s in \mathbb{R}^q . Given $p \in \Pi$, we call $B_s^{\Pi}(p)$ the two-dimensional ball with center p and radius s in Π , i.e. $B_s^{\Pi}(p) := B_s^q(p) \cap \Pi$. When the center is not specified, it is always meant to be the origin.

- Given a function $\Psi \in W^{1,2}(B_r^2(x))$ and $0 < s \leq r$, the notation $\Psi|_{\partial B_s^2(x)}$ always refers to the trace of Ψ on the circle $\partial B_s^2(x)$.
- Given $K \geq 1$, we define the following set of Beltrami coefficients:

$$\mathcal{E}_K := \left\{ \mu \in L^\infty(\mathbb{C}, \mathbb{C}) : \|\mu\|_{L^\infty} \leq \frac{K-1}{K+1} \right\}.$$

We let \mathcal{D}_K denote the set of K -quasiconformal homeomorphisms $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\varphi(0) = 0, \quad \min_{x \in \partial B_1^2} |\varphi(x)| = 1.$$

If $\varphi \in \mathcal{D}_K$, we have $\varphi \in W_{loc}^{1,2}(\mathbb{C})$ and $\partial_{\bar{z}}\varphi = \mu\partial_z\varphi$ for some $\mu \in \mathcal{E}_K$, in the weak sense; we refer the reader to [5, Chapter 4] for the basic theory of K -quasiconformal homeomorphisms in the plane. Moreover, it is immediate to check that φ is a linear map in \mathcal{D}_K if and only if $\varphi(e_1) = e'_1$ and $\varphi(e_2) = \lambda e'_2$, for suitable orthonormal bases (e_1, e_2) , (e'_1, e'_2) inducing the canonical orientation and a suitable $1 \leq \lambda \leq K$.

- We define

$$D(K) := \sup \left\{ |\varphi(x)|; x \in \overline{B}_1^2, \varphi \in \mathcal{D}_K \right\}, \quad s(K) := \inf \left\{ |\varphi^{-1}(y)|; |y| \geq \frac{1}{2}, \varphi \in \mathcal{D}_K \right\},$$

so that $\varphi(\overline{B}_1^2) \subseteq \overline{B}_{D(K)}^2$ and $\varphi(\overline{B}_{s(K)}^2) \subseteq \overline{B}_{1/2}^2$ for all $\varphi \in \mathcal{D}_K$. The fact that $D(K) < \infty$ and $s(K) > 0$ is guaranteed by Corollary A.4. We also set

$$\eta(K) := \frac{1}{4} \inf \left\{ |\varphi(x)|; x \in \partial B_{s(K)^2}^2, \varphi \in \mathcal{D}_K \right\} > 0.$$

- We let \mathcal{D}_K^Π denote the set of maps having the form $\Pi^* \circ R \circ \varphi$, where $\varphi \in \mathcal{D}_K$ and $R : \mathbb{R}^2 \rightarrow \Pi$ is a linear isometry. Given $0 < \delta < 1$, we call $\mathcal{R}_{K,\delta}^\Pi$ the set of maps in $W^{1,2}(B_1^2, \mathbb{R}^q)$ which are close to some $\psi \in \mathcal{D}_K^\Pi$ on the circles of radii $1, s(K), s(K)^2$, namely we set

$$\mathcal{R}_{K,\delta}^\Pi := \left\{ \Psi \in W^{1,2}(B_1^2, \mathbb{R}^q) : \min_{\psi \in \mathcal{D}_K^\Pi} \max_{r \in \{1, s(K), s(K)^2\}} \left\| \Psi|_{\partial B_r^2} - \psi(r \cdot) \right\|_{L^\infty(\partial B_1^2)} \leq \delta \right\}.$$

- Given $\Psi \in C^1(\Omega, \mathbb{R}^q)$, a ball $B_r^2(z) \subset\subset \Omega$ and a 2-plane Π , we define the *projected multiplicity* function

$$N(\Psi, B_r^2(z), \Pi) : \Pi \rightarrow \mathbb{N} \cup \{\infty\}, \quad N(\Psi, B_r^2(z), \Pi)(p) := \#((\Pi \circ \Psi)^{-1}(p) \cap B_r^2(z))$$

and, given $p \in \Pi$ and $t > 0$, we also define the *macroscopic multiplicity*

$$(3.1) \quad n(\Psi, B_r^2(z), B_t^\Pi(p)) := \left\lfloor \int_{B_t^\Pi(p)} N(\Psi, B_r^2(z), \Pi) + \frac{1}{2} \right\rfloor \in \mathbb{N}.$$

The mean appearing in (3.1) is finite by the area formula and $\lfloor \cdot \rfloor$ denotes the integer part. Note that, if the mean is close to an integer k , then the macroscopic multiplicity is precisely k . Note also that for any $p \in \mathbb{R}^q$ we have

$$n(\Psi, B_r^2(z), B_t^\Pi(\Pi(p))) = n\left(\frac{\Psi(z+r\cdot) - p}{t}, B_1^2, B_1^\Pi\right).$$

4. BACKGROUND ON PARAMETRIZED STATIONARY VARIFOLDS

Let $\mathcal{M}^m \subset \mathbb{R}^q$ be a (smooth, closed) embedded submanifold. Assume we have a smooth conformal immersion $\Phi : B_1^2 \rightarrow \mathcal{M}^m$, critical for the functional

$$(4.1) \quad \Phi \mapsto \int_{B_1^2} d\text{vol}_{g_\Phi} + \sigma^2 \int_{B_1^2} (1 + |A_{g_\Phi}|_{g_\Phi}^2)^2 d\text{vol}_{g_\Phi}$$

with respect to smooth variations (Φ_t) such that Φ_t is an immersion agreeing with Φ outside some compact set $F \subset B_1^2$ (independent of t). Here $g_\Phi := \Phi^*g_{\mathbb{R}^q}$ and A_{g_Φ} is the second fundamental form of Φ . Assume that the following *entropy condition*

$$(4.2) \quad \sigma^2 \log(\sigma^{-1}) \int_{B_1^2} (1 + |A|^2)^2 d\text{vol}_{g_\Phi} \leq \varepsilon \int_{B_1^2} d\text{vol}_{g_\Phi}$$

holds for some $\varepsilon > 0$. This condition plays a fundamental role in the viscosity approach presented in [13], and can be enforced ultimately owing to Struwe's monotonicity trick (see [13, Section II] and the references therein). Note that

$$g_\Phi = \frac{1}{2} |\nabla \Phi|^2 \delta, \quad \int_{B_1^2} d\text{vol}_{g_\Phi} = \frac{1}{2} \int_{B_1^2} |\nabla \Phi|^2$$

by conformality of Φ .

Given any $0 < \ell < 1$ and $p \in \mathcal{M}^m$, recall that $\mathcal{M}_{p,\ell}^m = \ell^{-1}(\mathcal{M}^m - p)$. The rescaled map

$$\Psi : B_1^2 \rightarrow \mathcal{M}_{p,\ell}^m, \quad \Psi := \ell^{-1}(\Phi - p)$$

is critical for the functional

$$(4.3) \quad \int_{B_1^2} d\text{vol}_{g_\Psi} + \tau^2 \int_{B_1^2} (\ell^2 + |A|^2)^2 d\text{vol}_{g_\Psi}, \quad \tau := \sigma \ell^{-2}$$

and, being $\tau^2 \log(\tau^{-1}) \leq \ell^{-4} \sigma^2 \log(\sigma^{-1})$, it satisfies

$$(4.4) \quad \tau^2 \log(\tau^{-1}) \int_{B_1^2} (\ell^2 + |A|^2)^2 d\text{vol}_{g_\Psi} \leq \varepsilon \int_{B_1^2} d\text{vol}_{g_\Psi},$$

where now A denotes the second fundamental form of Ψ in $\mathcal{M}_{p,\ell}^m$ and its norm is meant with respect to the induced metric g_Ψ .

In the sequel, we will establish many intermediate results on maps Ψ arising in this way, by means of compactness arguments. The starting point in these arguments is that, heuristically, if we have sequences $\Psi_k, p_k, \ell_k \rightarrow 0, \tau_k \rightarrow 0$ and $\varepsilon_k \rightarrow 0$, then by (4.3) and (4.4) Ψ_k should have a subsequential limit Ψ_∞ (in some weak sense) which is critical for the area functional in the tangent space $T_{p_\infty} \mathcal{M}^m$ (where p_∞ is a subsequential limit of the sequence p_k), i.e. Ψ_∞ should be a minimal parametrization.

The kind of limiting object that we get is specified by the following definition.

Definition 4.1. A triple (Ω, Φ, N) , with $\Omega \subseteq \mathbb{C}$ open, $\Phi \in W^{1,2}(\Omega, \mathbb{R}^q)$ weakly conformal and $N \in L^\infty(\Omega, \mathbb{N} \setminus \{0\})$, is a *local parametrized stationary varifold* if for almost every

$\omega \subset\subset \Omega$ the rectifiable 2-varifold

$$\mathbf{v}_\omega := (\Phi(\mathcal{G} \cap \omega), \theta_\omega), \quad \theta_\omega(p) := \sum_{x \in \mathcal{G} \cap \omega \cap \Phi^{-1}(p)} N(x)$$

is stationary in the open set $\mathbb{R}^q \setminus \Phi(\partial\omega)$, where \mathcal{G} denotes the set of Lebesgue points for both Φ and $d\Phi$. We also require the technical condition

$$\|\mathbf{v}_\Omega\|(B_s^q(p)) = \frac{1}{2} \int_{\Phi^{-1}(B_s^q(p))} N |\nabla\Phi|^2 d\mathcal{L}^2 = O(s^2),$$

uniformly in $p \in \mathbb{R}^q$.

We refer the reader to [12, Definition 2.1] for the notion of *almost every domain*, as well as to [12, Definitions 2.2 and 2.9] for another possible definition (in the global and local versions, respectively), whose equivalence with Definition 4.1 is detailed in [12, Remark 2.3]. The latter formulation will not be used here.

As already mentioned in the introduction, the main result of [12] is that Φ is harmonic (namely, it coincides a.e. with a harmonic map) and N is (a.e.) constant on those connected components of Ω where Φ is not itself (a.e.) constant.

Since these facts are crucially used in many intermediate steps towards the proof of Theorem 1.2, we give a precise statement that summarizes all the information we need to extract from the works [13] and [12].

Theorem 4.2 (limiting behaviour of almost flat critical immersions). *Assume that $\Psi_k \in C^2(\overline{B}_R^2, \mathcal{M}_{p_k, \ell_k}^m)$ is a sequence of conformal immersions such that Ψ_k is critical for the functional (4.3) on the interior B_R^2 (with τ_k, ℓ_k in place of τ, ℓ) and*

- $\Psi_k|_{\partial B_R^2} \rightarrow \gamma_\infty$ uniformly, for some $\gamma_\infty : \partial B_R^2 \rightarrow \mathbb{R}^q$,
- $\frac{1}{2} \int_{B_R^2} |\nabla\Psi_k|^2 \leq E$,
- $\tau_k^2 \log(\tau_k^{-1}) \int_{B_R^2} |A|^4 d\text{vol}_{g_{\Psi_k}} \rightarrow 0$,
- $\ell_k, \tau_k \rightarrow 0$.

Then, up to subsequences, $\Psi_k \rightharpoonup \Psi_\infty$ in $W^{1,2}(B_R^2, \mathbb{R}^q)$, for some Ψ_∞ which is continuous (in the interior), has trace γ_∞ , and satisfies the convex hull property, namely

$$\Psi_\infty(\overline{\omega}) \subseteq \text{co}(\Psi_\infty(\partial\omega)) \quad \text{for all } \omega \subset\subset B_R^2.$$

The image measures $(\Psi_k)_* (\frac{1}{2} |\nabla\Psi_k|^2 \mathcal{L}^2)$ in \mathbb{R}^q form a tight sequence.

Given $\omega \subset\subset B_R^2$ with $\Psi_\infty(\overline{\omega}) \subseteq \mathbb{R}^q \setminus \gamma_\infty(\partial B_R^2)$, there exists a quasiconformal homeomorphism φ_∞ of \mathbb{R}^2 and a locally constant multiplicity $N_\infty \in L^\infty(\omega, \mathbb{Z}^+)$ such that the 2-varifolds induced by $\Psi_k|_\omega$ subsequentially converge on $\mathbb{R}^q \setminus \Psi_\infty(\partial\omega)$ to a (local) parametrized stationary varifold

$$(\varphi_\infty(\omega), \Psi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1}).$$

in the varifold sense, namely in duality with $C_c^0((\mathbb{R}^q \setminus \Phi_\infty(\partial\omega)) \times \text{Gr}(2, \mathbb{R}^q))$. Also, on ω we have the convergence of Radon measures

$$(4.5) \quad \frac{1}{2} |\nabla \Psi_k|^2 \mathcal{L}^2 \xrightarrow{*} N_\infty |\partial_1 \Psi_\infty \wedge \partial_2 \Psi_\infty| \mathcal{L}^2.$$

Setting $\Gamma_\infty := \gamma_\infty(\partial B_R^2)$, we have $N_\infty \leq \frac{E}{\pi \text{dist}(\Psi_\infty(\bar{\omega}), \Gamma_\infty)^2}$ a.e. and the distortion constant of φ_∞ is bounded by $\left(\frac{E}{\pi \text{dist}(\Psi_\infty(\bar{\omega}), \Gamma_\infty)^2}\right)^2$.

Proof of Theorem 4.2. The proof is essentially already contained in [13] and [12], so we just present the required adaptations.

Up to subsequences, we can assume that Ψ_k has a weak limit Ψ_∞ in $W^{1,2}(B_R^2, \mathbb{R}^q)$, with trace γ_∞ , and that the varifolds \mathbf{v}_k induced by Ψ_k converge to a varifold \mathbf{v}_∞ in \mathbb{R}^q .

The arguments used in [13, Section III] and in [12, Section 2] show that Ψ_∞ has a continuous representative (on the interior B_R^2), satisfying the convex hull property. Also, from [13, Section III] we have that

$$(4.6) \quad \mathbf{v}_\infty \text{ is stationary in } U := \mathbb{R}^q \setminus \gamma_\infty(\partial B_R^2)$$

and is an integer rectifiable varifold. We claim that the measures $\|\mathbf{v}_k\| = (\Psi_k)_* \left(\frac{1}{2} |\nabla \Psi_k|^2\right)$ on \mathbb{R}^q form a tight sequence. If this were not true, up to subsequences we could find points $q_k \in \mathbb{R}^q$ with $|q_k| \rightarrow \infty$ and such that the argument of [13, Lemma III.3] applies (with q_k in place of q), on the region $\mathbb{R}^q \setminus \Psi_k(\partial B_R^2)$. Hence,

$$\liminf_{k \rightarrow \infty} \|\mathbf{v}_k\|(B_1^q(q_k)) > 0.$$

So the varifolds $\mathbf{v}_k - q_k$ converge subsequentially to a nontrivial varifold \mathbf{v}'_∞ in \mathbb{R}^q , stationary on $U' := \mathbb{R}^q$: indeed, the proof of (4.6) can be repeated with $\Psi_k - q_k$ in place of Ψ_k (and U' in place of U), using the fact that, for all $s > 0$, the image of $\Psi_k|_{\partial B_R^2} - q_k$ is eventually disjoint from B_s^q . Its total mass $\|\mathbf{v}'_\infty\|(\mathbb{R}^q)$ must be bounded by $\liminf_{k \rightarrow \infty} \|\mathbf{v}_k\|(\mathbb{R}^q) \leq E$; however, the monotonicity formula implies that $\|\mathbf{v}'_\infty\|(\mathbb{R}^q) = \infty$, a contradiction.

We also claim that the bubbling set is empty in our setting. Indeed, by tightness of the measures $\|\mathbf{v}_k\|$, a bubbling point would produce, in the limit, a nontrivial stationary varifold in \mathbb{R}^q . Again, its mass would be bounded by E , contradicting the monotonicity formula.

Now [13, Lemma III.5] gives, up to further subsequences, the limit

$$\nu_k := \frac{1}{2} |\nabla \Psi_k|^2 \xrightarrow{*} \nu_\infty, \quad \text{with } \nu_\infty = m \mathcal{L}^2$$

in the sense of Radon measures (i.e. in duality with $C_c^0(B_R^2)$). The function $m(z) \geq 0$ equals $N_\infty(z) |\partial_1 \Phi_\infty \wedge \partial_2 \Phi_\infty|(z)$ a.e., for a positive integer $N_\infty(z)$ which is bounded by the density of \mathbf{v}_∞ at $\Psi_\infty(z)$ whenever $\Psi_\infty(z) \in U$.

Let $\omega \subset\subset B_R^2$ be such that $\Psi_\infty(\bar{\omega}) \subseteq \mathbb{R}^q \setminus \Gamma_\infty$. Defining $s := \text{dist}(\Psi_\infty(\bar{\omega}), \Gamma_\infty) > 0$, note that

$$B_s^q(q) \subseteq U \quad \text{for all } q \in \Psi_\infty(\bar{\omega}).$$

Hence, by the monotonicity formula, the density of \mathbf{v}_∞ at such points q is bounded by $\frac{E}{\pi s^2}$. This gives the upper bound for N_∞ . As explained in detail in [12, Section 4], there exists a quasiconformal homeomorphism φ_∞ of the plane, with distortion constant bounded by the square of the (essential) supremum of $N_\infty|_\omega$, such that $\Psi_\infty \circ \varphi_\infty^{-1}$ is weakly conformal on $\varphi_\infty(\omega)$. Finally, it is the main outcome of [13] that the 2-varifolds induced by $\Phi_k|_\omega$ converge to the (local) parametrized stationary varifold $(\varphi_\infty(\omega), \Psi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$ (whose mass measure is bounded above by $\|\mathbf{v}_\infty\|$), in the complement of $\Psi_\infty(\partial\omega)$.² ■

Theorem 4.3 (regularity of parametrized stationary varifolds). *In the situation of Theorem 4.2, $\Psi_\infty \circ \varphi_\infty^{-1} : \varphi_\infty(\omega) \rightarrow \mathbb{R}^q$ is harmonic. Also, if ω is connected and $\Psi_\infty|_\omega$ is not constant, N_∞ equals a constant integer (a.e.) on ω and $\Psi_\infty \circ \varphi_\infty^{-1}$ is a minimal branched immersion.*

Proof of Theorem 4.3. This is a special case of the main theorem in [12], namely [12, Theorem 5.7]. ■

5. TWO LEMMAS ON HARMONIC MAPS

Lemma 5.1 (uniform convergence for Dirichlet problem with variable domain).

Let $\gamma_k \in C^0(\partial B_1^2, \mathbb{R}^2)$ be a sequence of Jordan curves converging (in C^0) to a Jordan curve γ_∞ and let $f_k \in C^0(\partial B_1^2)$ be a sequence converging uniformly to a function f_∞ . Let D_k be the domain bounded by γ_k , let $u_k \in C^0(\overline{D}_k)$ be the harmonic extension of $f_k \circ \gamma_k^{-1}$, and similarly define D_∞ and u_∞ . Then $u_k \rightarrow u_\infty$ in $C_{loc}^0(D_\infty)$. Moreover, if $y_k \rightarrow y_\infty$ with $y_k \in \overline{D}_k$ and $y_\infty \in \overline{D}_\infty$, then $u_k(y_k) \rightarrow u_\infty(y_\infty)$.

Note that such harmonic extensions exist and are unique, since by Carathéodory's theorem there exist homeomorphisms $\overline{B}_1^2 \rightarrow \overline{D}_k$ restricting to biholomorphisms $B_1^2 \rightarrow D_k$ (and similarly for D_∞), allowing to reduce matters to the well-known existence and uniqueness of the harmonic extension on the unit disk.

Proof of Lemma 5.1. Since the functions f_k are equibounded, from the maximum principle and interior estimates it follows that the functions u_k are equibounded in $C^2(\overline{\omega})$, for any $\omega \subset\subset D_\infty$, and hence by Ascoli–Arzelà theorem the convergence $u_k \rightarrow u_\infty$ in $C_{loc}^0(D_\infty)$ follows from the second claim.

It suffices to show that the second claim holds for a subsequence: once this is done, it can be obtained for the full sequence by a standard contradiction argument (given a sequence $y_k \rightarrow y_\infty$, if $u_k(y_k)$ does not converge to $u_\infty(y_\infty)$, we can find a subsequence such that it converges to a different value; then we reach a contradiction along a further subsequence where the second claim holds).

Up to removing a finite set of indices, we can suppose that there is a point p such that $p \in D_k$ for all $k \in \mathbb{N} \cup \{\infty\}$. By Carathéodory's theorem, we can find homeomorphisms

²The convergence actually holds on all of \mathbb{R}^q (or, more precisely, on $\mathbb{R}^q \times \text{Gr}(2, \mathbb{R}^q)$) if $\nu_\infty(\partial\omega) = 0$.

$v_k : \overline{B}_1^2 \rightarrow \overline{D}_k$ restricting to biholomorphisms from B_1^2 to D_k , so that $v_k|_{\partial B_1^2} = \gamma_k \circ \beta_k$, for suitable homeomorphisms $\beta_k : \partial B_1^2 \rightarrow \partial B_1^2$ (for all $k \in \mathbb{N}$), and $v_k(0) = p$.

Since the maps v_k and v_k^{-1} are equibounded and harmonic, we can assume that

$$(5.1) \quad v_k \rightarrow v_\infty, \quad \zeta_k := v_k^{-1} \rightarrow \zeta_\infty$$

in $C_{loc}^\infty(B_1^2)$ and $C_{loc}^\infty(D_\infty)$, respectively. Note that v_∞ is a holomorphic map taking values into \overline{D}_∞ , while ζ_∞ is holomorphic and takes values into B_1^2 (by the maximum principle, since $\zeta_\infty(p) = 0$ and $|\zeta_\infty| \leq 1$). So for any $w \in D_\infty$ the set $\{\zeta_k(w) \mid k \in \mathbb{N}\} \cup \{\zeta_\infty(w)\} \subset B_1^2$ is compact and we infer

$$(5.2) \quad v_\infty \circ \zeta_\infty(w) = \lim_{k \rightarrow \infty} v_k \circ \zeta_k(w) = w.$$

Hence v_∞ is surjective and thus an open map. So $v_\infty(B_1^2) = D_\infty$ and, by [16, Theorem 10.43] (applied with $f := v_\infty - w$, $g := v_k - w$, for a fixed $w \in D_\infty$ and an arbitrary circle $\partial B_r^2 \subseteq B_1^2$ avoiding $v_\infty^{-1}(w)$, with k large enough), it is also injective. By Carathéodory's theorem, it extends continuously to a homeomorphism (still denoted v_∞) from \overline{B}_1^2 to \overline{D}_∞ and we have $v_\infty|_{\partial B_1^2} = \gamma_\infty \circ \beta_\infty$ for a suitable homeomorphism $\beta_\infty : \partial B_1^2 \rightarrow \partial B_1^2$.

Up to subsequences, applying Helly's selection principle to suitable lifts $\overline{\beta}_k : \mathbb{R} \rightarrow \mathbb{R}$, we can assume that $\beta_k \rightarrow \widetilde{\beta}_\infty$ everywhere, for some order-preserving $\widetilde{\beta}_\infty$.³ On the other hand, since $\sup_k \int_{B_1^2} |v_k'|^2 = \sup_k \mathcal{L}^2(D_k)$ is finite, we have weak convergence $v_k \rightharpoonup v_\infty$ in $W^{1,2}(B_1^2)$ and thus weak convergence $\gamma_k \circ \beta_k \rightharpoonup \gamma_\infty \circ \beta_\infty$ in $L^2(\partial B_1^2)$. The everywhere convergence $\gamma_k \circ \beta_k \rightarrow \gamma_\infty \circ \widetilde{\beta}_\infty$ implies $\gamma_\infty \circ \beta_\infty = \gamma_\infty \circ \widetilde{\beta}_\infty$ a.e. and thus $\beta_\infty = \widetilde{\beta}_\infty$ a.e. In particular, β_∞ is also order-preserving. Since β_∞ is continuous and both maps are order-preserving, we conclude that $\beta_\infty = \widetilde{\beta}_\infty$ everywhere. Using again the continuity of β_∞ , as well as the everywhere convergence of the order-preserving maps $\beta_k \rightarrow \beta_\infty$, we also get that $\beta_k \rightarrow \beta_\infty$ uniformly.

Being v_k the harmonic extension of $\gamma_k \circ \beta_k$ (for $k \in \mathbb{N} \cup \{\infty\}$), we conclude that $v_k \rightarrow v_\infty$ in $C^0(\overline{B}_1^2)$. Let $U_k \in C^0(\overline{B}_1^2)$ be the harmonic extension of $f_k \circ \beta_k$ and note that $U_k \rightarrow U_\infty$ in $C^0(\overline{B}_1^2)$. By conformal invariance, $u_k := U_k \circ v_k^{-1}$ is the harmonic extension of $f_k \circ \gamma_k^{-1}$ on D_k (for $k \in \mathbb{N} \cup \{\infty\}$).

Finally, we claim that in the situation of the second claim we have $v_k^{-1}(y_k) \rightarrow v_\infty^{-1}(y_\infty)$. This easily follows from the injectivity of v_∞ : if we had $|v_k^{-1}(y_k) - v_\infty^{-1}(y_\infty)| \geq \varepsilon$ along some subsequence (for some $\varepsilon > 0$), we would have a subsequential limit point $x_\infty \in \overline{B}_1^2$ with $|x_\infty - v_\infty^{-1}(y_\infty)| \geq \varepsilon$ and $v_\infty(x_\infty) = \lim_{k \rightarrow \infty} y_k = y_\infty$, which is a contradiction. Hence,

$$(5.3) \quad u_k(y_k) = U_k(v_k^{-1}(y_k)) \rightarrow U_\infty(v_\infty^{-1}(y_\infty)) = u_\infty(y_\infty),$$

as desired. ■

³The map $\widetilde{\beta}_\infty$ could also be order-reversing: this happens precisely if β_k reverses the orientation along the subsequence. For simplicity, we assume $\beta_k, \widetilde{\beta}_\infty$ to be order-preserving (the other case being analogous).

Remark 5.2. In the situation of Lemma 5.1, if $D_k \supseteq D$ for all $k \in \mathbb{N} \cup \{\infty\}$ then $u_k \rightarrow u_\infty$ uniformly on \bar{D} . Indeed, if this were not true, then we could find points $y_k \in \bar{D} \subseteq \bar{D}_k$ such that $|u_k(y_k) - u_\infty(y_k)| \geq \varepsilon$ (along a subsequence) for some $\varepsilon > 0$. Assuming without loss of generality $y_k \rightarrow y_\infty$, we would get

$$\liminf_{k \rightarrow \infty} |u_k(y_k) - u_\infty(y_\infty)| \geq \varepsilon$$

by continuity of u_∞ on \bar{D} . This would however contradict the last part of Lemma 5.1.

Lemma 5.3 (injectivity under a boundary constraint). *Given $K \geq 1$ and $s, \varepsilon > 0$, there exists a constant $0 < \delta_0 < \varepsilon$, depending only on q, K, s, ε , with the following property: whenever*

- $\Psi \in W^{1,2} \cap C^0(\bar{B}_1^2, \mathbb{R}^q)$ has $\left\| \Psi|_{\partial B_1^2} - \psi(s \cdot) \right\|_{C^0(\partial B_1^2)} \leq \delta_0$ for some $\psi \in \mathcal{D}_K^\Pi$,
- $\Psi \circ \varphi^{-1}$ is harmonic and weakly conformal on $\varphi(B_1^2)$, where $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a K -quasiconformal homeomorphism,⁴

then $\Pi \circ \Psi \circ \varphi^{-1}$ is a diffeomorphism from $\varphi(\bar{B}_{1/2}^2)$ onto its image and

$$(5.4) \quad \text{dist}(\Pi, \Pi(x)) < \varepsilon, \quad \Pi(x) := 2\text{-plane spanned by } \nabla(\Psi \circ \varphi^{-1})(x),$$

for all $x \in \varphi(\bar{B}_{1/2}^2)$. In particular, $\Pi \circ \Psi$ is injective on $\bar{B}_{1/2}^2$.

Proof of Lemma 5.3. Assume by contradiction that, for a sequence $\delta_k \downarrow 0$, there exist maps $\Psi_k : B_1^2 \rightarrow \mathbb{R}^q$, planes Π_k , homeomorphisms $\varphi_k : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and coefficients μ_k such that the claim fails with $\delta_0 = \delta_k$. By Corollary A.4, up to subsequences we have $\Pi_k \rightarrow \Pi_\infty$ and $\Psi_k|_{\partial B_1^2} \rightarrow \Gamma$ (uniformly), where $\Gamma : \partial B_1^2 \rightarrow \mathbb{R}^q$ is the restriction of a map in $\mathcal{D}_K^{\Pi_\infty}$.

We can assume that $\varphi_k \in \mathcal{D}_K$ (replacing φ_k with $\frac{\varphi_k - \varphi_k(0)}{\min_{\partial B_1^2} |\varphi_k - \varphi_k(0)|}$). By Corollary A.4, we can assume that $\varphi_k \rightarrow \varphi_\infty$ and $\varphi_k^{-1} \rightarrow \varphi_\infty^{-1}$ in $C_{loc}^0(\mathbb{R}^2)$, for some homeomorphism $\varphi_\infty : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

By harmonicity, up to subsequences we get $\Theta_k := \Psi_k \circ \varphi_k^{-1} \rightarrow \Theta_\infty$ in $C_{loc}^2(\varphi_\infty(B_1^2))$, for some $\Theta_\infty : \varphi_\infty(B_1^2) \rightarrow \mathbb{R}^q$, so that Θ_∞ is weakly conformal and harmonic.

On the other hand, by Lemma 5.1 applied to the sequence of harmonic maps Θ_k on the Jordan domains $\varphi_k(B_1^2)$, Θ_∞ is the harmonic extension of $\Gamma \circ \varphi_\infty^{-1}$ and $\Psi_k \rightarrow \Theta_\infty \circ \varphi_\infty =: \Psi_\infty$ in $C^0(\bar{B}_1^2)$: see Remark 5.2. By the maximum principle we have $\Pi_\infty^\perp \circ \Theta_\infty = 0$ and thus $\Pi_\infty \circ \Theta_\infty$ is either holomorphic or antiholomorphic on $\varphi_\infty(B_1^2)$ (once Π_∞ is identified with \mathbb{C}).

Now, given two Jordan domains $U, V \subset \mathbb{C}$, if a holomorphic map $h : U \rightarrow \mathbb{C}$ extends to a continuous map $h : \bar{U} \rightarrow \mathbb{C}$ mapping ∂U onto ∂V homeomorphically, then h maps U

⁴The maps ψ and φ are not necessarily related to each other.

diffeomorphically onto V .⁵ Being $\Pi_\infty \circ \Theta_\infty|_{\partial\varphi_\infty(B_1^2)} = \Pi_\infty \circ \Gamma \circ \varphi_\infty^{-1}$ a Jordan curve, we deduce that $\Pi_\infty \circ \Theta_\infty$ is a diffeomorphism from $\varphi_\infty(B_1^2)$ onto its image.

Fix now a compact neighborhood F of $\varphi_\infty(\overline{B}_{1/2}^2)$ in $\varphi_\infty(B_1^2)$, with smooth boundary. Since $\Theta_k \rightarrow \Theta_\infty$ in $C_{loc}^1(\varphi_\infty(B_1^2))$, we obtain that eventually $\Pi_k \circ \Theta_k$ is a diffeomorphism of F onto its image, with

$$\text{dist}(\Pi_k, \Pi_k(x)) < \varepsilon, \quad x \in F.$$

The fact that eventually $\varphi_k(\overline{B}_{1/2}^2) \subseteq F$ yields the desired contradiction. \blacksquare

6. TECHNICAL ITERATION LEMMAS

6.1. Informal discussion of the results. Since the intermediate results contained in this section have rather involved statements, with several different constants and thresholds appearing along the way, we find it helpful to provide an informal explanation of the meaning of these statements and constants, as well as a rough sketch of the underlying ideas in the proofs.

This section contains four important intermediate results, namely Lemmas 6.2, 6.3, 6.5 and 6.6, which all invoke Theorem 4.2 (except for Lemma 6.6) by means of a compactness-and-contradiction argument. All statements are about a conformal immersion $\Psi : \overline{B}_r^2(z) \rightarrow \mathcal{M}_{p,\ell}^m$, critical for the functional (4.3) (on the interior). For simplicity, in this discussion we assume $z = 0$ and $r = 1$. The first three statements require the following:

- (i) a control of the shape of the images of three circles, dictated by a distortion constant K ; namely we require that

$$\Psi \in \mathcal{R}_{K,\delta_0}^\Pi$$

for some 2-plane Π and some small δ_0 ; recall from Section 3 that this means (assuming $\Pi = \mathbb{R}^2 \subseteq \mathbb{R}^q$, up to rotations of \mathbb{R}^q) that Ψ is C^0 -close to a K -quasiconformal homeomorphism $\varphi \in \mathcal{D}_K$, $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \subseteq \mathbb{R}^q$ on the three circles ∂B_1^2 , $\partial B_{s(K)}^2$, $\partial B_{s(K)^2}^2$ (it would be far too restrictive to ask for C^0 -closeness on all of B_1^2);

- (ii) an upper bound E on the Dirichlet energy $\frac{1}{2} \int_{B_1^2} |\nabla \Psi|^2$;
- (iii) an upper bound V on the area (divided by π) of the immersed surface $\Psi(B_1^2) \cap B_1^q$, taking into account multiplicity; namely,

$$\int_{\Psi^{-1}(B_1^q)} d\text{vol}_{g_\Psi} = \frac{1}{2} \int_{\Psi^{-1}(B_1^q)} |\nabla \Psi|^2 \leq V\pi,$$

⁵Indeed, $h|_U$ must be an open map, hence $h(\overline{U}) \setminus \partial V$ is closed and open in $\mathbb{C} \setminus \partial V$ and it follows that $h(U) = V$. We can find biholomorphisms $u : B_1^2 \rightarrow U$ and $v : B_1^2 \rightarrow V$ extending to homeomorphisms of the closures. The map $g := v^{-1} \circ h \circ u$ satisfies $g(B_1^2) \subseteq B_1^2$ and maps ∂B_1^2 to itself homeomorphically. Given $w \in B_1^2$, for $r < 1$ close enough to 1 the loop $g(re^{i\theta}) - w$ is homotopic to $g(e^{i\theta})$ in $\mathbb{C} \setminus \{0\}$, so the classical argument principle gives $\#g^{-1}(w) = 1$.

where g_Ψ is the pullback of the Euclidean metric, which equals $\frac{1}{2}|\nabla\Psi|^2\delta$ by conformality; this upper bound will give a crucial improvement on the last conclusions of Theorem 4.2, as discussed below.

Also, in the same spirit as Theorem 4.2, these lemmas assume $\tau, \ell \ll 1$ and

$$\tau^2 \log(\tau^{-1}) \int_{B_1^2} |A|^4 d \operatorname{vol}_{g_\Psi} \ll \int_{B_1^2} d \operatorname{vol}_{g_\Psi}.$$

In Lemmas 6.3 and 6.5, the closeness in (i) is measured by a threshold δ_0 (which will be specified according to Lemma 5.3), while other closeness or smallness constraints will be measured by thresholds $\varepsilon_0, \varepsilon'_0, \varepsilon''_0$ in Lemmas 6.2, 6.3, 6.5, respectively.

Observe that the hypotheses guarantee that $\Pi \circ \Psi$ maps $B_{s(K)}^2$ to a subset of $B_{1/2}^\Pi$ and ∂B_1^2 to a subset of $\Pi \setminus B_1^\Pi$ (approximately), hence $\Psi(B_{s(K)}^2)$ is far away from $\Psi(\partial B_1^2)$. Hence, when arguing by contradiction, we can apply the last part of Theorem 4.2 and obtain in the limit a parametrized stationary varifold close to $\Psi|_\omega$ (we will choose either $\omega := B_{s(K)}^2$ or the smaller domain $\omega := B_{s(K)^2}^2$). The reason to impose the geometric control on three circles, rather than two, is merely technical and is convenient for the proofs.

Lemma 6.2 says that the *projected multiplicity* $N(\Psi, B_{s(K)^2}^2, \Pi)$ (introduced in Section 3) issued by Ψ from the ball $B_{s(K)^2}^2$ has an average close to a positive integer k , on the ball $B_{\eta(K)}^\Pi$. It also asserts that this holds for 2-planes Π' close enough to Π . As a consequence, the corresponding *macroscopic multiplicity* will be precisely k .

Observe that the hypotheses guarantee that $\Pi \circ \Psi$ maps $B_{s(K)^2}^2$ approximately to a superset of $B_{\eta(K)}^\Pi$ (see Section 3 for the definition of these geometrical constants). Hence, arguing by contradiction, we obtain in the limit a (parametrized) stationary varifold which is close, in the varifold sense, to $\Psi(B_{s(K)^2}^2)$. The constraints on Ψ force this limiting varifold to lie on a 2-plane, so by the constancy theorem it has constant integer multiplicity on $B_{\eta(K)}^\Pi$, giving a contradiction. Note that the volume constraint V is not used here.

As already mentioned in the introduction, we would now like to find a decreasing sequence of radii $r_0 := 1, \dots, r_k \approx \sqrt{\tau}$, with r_j comparable to r_{j+1} , such that the maps $\Psi(r_j \cdot)$ satisfy the same assumptions (with different scales $\ell_0 := \ell, \dots, \ell_k$ in the target). The strategy to get Theorem 1.2 is then to show that the corresponding macroscopic multiplicities n_j *do not change* from one scale to the next one: $n_0 = n_1 = \dots = n_k$. At the smallest scale, we will be able to say that the immersed surface $\ell_k^{-1}\Psi(B_{r_k}^2)$ has small second fundamental form in L^4 , implying a strong graphical control that allows to conclude $n_k = 1$ and thus $n_0 = 1$. In the situation where we will apply this strategy (namely, in Section 7), upon careful selection of the center z , it will be easy to impose the “maximal” bounds

$$(\ell')^{-2} \int_{\Psi^{-1}(B_{\ell'}^q)} d \operatorname{vol}_{g_\Psi} \leq V\pi, \quad \tau^2 \log(\tau^{-1}) \int_{B_{r'}^2} |A|^4 d \operatorname{vol}_{g_\Psi} \ll \int_{B_{r'}^2} d \operatorname{vol}_{g_\Psi},$$

for all $0 < \ell' < 1$ and $0 < r' < 1$, by means of covering arguments. However, we *cannot* a priori impose similar bounds on the Dirichlet energy and on the shape of the images of small circles (items (ii) and (i)). Note that if $(\ell')^{-1}\Psi(r' \cdot)$ satisfies (i), then we can bound

the Dirichlet energy of this rescaled map on the ball $B_{s(K)}^2$, in terms of V , as Ψ maps $B_{s(K)r'}^2$ into $B_{\ell'}^q$ (approximately). So (i) would give (ii) for free (on a smaller domain ball), with a uniform bound (depending on K, V) in place of E .

Lemma 6.3 is the main technical workhorse, and essentially says that we can circumvent this difficulty: namely, the hypotheses (i)–(iii) are still satisfied for a smaller radius $r' \leq \frac{r}{2}$ in the domain, with a smaller scale $\ell'\ell \leq \frac{\ell}{2}$ in the codomain. Note that the reference point p also changes; this will in principle destroy the maximal volume bound, but we can still recover (iii) in the new situation, exploiting the fact that the multiplicity is quantized in the limit (see the proof of Lemma 6.3 and Definition 6.4 for the details).

The idea of the proof of Lemma 6.3 is that, up to a quasiconformal homeomorphism φ , Ψ is close (in the weak $W^{1,2}$ -topology) to a conformal harmonic map with small oscillation with respect to Π . Hence, by Lemma 5.3, it will be arbitrarily close to an affine injective conformal map L on smaller and smaller balls $B_{r'}^2$. If φ were the identity, given a (finite) collection of circles centered at 0 we would get C^0 -closeness of $\Psi(r'\cdot)$ to L on all these circles, for some r' small, and we would be done.

The important observation now is that the distortion constant of φ_∞ can be bounded solely in terms of V : indeed, as in the proof of Theorem 4.2, $\Psi(B_1^2) \cap B_1^q$ is close to a stationary varifold \mathbf{v} (in B_1^q), whose density on $B_{1/2}^q$ is bounded in terms of V . Since $\Psi(B_{s(K)}^2) \subseteq B_{1/2}^q$ (approximately), the upper bound on the distortion constant given by Theorem 4.2 can be improved to a constant $K'(V)$ depending *only* on V . Hence, we get (i) also for a smaller radius r' (with $K'(V)$ replacing K) and, as already said, this gives also (ii) with a bound $E'(V)$ in place of E . Our sequence of radii is now obtained by iterated application of Lemma 6.3 with parameters $K'(V), E'(V), V$.

Given constants K'', E'' and V , which will be chosen when applying these results in Section 7, we then fix $K_0 := \max\{K'(V), K''\}$ and $E_0 := \max\{E'(V), E''\}$, so that all the statements apply for all radii r_0, r_1, \dots, r_k .

Lemma 6.5 says that the macroscopic multiplicity does not change after applying Lemma 6.3, namely when replacing the domain and codomain scales r, ℓ with r', ℓ' (and p, Π with p', Π'). Its proof uses Lemma 5.3 to claim that Ψ is approximately a graph over Π , and then applies the constancy theorem (in the limiting situation).

Finally, as it will be clear along the proof of Theorem 7.1, Lemma 6.6 concerns the behaviour of a conformal immersion $\Phi : B_1^2 \rightarrow \mathcal{M}^m$ at a scale (comparable to) $\ell := \sqrt{\sigma}$ in the codomain, when Φ is critical for (4.1). Assume that $\Phi(B_r^2)$ has diameter approximately ℓ^2 , and assume the smallness

$$(6.1) \quad \sigma^2 \int_{B_r^2} |A|^4 d\text{vol}_{g_\Phi} \ll \int_{B_r^2} d\text{vol}_{g_\Phi}$$

and the bound $\int_{B_r^2} d\text{vol}_{g_\Phi} \leq C\ell^2$. When dilating the codomain by a factor ℓ^{-1} , (6.1) becomes $\int_{B_r^2} |A_\Psi|^4 d\text{vol}_{g_\Psi} \ll 1$, for $\Psi := \ell^{-1}(\Phi - \Phi(0))$. As Ψ is conformal, we have $\Delta\Psi = 2H_\Psi e^{2\lambda}$ (where e^λ is the conformal factor). Thus, we get that $\Delta\Psi$ is small in L^4

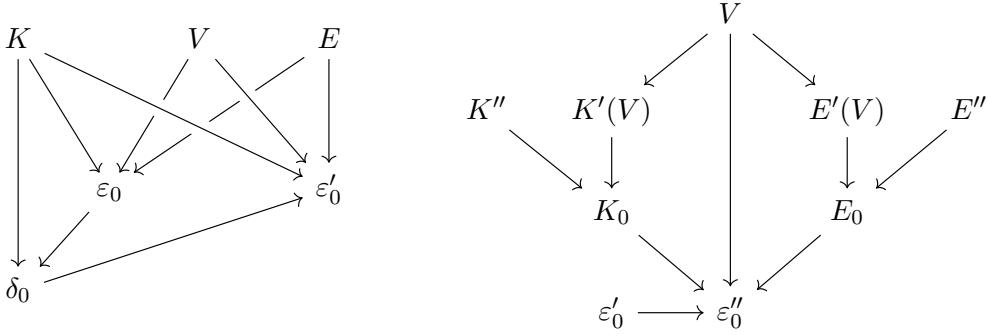
provided we can obtain an upper bound on λ ; once this is done, by Sobolev's embedding we obtain a C^1 -control on Ψ , which implies that the macroscopic multiplicity is 1 at this scale.

In order to bound λ , we use a result by Hélein (belonging to a broad class of phenomena of integrability by compensation, whose study dates back to the discovery of Wente's inequality), guaranteeing the existence of an orthonormal frame $\{e_1(z), e_2(z)\}$ for the tangent space of the immersed surface Ψ , with a bound on $\|\nabla e_i\|_{L^2}$ depending only the L^2 -norm of the second fundamental form. Then we show that

$$-\Delta\lambda = \partial_1 e_1 \cdot \partial_2 e_2 - \partial_2 e_1 \cdot \partial_1 e_2$$

and we compare λ with the solution μ to the same equation, with zero boundary conditions on a ball. A pointwise bound for μ now follows from Wente's inequality, from which one easily deduces the desired upper bound for λ . Although not necessary, we will also show how to obtain a pointwise lower bound on λ in this situation.

While reading Sections 6 and 7, it can be useful to refer to the following diagrams, illustrating how the constants depend on each other:



(an arrow $A \rightarrow B$ means that B depends on A).

6.2. Rigorous statements and proofs. We now make the above discussion rigorous. In a first reading, it can be helpful to pretend that all quasiconformal homeomorphisms appearing in the proofs coincide with the identity.

Definition 6.1. Given $V > 0$ with $V = [V] + \frac{1}{2}$, we define the constants

$$K'(V) := (4V)^2, \quad E'(V) := 2\pi K'(V)D(K'(V))^2.$$

In the sequel, it will be convenient to assume always that $V \in \mathbb{N} + \frac{1}{2}$, so that $V = [V] + \frac{1}{2}$.

Lemma 6.2 (almost integrality of averaged projected multiplicity). *There exists $0 < \varepsilon_0 < \eta(K)$, depending on $E, V > 0$, $K \geq 1$ and \mathcal{M}^m , such that whenever $\Psi \in C^2(\overline{B}_r^2(z), \mathcal{M}_{p,\ell}^m)$ is a conformal immersion, critical for the functional (4.3) on $B_r^2(z)$, and Π, Π' are 2-planes satisfying*

- $\Psi(z + r \cdot) \in \mathcal{R}_{K, \varepsilon_0}^\Pi$,
- $\frac{1}{2} \int_{B_r^2(z)} |\nabla \Psi|^2 \leq E$,

- $\int_{\Psi^{-1}(B_1^q)} d \operatorname{vol}_{g_\Psi} = \frac{1}{2} \int_{\Psi^{-1}(B_1^q)} |\nabla \Psi|^2 \leq V\pi,$
- $\tau^2 \log(\tau^{-1}) \int_{B_r^2(z)} |A|^4 d \operatorname{vol}_{g_\Psi} \leq \varepsilon_0$ for some $0 < \tau \leq \varepsilon_0,$
- $\operatorname{dist}(\Pi, \Pi') \leq \varepsilon_0$ and $0 < \ell \leq \varepsilon_0,$

then the projected multiplicity $N(\Psi, B_{s(K)2r}^2(z), \Pi)$ satisfies

$$(6.2) \quad \operatorname{dist} \left(\int_{B_{\eta(K)}^\Pi} N(\Psi, B_{s(K)2r}^2(z), \Pi), \mathbb{Z}^+ \right) < \frac{1}{8},$$

$$(6.3) \quad \left| \int_{B_{\eta(K)}^\Pi} N(\Psi, B_{s(K)2r}^2(z), \Pi) - \int_{B_{\eta(K)}^{\Pi'}} N(\Psi, B_{s(K)2r}^2(z), \Pi') \right| < \frac{1}{8},$$

where \mathbb{Z}^+ is the set of positive integers.

Proof of Lemma 6.2. We can assume $z = 0$ and $r = 1$. Suppose by contradiction that there exist sequences $\varepsilon_k \downarrow 0$, τ_k , ℓ_k , points p_k , maps Ψ_k and planes Π_k, Π'_k making the claim false for $\varepsilon_0 = \varepsilon_k$. Up to subsequences, we can assume that $\Pi_k, \Pi'_k \rightarrow \Pi_\infty$, that Ψ_k has a weak limit Ψ_∞ in $W^{1,2}(B_1^2, \mathbb{R}^q)$, with traces $\Psi_\infty|_{\partial B_s^2} = \psi(s \cdot)$ for some $\psi \in \mathcal{D}_K^{\Pi_\infty}$ and all $s \in \{1, s(K), s(K)^2\}$ (thanks to Corollary A.4), and that the varifolds \mathbf{v}_k induced by Ψ_k converge to a varifold \mathbf{v}_∞ in \mathbb{R}^q .

We now invoke Theorem 4.2. Recalling the definition of $\mathcal{D}_K^{\Pi_\infty}$ and $s(K)$ from Section 3, the convex hull property satisfied by Ψ_∞ gives

$$(6.4) \quad \Psi_\infty(\overline{B}_{s(K)}^2) \subseteq \operatorname{co} \left(\Psi_\infty(\partial B_{s(K)}^2) \right) = \operatorname{co} \left(\psi(\partial B_{s(K)}^2) \right) \subseteq \overline{B}_{1/2}^q,$$

so that, being $\Gamma_\infty = \Psi_\infty(\partial B_1^2) = \psi(\partial B_1^2)$ disjoint from B_1^q (as $|\psi(x)| \geq 1$ for $x \in \partial B_1^2$, by definition of \mathcal{D}_K),

$$(6.5) \quad \operatorname{dist}(\Psi_\infty(x), \Gamma_\infty) \geq \frac{1}{2} \quad \text{for } x \in \overline{B}_{s(K)}^2.$$

Theorem 4.2 gives the varifold convergence $\mathbf{v}'_k \xrightarrow{*} \mathbf{v}'_\infty$ and $\mathbf{v}''_k \xrightarrow{*} \mathbf{v}''_\infty$ as $k \rightarrow \infty$, as well as the tightness of the sequences of mass measures $\|\mathbf{v}'_k\|$ and $\|\mathbf{v}''_k\|$, where \mathbf{v}'_k and \mathbf{v}''_k are the varifolds issued by $\Psi_k|_{B_{s(K)}^2}$ and $\Psi_k|_{B_{s(K)^2}^2}$ respectively, while \mathbf{v}'_∞ and \mathbf{v}''_∞ are the ones issued by $(\varphi_\infty(B_{s(K)}^2), \Psi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$ and $(\varphi_\infty(B_{s(K)^2}^2), \Psi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$.⁶

Although not needed in the present proof, let us remark the following improvement on the last statement in Theorem 4.2, which will be used in the proof of Lemma 6.3: we have

$$N_\infty \leq \frac{V\pi}{\pi \left(\frac{1}{2}\right)^2} = 4V$$

and the distortion constant of φ_∞ is bounded by $K'(V) = (4V)^2$. Indeed, since \mathbf{v}_∞ is stationary in B_1^q and $\|\mathbf{v}_\infty\|(B_1^q) \leq V\pi$, by the monotonicity formula its density is bounded

⁶The fact that one can choose the same multiplicity N_∞ and the same quasiconformal homeomorphism $\varphi_\infty : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for both domains is evident from the proof of Theorem 4.2.

by $\frac{V\pi}{\pi(1-|p|)^2}$ at any $p \in B_1^q$. In particular, (6.4) gives an upper bound $4V$ at points in $\Psi_\infty(B_{s(K)}^2)$, which implies our claim.

The support of \mathbf{v}''_∞ is contained in the plane Π_∞ , by the convex hull property enjoyed by Ψ_∞ and the fact that Ψ_∞ maps $\partial B_{s(K)}^2$ to Π_∞ . Since $\Psi_\infty(\partial B_{s(K)}^2)$ does not intersect $B_{\eta(K)}^{\Pi_\infty}$, the varifold \mathbf{v}''_∞ is stationary here and thus, by the constancy theorem [18, Theorem 41.1], it has a constant density $\nu \in \mathbb{N}$. We must have $\nu > 0$, since $\Psi_\infty(B_{s(K)}^2)$ is a superset of $B_{\eta(K)}^{\Pi_\infty}$ by Lemma A.1 (applied to $\eta(K)^{-1}\Psi_\infty(s(K)^2 \cdot)$). The area formula and the tightness of $\|\mathbf{v}''_k\|$ then give

$$\int_{B_{\eta(K)}^{\Pi_k}} N(\Psi_k, B_{s(K)}^2, \Pi_k) = \frac{\|(\Pi_k)_* \mathbf{v}''_k\| (B_{\eta(K)}^{\Pi_k})}{\pi\eta(K)^2} \rightarrow \frac{\|(\Pi_\infty)_* \mathbf{v}''_\infty\| (B_{\eta(K)}^{\Pi_\infty})}{\pi\eta(K)^2} = \nu.$$

Similarly, $\int_{B_{\eta(K)}^{\Pi'_k}} N(\Psi_k, B_{s(K)}^2, \Pi'_k) \rightarrow \nu$ as $k \rightarrow \infty$. Hence the claim is eventually true, yielding the desired contradiction. \blacksquare

We now specify δ_0 so that Lemma 5.3 applies, with $\varepsilon := \varepsilon_0$ and $s := s(K)$. Note that $\delta_0 < \varepsilon_0 < \eta(K)$ and that ε_0 and δ_0 still depend on V , K and E .

Lemma 6.3 (existence of a smaller good scale). *Given $E > 0$ and $K \geq 1$ there exists a constant $0 < \varepsilon'_0 < \varepsilon_0$ (depending on E, V, K, \mathcal{M}^m) with the following property: if a conformal immersion $\Psi \in C^2(\overline{B}_r^2(z), \mathcal{M}_{p,\ell}^m)$ is critical for the functional (4.3) (on the interior) and satisfies*

- $\Psi(z + r \cdot) \in \mathcal{R}_{K,\delta_0}^\Pi$,
- $\frac{1}{2} \int_{B_r^2(z)} |\nabla \Psi|^2 \leq E$,
- $\frac{1}{\pi} \int_{\Psi^{-1}(B_1^q)} d\text{vol}_{g_\Psi}, \frac{1}{\pi\eta(K)^2} \int_{\Psi^{-1}(B_{\eta(K)}^q)} d\text{vol}_{g_\Psi} \leq V$,
- $\tau^2 \log(\tau^{-1}) \int_{B_r^2(z)} |A|^4 d\text{vol}_{g_\Psi} \leq \varepsilon'_0$ for some $0 < \tau \leq \varepsilon'_0$,
- $0 < \ell \leq \varepsilon'_0$,

then there exist a new point $p' \in \mathcal{M}_{p,\ell}^m$, new scales r', ℓ' and a new 2-plane Π' with

- $\varepsilon'_0 r < r' < s(K)r$,
- $\varepsilon'_0 < \ell' < \frac{1}{2}$,
- $\text{dist}(\Pi, \Pi') < \varepsilon_0$,
- $(\ell')^{-1}(\Psi(z + r' \cdot) - p') \in \mathcal{R}_{K'(V),\delta_0}^{\Pi'}$,
- $\frac{1}{2} \int_{B_{r'}^2(z)} |\nabla \Psi'|^2 < E'(V)$, for $\Psi' := (\ell')^{-1}(\Psi - p')$ (defined on $B_{r'}^2(z)$),
- $\frac{1}{\pi} \int_{(\Psi')^{-1}(B_1^q)} d\text{vol}_{g_{\Psi'}}, \frac{1}{\pi\eta(K)^2} \int_{(\Psi')^{-1}(B_{\eta(K)}^q)} d\text{vol}_{g_{\Psi'}} < \left[\left(\frac{\eta(K)}{\eta(K) - \varepsilon_0} \right)^2 V \right] + \frac{1}{2}$.

Proof of Lemma 6.3. We can assume $z = 0$ and $r = 1$. By contradiction, suppose that there is a sequence $\varepsilon_k \downarrow 0$ such that the claim fails (with $\varepsilon'_0 = \varepsilon_k$) for all radii $\varepsilon_k < r' < s(K)$, for some Ψ_k and Π_k satisfying all the hypotheses. As observed in the proof of Lemma 6.2, up to subsequences we get a limiting local parametrized stationary varifold $(\Omega_\infty, \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$ in \mathbb{R}^q , where $\Theta_\infty = \Psi_\infty \circ \varphi_\infty^{-1}$ and $\Omega_\infty = \varphi_\infty(B_{s(K)}^2)$ for a suitable $K'(V)$ -quasiconformal homeomorphism φ_∞ of the plane. By Theorem 4.3, Θ_∞ is harmonic. Also, it takes values in the tangent space T at p_∞ (translated to the origin).

Moreover, assuming also that $\Pi_k \rightarrow \Pi_\infty$ and $p_k \rightarrow p_\infty$, by Corollary A.4 we still have $\Psi_\infty \in \mathcal{R}_{K, \delta_0}^{\Pi_\infty}$: indeed, if $\psi_k \in \mathcal{D}_K^{\Pi_k}$ are such that $\left\| \Psi_k|_{\partial B_s^2}(s \cdot) - \psi_k(s \cdot) \right\|_{L^\infty(\partial B_1^2)} \leq \delta_0$ for $s = 1, s(K), s(K)^2$, then there exists $\psi_\infty \in \mathcal{D}_K^{\Pi_\infty}$ such that $\psi_k \rightarrow \psi_\infty$ (up to subsequences), uniformly on the three circles; for any bounded measurable function $\chi : \partial B_1^2 \rightarrow \mathbb{R}^q$ with $\|\chi\|_{L^1} \leq 1$, weak convergence of the traces $\Psi_k|_{\partial B_s^2} \rightharpoonup \Psi_\infty|_{\partial B_s^2}$ in L^2 gives

$$\int_{\partial B_1^2} \chi \cdot (\Psi_\infty|_{\partial B_s^2}(s \cdot) - \psi_\infty(s \cdot)) = \lim_{k \rightarrow \infty} \int_{\partial B_1^2} \chi \cdot (\Psi_k|_{\partial B_s^2}(s \cdot) - \psi_k(s \cdot)) \leq \delta_0$$

for $s = 1, s(K), s(K)^2$; thus, being χ arbitrary, the desired inequality holds also for $k = \infty$.

We can assume that $\varphi_\infty(0) = 0$. By definition of δ_0 and Lemma 5.3, applied to $\Psi_\infty(s(K) \cdot)$ and $\varphi_\infty(s(K) \cdot)$, Θ_∞ is a diffeomorphism from $\varphi_\infty(\overline{B}_{s(K)/2}^2)$ onto its image and the differential $\nabla \Theta_\infty(0)$ is a conformal linear map of full rank, spanning a plane Π' with $\text{dist}(\Pi_\infty, \Pi') < \varepsilon_0$.

Since $s(K)^2 \leq \frac{s(K)}{2}$, the varifolds \mathbf{v}_k induced by $\Psi_k|_{B_{s(K)^2}^2}$ converge to \mathbf{v}_∞ , induced by $(\varphi_\infty(B_{s(K)^2}^2), \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$. Using Lemma A.1, applied to $\eta(K)^{-1} \Pi_\infty \circ \Psi_\infty(s(K)^2 \cdot)$, and the fact that $\delta_0 < \eta(K)$, we deduce the existence of a point $y \in B_{s(K)^2}^2$ such that $\Pi_\infty \circ \Psi_\infty(y) = 0$. By the convex hull property enjoyed by Ψ_∞ , it follows that

$$|\Psi_\infty(y)| = \left| \Pi_\infty^\perp \circ \Psi_\infty(y) \right| \leq \delta_0,$$

as $\Psi_\infty(\partial B_{s(K)^2}^2) \subseteq \{p : |\Pi_\infty^\perp(p)| \leq \delta_0\}$. Since $\|\mathbf{v}_\infty\|(B_{\eta(K)}^q) \leq V\pi\eta(K)^2$, the stationarity of \mathbf{v}_∞ on $B_{\eta(K)}^q$ implies that its density at $\Psi_\infty(y)$ is at most

$$(6.6) \quad \frac{V\pi\eta(K)^2}{\pi(\eta(K) - \delta_0)^2} \leq \left(\frac{\eta(K)}{\eta(K) - \varepsilon_0} \right)^2 V.$$

Being \mathbf{v}_∞ stationary in the embedded surface $\Theta_\infty(\varphi_\infty(B_{s(K)^2}^2))$, the constancy theorem gives that its density θ is a constant integer here. Thus we have

$$(6.7) \quad \|\mathbf{v}_\infty\|(\overline{B}_t^q(p'_\infty)) < \left(\left\lfloor \left(\frac{\eta(K)}{\eta(K) - \varepsilon_0} \right)^2 V \right\rfloor + \frac{1}{2} \right) \pi t^2, \quad p'_\infty := \Theta_\infty(0) \in T,$$

for all $t > 0$ small enough. Fix now any $r' < s(K)$ such that we have the strong convergence $\Psi_k(r' \cdot) \rightarrow \Psi_\infty(r' \cdot)$ in $C^0(\partial B_1^2 \cup \partial B_{s(K)}^2 \cup \partial B_{s(K)^2}^2)$ along a subsequence.⁷ Note that

⁷This can be obtained by applying e.g. [12, Lemma A.5] to the weakly converging \mathbb{R}^{3q} -valued maps

$$(\Psi_k, \Psi_k(s(K) \cdot), \Psi_k(s(K)^2 \cdot)) \rightharpoonup (\Psi_\infty, \Psi_\infty(s(K) \cdot), \Psi_\infty(s(K)^2 \cdot)).$$

$\lambda^{-1}\varphi_\infty(r'\cdot) \in \mathcal{D}_{K'(V)}$, where $\lambda := \min_{|x|=r'} |\varphi_\infty(x)|$. Also, the fact that $\Psi_\infty = \Theta_\infty \circ \varphi_\infty$ and the smoothness of Θ_∞ give

$$(6.8) \quad |\Psi_\infty(r'x) - \Psi_\infty(0) - \langle \nabla \Theta_\infty(0), \varphi_\infty(r'x) \rangle| < \frac{\delta_0 |\nabla \Theta_\infty(0)|}{\sqrt{2}D(K'(V))} |\varphi_\infty(r'x)| \leq \delta_0 \ell'$$

if r' is chosen small enough, where $\ell' := \frac{|\nabla \Theta_\infty(0)|}{\sqrt{2}} \lambda$ and $x \in \overline{B}_1^2$. This implies

$$(\ell')^{-1}(\Psi_k(r'\cdot) - p'_\infty) \in \mathcal{R}_{K'(V), \delta_0}^{\Pi'}$$

by conformality of $\nabla \Theta_\infty(0)$. Shrinking r' , we can also ensure that $\ell' < \frac{1}{2}$, as well as

$$(6.9) \quad \int_{B_{r'}^2} N_\infty |\partial_1 \Psi_\infty \wedge \partial_2 \Psi_\infty| \leq \frac{K'(V)}{2} \int_{B_{D(K'(V))\lambda}^2} |\nabla \Theta_\infty|^2 < K'(V)(D(K'(V))\lambda)^2 \pi |\nabla \Theta_\infty(0)|^2.$$

Calling \mathbf{v}'_∞ the varifold induced by $(\varphi_\infty(B_{r'}^2), (\ell')^{-1}(\Theta_\infty - p'_\infty), N_\infty \circ \varphi_\infty^{-1})$, in view of (6.7) we can even guarantee that

$$\frac{\|\mathbf{v}'_\infty\|(\overline{B}_1^q)}{\pi}, \frac{\|\mathbf{v}'_\infty\|(\overline{B}_{\eta(K)}^q)}{\pi \eta(K)^2} < \left\lfloor \left(\frac{\eta(K)}{\eta(K) - \varepsilon_0} \right)^2 V \right\rfloor + \frac{1}{2}.$$

Calling p'_k the closest point to p'_∞ in $\mathcal{M}_{p_k, \ell_k}^m$ (eventually defined and converging to p'_∞ , since $\mathcal{M}_{p_k, \ell_k}^m \rightarrow T$), thanks to (6.8) and $\lambda^{-1}\varphi_\infty(r'\cdot) \in \mathcal{D}_{K'(V)}$, eventually we have

$$(\ell')^{-1}(\Psi_k(r'\cdot) - p'_k) \in \mathcal{R}_{K'(V), \delta_0}^{\Pi'}.$$

Moreover, (6.9) and (4.5) give

$$\frac{1}{2} \int_{B_{r'}^2(z)} |\nabla \Psi_k|^2 \rightarrow \int_{B_{r'}^2(z)} N_\infty |\partial_1 \Psi_\infty \wedge \partial_2 \Psi_\infty| < (\ell')^2 E'(V).$$

From the convergence of the varifolds \mathbf{v}'_k induced by $(\ell')^{-1}(\Psi_k - p'_k)|_{B_{r'}^2}$ to \mathbf{v}'_∞ we get

$$\limsup_{k \rightarrow \infty} \frac{\|\mathbf{v}'_k\|(\overline{B}_1^q)}{\pi}, \limsup_{k \rightarrow \infty} \frac{\|\mathbf{v}'_k\|(\overline{B}_{\eta(K)}^q)}{\pi \eta(K)^2} < \left\lfloor \left(\frac{\eta(K)}{\eta(K) - \varepsilon_0} \right)^2 V \right\rfloor + \frac{1}{2}.$$

So eventually $(\ell')^{-1}(\Psi_k(r'\cdot) - p'_k)$ satisfies all the conclusions. This yields the desired contradiction. \blacksquare

Definition 6.4. Given constants $K'' \geq 1$ and $E'' > 0$, we define $K_0 := \max\{K'(V), K''\}$ and $E_0 := \max\{E'(V), E''\}$. We also let $s_0 := s(K_0)$ and $\eta_0 := \eta(K_0)$.

We fix ε_0 (and thus δ_0) and ε'_0 so that Lemmas 6.2 and 6.3 apply with $K := K_0$, $E := E_0$. Since ε_0 depends on V , we can assume that it is chosen so small that

$$(6.10) \quad \left\lfloor \left(\frac{\eta_0}{\eta_0 - \varepsilon_0} \right)^2 V \right\rfloor + \frac{1}{2} = \lfloor V \rfloor + \frac{1}{2} = V.$$

This makes the last conclusion of Lemma 6.3 match one of the hypotheses, making it possible to iterate that result. On the other hand, the constants V , K'' , E'' (upon which all the aforementioned constants depend) will be fixed only in Section 7.

Lemma 6.5 ($n(\cdot)$ does not change from one scale to the next one). *There exists a constant $0 < \varepsilon_0'' \leq \frac{(\varepsilon_0')^2}{2} < \varepsilon_0'$ with the following property: if a conformal immersion $\Psi \in C^2(\overline{B}_r^2(z), \mathcal{M}_{p,\ell}^m)$ satisfies the hypotheses of the previous lemma (with $\varepsilon_0'', E_0, K_0$ in place of ε_0', E, K), then the new point p' , the new radius r' and the new scale ℓ' provided by Lemma 6.3 satisfy*

$$(6.11) \quad n(\Psi, B_{s_0' r}^2(z), B_{\eta_0'}^\Pi) = n(\Psi - p', B_{s_0' r'}^2(z), B_{\eta_0' \ell'}^\Pi) = n(\Psi - p', B_{s_0' r'}^2(z), B_{\eta_0' \ell'}^{\Pi'}).$$

Proof of Lemma 6.5. The second equality in (6.11) follows immediately from Lemma 6.2 (applied with $(\ell')^{-1}(\Psi - p')$ on $B_{r'}^2$), which gives

$$n(\Psi - p', B_{s_0' r'}^2, B_{\eta_0' \ell'}^\Pi) = n(\Psi - p', B_{s_0' r'}^2, B_{\eta_0' \ell'}^{\Pi'})$$

since $\text{dist}(\Pi', \Pi) < \varepsilon_0$.

Assume again $z = 0, r = 1$ and, by contradiction, that the first equality in (6.11) fails, so that we have again two sequences $\varepsilon_k \downarrow 0$ and Ψ_k . We can assume that $\Pi_k \rightarrow \Pi_\infty, p'_k \rightarrow p'_\infty, \ell'_k \rightarrow \ell'_\infty$ and $r'_k \rightarrow r'_\infty$, with $p'_\infty \in \mathcal{M}^m, \varepsilon_0' \leq \ell'_\infty \leq \frac{1}{2}$ and $\varepsilon_0' \leq r'_\infty \leq s_0$. Moreover, as in the proof of Lemma 6.3, up to further subsequences we get a limiting local parametrized stationary varifold $(\Omega_\infty, \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$ in \mathbb{R}^q , with $\Omega_\infty = \varphi_\infty(B_{s_0}^2)$. From Theorem 4.3 we know that Θ_∞ is harmonic and $N_\infty = \nu$ is constant, so Lemma 5.3 gives that $\Pi_\infty \circ \Theta_\infty$ is a diffeomorphism from $\varphi_\infty(\overline{B}_{s_0/2}^2)$ onto its image.

Calling \mathbf{v}_k the varifold issued by $\Psi_k|_{B_{s_0}^2}$ and \mathbf{v}_∞ the one issued by $(\varphi_\infty(B_{s_0}^2), \Theta_\infty, \nu)$, we have the varifold convergence $\mathbf{v}_k \xrightarrow{*} \mathbf{v}_\infty$ as $k \rightarrow \infty$. The area formula gives

$$\int_{B_{\eta_0}^{\Pi_k}} N(\Psi_k, B_{s_0}^2, \Pi_k) = \frac{\|(\Pi_k)_* \mathbf{v}_k\| (B_{\eta_0}^{\Pi_k})}{\pi \eta_0^2} \rightarrow \frac{\|(\Pi_\infty)_* \mathbf{v}_\infty\| (B_{\eta_0}^{\Pi_\infty})}{\pi \eta_0^2} = \nu,$$

since $(\Pi_\infty)_* \mathbf{v}_\infty$ equals an open superset of $B_{\eta_0}^{\Pi_\infty}$ in Π_∞ (by Lemma A.1), equipped with the constant integer multiplicity ν . Hence, $n(\Psi_k, B_{s_0}^2, B_{\eta_0}^{\Pi_k}) = \nu$ eventually.

Similarly, calling \mathbf{v}'_k the varifold induced by $\Psi_k|_{B_{s_0' r'_k}^2}$ and \mathbf{v}'_∞ the varifold induced by $(\varphi_\infty(B_{s_0' r'_\infty}^2), \Theta_\infty, \nu)$, we have $\mathbf{v}'_k \xrightarrow{*} \mathbf{v}'_\infty$ as $k \rightarrow \infty$, as is readily seen by approximating with domains which do not vary along the sequence. Since $(\ell'_\infty)^{-1}(\Psi_\infty(r'_\infty \cdot) - p'_\infty) \in \mathcal{R}_{K_0, \delta_0}^{\Pi_\infty}$, again $(\Pi_\infty)_* \mathbf{v}'_\infty$ equals a superset of $B_{\eta_0 \ell'_\infty}^{\Pi_\infty}$ in Π_∞ , with constant density ν . This gives again

$$\int_{B_{\eta_0 \ell'_k}^{\Pi_k}(q_k)} N(\Psi_k, B_{s_0' r'_k}^2, \Pi_k) = \frac{\|(\Pi_k)_* \mathbf{v}'_k\| (B_{\eta_0 \ell'_k}^{\Pi_k}(q_k))}{\pi \eta_0^2 (\ell'_k)^2} \rightarrow \frac{\|(\Pi_\infty)_* \mathbf{v}'_\infty\| (B_{\eta_0 \ell'_\infty}^{\Pi_\infty}(q_\infty))}{\pi \eta_0^2 (\ell'_\infty)^2} = \nu,$$

where $q_k := \Pi_k(p'_k)$ for $k \in \mathbb{N} \cup \{\infty\}$. Hence, $n(\Psi_k - p'_k, B_{s_0' r'_k}^2, B_{\eta_0 \ell'_k}^{\Pi_k}) = \nu$ eventually. So the first equality in (6.11) holds eventually, giving the desired contradiction. \blacksquare

Lemma 6.6 ($n(\cdot) = 1$ when $\tau = 1$). *Assume that $\Psi \in C^\infty(\overline{B}_r^2(z), \mathcal{M}_{p,\ell}^m)$ is a conformal immersion and Π is a 2-plane with $\Psi(z + r \cdot) \in \mathcal{R}_{K_0, \delta_0}^\Pi$ and $\frac{1}{2} \int_{B_r^2(z)} |\nabla \Psi|^2 \leq E$. If*

$\int_{B_1^2} |A|^4 d \text{vol}_{g_\Psi}$ and ℓ are sufficiently small, then $\Pi \circ \Psi$ is a diffeomorphism from $\overline{B}_{s_0^2 r}^2(z)$ onto its image.

Proof of Lemma 6.6. We can suppose $z = 0$, $r = 1$. Assume by contradiction that the claim does not hold, for a sequence of 2-planes $\Pi_k \rightarrow \Pi_\infty$ and immersions $\Psi_k : \overline{B}_1^2 \rightarrow \mathcal{M}_{p_k, \ell_k}^m$ with $\ell_k \rightarrow 0$ and second fundamental forms A_k satisfying

$$(6.12) \quad \int_{B_1^2} |A_k|^4 d \text{vol}_{g_{\Psi_k}} \rightarrow 0.$$

Let $\lambda_k \in C^\infty(\overline{B}_1^2)$ be defined by $|\partial_1 \Psi_k| = |\partial_2 \Psi_k| =: e^{\lambda_k}$ and let $A_{p, \ell}$ and \tilde{A}_k denote the second fundamental forms of $\mathcal{M}_{p, \ell}^m \subseteq \mathbb{R}^q$ and of the immersion Ψ_k in \mathbb{R}^q respectively, so that $\tilde{A}_k = A_{p_k, \ell_k} + A_k$. Note that

$$(6.13) \quad \|A_{p_k, \ell_k}\|_{L^\infty} \leq C(\mathcal{M}^m) \ell_k \rightarrow 0,$$

so that

$$(6.14) \quad \int_{B_1^2} |\tilde{A}_k|^4 d \text{vol}_{g_{\Psi_k}} \rightarrow 0.$$

With a slight abuse of notation, let us drop the dependence on k in the subsequent computations. We define the orthonormal frame

$$(6.15) \quad \tilde{e}_1 := e^{-\lambda} \partial_1 \Psi, \quad \tilde{e}_2 := e^{-\lambda} \partial_2 \Psi$$

for the tangent space of the immersed surface Ψ . It is straightforward to check that the map $\tilde{e}_1 \wedge \tilde{e}_2 : \overline{B}_1^2 \rightarrow \Lambda_2 \mathbb{R}^q$ has $|\nabla(\tilde{e}_1 \wedge \tilde{e}_2)| = e^\lambda |\tilde{A}|$, so

$$(6.16) \quad \int_{B_1^2} |\nabla(\tilde{e}_1 \wedge \tilde{e}_2)|^2 d\mathcal{L}^2 = \int_{B_1^2} e^{2\lambda} |\tilde{A}|^2 d\mathcal{L}^2 = \int_{B_1^2} |\tilde{A}|^2 d \text{vol}_{g_\Psi} \rightarrow 0$$

by Hölder's inequality, since $\int_{B_1^2} d \text{vol}_{g_\Psi} \leq E$. We identify the Grassmannian $\text{Gr}(2, \mathbb{R}^q)$ of 2-planes in \mathbb{R}^q with a submanifold of the projectivization of $\Lambda_2 \mathbb{R}^q$, by means of Plücker's embedding. For k large enough [3, Lemma 5.1.4] applies and provides a rotated frame (e_1, e_2) , given by

$$(6.17) \quad e_{\mathbb{C}} := e_1 + i e_2 = e^{i\theta} \tilde{e}_{\mathbb{C}}, \quad \tilde{e}_{\mathbb{C}} := \tilde{e}_1 + i \tilde{e}_2,$$

for a suitable real function $\theta \in W^{1,2}(B_1^2)$ minimizing $\int_{B_1^2} |\nabla \theta + \tilde{e}_1 \cdot \nabla \tilde{e}_2|^2$ (in particular, θ and $e_{\mathbb{C}}$ are smooth functions on \overline{B}_1^2) and with $\|\nabla e_{\mathbb{C}}\|_{L^2}^2$ becoming arbitrarily small as $k \rightarrow \infty$. We will assume in the sequel that $\|\nabla e_{\mathbb{C}}\|_{L^2}^2 \leq 1$. Observe that, whenever $\alpha, \beta \in C^1(\overline{B}_1^2)$,

$$\begin{aligned} \partial_1 \alpha \partial_2 \beta - \partial_2 \alpha \partial_1 \beta &= \frac{1}{4} (\partial_1 \alpha + \partial_2 \beta)^2 + \frac{1}{4} (\partial_2 \alpha - \partial_1 \beta)^2 - \frac{1}{4} (\partial_1 \alpha - \partial_2 \beta)^2 - \frac{1}{4} (\partial_2 \alpha + \partial_1 \beta)^2 \\ &= |\partial_z(\alpha + i\beta)|^2 - |\partial_{\bar{z}}(\alpha + i\beta)|^2. \end{aligned}$$

Hence, being $\tilde{e}_1 + i\tilde{e}_2 = 2e^{-\lambda}\partial_{\bar{z}}\Psi$ and $\partial_z\Psi \cdot \partial_z\Psi = \partial_{\bar{z}}\Psi \cdot \partial_{\bar{z}}\Psi = 0$ by conformality, we get

$$\begin{aligned}
 & -(\partial_1\tilde{e}_1 \cdot \partial_2\tilde{e}_2 - \partial_2\tilde{e}_1 \cdot \partial_1\tilde{e}_2) \\
 &= 4\partial_{\bar{z}}(e^{-\lambda}\partial_{\bar{z}}\Psi) \cdot \partial_z(e^{-\lambda}\partial_z\Psi) - 4\partial_{\bar{z}}(e^{-\lambda}\partial_z\Psi) \cdot \partial_z(e^{-\lambda}\partial_{\bar{z}}\Psi) \\
 &= 4e^{-2\lambda}(\partial_{\bar{z}\bar{z}}^2\Psi \cdot \partial_{zz}^2\Psi - \partial_{\bar{z}z}^2\Psi \cdot \partial_{z\bar{z}}^2\Psi - \partial_{\bar{z}}\lambda\partial_{\bar{z}}\Psi \cdot \partial_{zz}^2\Psi - \partial_z\lambda\partial_z\Psi \cdot \partial_{\bar{z}\bar{z}}^2\Psi) \\
 &\quad + 2e^{-2\lambda}\partial_{\bar{z}}\lambda\partial_{\bar{z}}(\partial_z\Psi \cdot \partial_z\Psi) + 2e^{-2\lambda}\partial_z\lambda\partial_z(\partial_{\bar{z}}\Psi \cdot \partial_{\bar{z}}\Psi) \\
 &= 4e^{-2\lambda}(\partial_{\bar{z}\bar{z}}^2\Psi \cdot \partial_{zz}^2\Psi - \partial_{\bar{z}z}^2\Psi \cdot \partial_{z\bar{z}}^2\Psi - \partial_{\bar{z}}\lambda\partial_{\bar{z}}\Psi \cdot \partial_{zz}^2\Psi - \partial_z\lambda\partial_z\Psi \cdot \partial_{\bar{z}\bar{z}}^2\Psi).
 \end{aligned}$$

On the other hand we have

$$2e^{2\lambda}\partial_z\lambda = \partial_z(e^{2\lambda}) = \partial_z(2\partial_{\bar{z}}\Psi \cdot \partial_z\Psi) = \partial_{\bar{z}}(\partial_z\Psi \cdot \partial_z\Psi) + 2\partial_{\bar{z}}\Psi \cdot \partial_{zz}^2\Psi = 2\partial_{\bar{z}}\Psi \cdot \partial_{zz}^2\Psi,$$

$$\begin{aligned}
 \Delta(e^{2\lambda}) &= 4\partial_{\bar{z}\bar{z}}^2(2\partial_{\bar{z}}\Psi \cdot \partial_z\Psi) = 8\partial_{\bar{z}}(\partial_z\Psi \cdot \partial_{zz}^2\Psi) + 4\partial_{\bar{z}\bar{z}}^2(\partial_z\Psi \cdot \partial_z\Psi) \\
 &= 8(\partial_{\bar{z}\bar{z}}^2\Psi \cdot \partial_{zz}^2\Psi - \partial_{\bar{z}z}^2\Psi \cdot \partial_{z\bar{z}}^2\Psi) + 4\partial_{\bar{z}\bar{z}}^2(\partial_z\Psi \cdot \partial_z\Psi) \\
 &= 8(\partial_{\bar{z}\bar{z}}^2\Psi \cdot \partial_{zz}^2\Psi - \partial_{\bar{z}z}^2\Psi \cdot \partial_{z\bar{z}}^2\Psi),
 \end{aligned}$$

so we arrive at

$$(6.18) \quad \partial_1\tilde{e}_1 \cdot \partial_2\tilde{e}_2 - \partial_2\tilde{e}_1 \cdot \partial_1\tilde{e}_2 = -\frac{\Delta(e^{2\lambda})}{2e^{2\lambda}} + 8\partial_{\bar{z}}\lambda\partial_z\lambda = -\Delta\lambda.$$

Alternatively, since the projections of $\partial_j\tilde{e}_1$ and $\partial_k\tilde{e}_2$ onto the tangent space of the immersion Ψ are orthogonal (being the projection of $\partial_j\tilde{e}_1$ a multiple of \tilde{e}_2 and the projection of $\partial_k\tilde{e}_2$ a multiple of \tilde{e}_1), we have

$$\partial_1\tilde{e}_1 \cdot \partial_2\tilde{e}_2 - \partial_2\tilde{e}_1 \cdot \partial_1\tilde{e}_2 = e^{2\lambda}(\tilde{A}(\tilde{e}_1, \tilde{e}_1) \cdot \tilde{A}(\tilde{e}_2, \tilde{e}_2) - \tilde{A}(\tilde{e}_1, \tilde{e}_2) \cdot \tilde{A}(\tilde{e}_2, \tilde{e}_1)) = e^{2\lambda}K,$$

by Gauss' formula, K denoting the Gaussian curvature of the immersed surface. But, by the well-known formula for the curvature of a conformal metric, we have $K = -e^{-2\lambda}\Delta\lambda$, which gives again (6.18). Moreover,

$$\begin{aligned}
 \partial_1e_1 \cdot \partial_2e_2 - \partial_2e_1 \cdot \partial_1e_2 &= \Im \langle \nabla\bar{e}_{\mathbb{C}}; \nabla e_{\mathbb{C}} \rangle = \Im \langle \nabla\bar{e}_{\mathbb{C}} - i\bar{e}_{\mathbb{C}} \otimes \nabla\theta; \nabla\tilde{e}_{\mathbb{C}} + i\tilde{e}_{\mathbb{C}} \otimes \nabla\theta \rangle \\
 &= \Im \langle \nabla\bar{e}_{\mathbb{C}}; \nabla\tilde{e}_{\mathbb{C}} \rangle = \partial_1\tilde{e}_1 \cdot \partial_2\tilde{e}_2 - \partial_2\tilde{e}_1 \cdot \partial_1\tilde{e}_2,
 \end{aligned}$$

since $\langle \bar{e}_{\mathbb{C}} \otimes \nabla\theta; \tilde{e}_{\mathbb{C}} \otimes \nabla\theta \rangle$ is real and $\langle -i\bar{e}_{\mathbb{C}} \otimes \nabla\theta; \nabla\tilde{e}_{\mathbb{C}} \rangle = \overline{\langle \nabla\bar{e}_{\mathbb{C}}; i\tilde{e}_{\mathbb{C}} \otimes \nabla\theta \rangle}$. Thus, calling $\mu \in C^\infty(\bar{B}_1^2)$ the solution to

$$\begin{cases} -\Delta\mu = \partial_1e_1 \cdot \partial_2e_2 - \partial_2e_1 \cdot \partial_1e_2 & \text{on } B_1^2 \\ \mu = 0 & \text{on } \partial B_1^2, \end{cases}$$

we obtain that $\lambda - \mu$ is harmonic and, by Wente's inequality,

$$(6.19) \quad \|\mu\|_{L^\infty} \leq C(q) \left(\|\nabla e_1\|_{L^2}^2 + \|\nabla e_2\|_{L^2}^2 \right) \leq C(q).$$

Since $\lambda < e^{2\lambda}$, for all $x \in \bar{B}_{3/4}^2$ we get

$$(6.20) \quad (\lambda - \mu)(x) = \int_{B_{1/4}^2(x)} (\lambda - \mu) \leq \int_{B_{1/4}^2(x)} e^{2\lambda} + \|\mu\|_{L^\infty} \leq \frac{E}{\mathcal{L}^2(B_{1/4}^2)} + C(q).$$

Together with (6.19), this gives an upper bound for λ on $B_{3/4}^2$, depending only on E, q . Although this is sufficient for the present purposes, one can also get a lower bound for λ on $B_{s_0}^2$. Indeed, calling M the right-hand side of (6.20), we obtain that $M - (\lambda - \mu)$ is a nonnegative harmonic function on $B_{3/4}^2$. Moreover, the length of the curve $\Psi|_{\partial B_{s_0}^2}$ is

$$(6.21) \quad \int_{\partial B_{s_0}^2} e^\lambda \geq 2\pi\eta_0,$$

since the composition of $\Pi \circ \Psi|_{\partial B_{s_0}^2}$ with the radial projection onto $\partial B_{\eta_0}^\Pi$ (which does not increase the length) is surjective (being a generator of the fundamental group of $\partial B_{\eta_0}^\Pi$). Hence, there exists some $x \in \partial B_{s_0}^2$ such that $\lambda(x) \geq \log(s_0^{-1}\eta_0)$. We deduce that

$$(6.22) \quad \inf_{B_{s_0}^2} (M - (\lambda - \mu)) \leq M + C(q) - \log(s_0^{-1}\eta_0)$$

and so, by Harnack's inequality, the supremum of $M - (\lambda - \mu)$ on $B_{s_0}^2$ is bounded by a constant depending only on E, s_0, η_0, q . This, together with (6.20) and (6.19), gives

$$(6.23) \quad \|\lambda\|_{L^\infty(B_{s_0}^2)} \leq C(E, s_0, \eta_0, q).$$

The mean curvature of the immersion Ψ is $\tilde{H} = \frac{1}{2e^{2\lambda}}(\tilde{A}(\partial_1\Psi, \partial_1\Psi) + \tilde{A}(\partial_2\Psi, \partial_2\Psi)) = -\frac{\Delta\Psi}{2e^{2\lambda}}$ (note that $\Delta\Psi$ is already orthogonal to the tangent space of the immersion, since $\partial_z\Psi \cdot \Delta\Psi = 4\partial_z\Psi \cdot \partial_{\bar{z}\bar{z}}^2\Psi = 2\partial_{\bar{z}}(\partial_z\Psi \cdot \partial_z\Psi) = 0$). So we get

$$(6.24) \quad \begin{aligned} \int_{B_{3/4}^2} |\Delta\Psi_k|^4 d\mathcal{L}^2 &= 16 \int_{B_{3/4}^2} |\tilde{H}_k|^4 e^{6\lambda_k} d\text{vol}_{g_{\Psi_k}} \\ &\leq C(E, q) \int_{B_{3/4}^2} |\tilde{A}_k|^4 d\text{vol}_{g_{\Psi_k}} \rightarrow 0. \end{aligned}$$

Since $s_0 \leq \frac{1}{2}$, this implies that (Ψ_k) is a bounded sequence in $W^{2,4}(B_{s_0}^2)$ (by Lemma A.2 applied to $\Psi_k(\frac{3}{4}\cdot)$), so by the compact embedding $W^{2,4}(B_{s_0}^2) \hookrightarrow C^1(\bar{B}_{s_0}^2)$ we obtain a strong limit Ψ_∞ in $C^1(\bar{B}_{s_0}^2)$, up to subsequences. Thus Ψ_∞ is weakly conformal and, by (6.24), it is also harmonic. Lemma 5.3 applies (with $\Psi_\infty(s_0\cdot)$ and $\text{id}_{\mathbb{R}^2}$ in place of Ψ and φ) and gives that $\Pi_\infty \circ \Psi_\infty$ is a diffeomorphism from $\bar{B}_{s_0/2}^2 \supseteq \bar{B}_{s_0}^2$ onto its image, hence the same is eventually true for $\Pi_k \circ \Psi_k$, giving the desired contradiction. \blacksquare

7. MULTIPLICITY ONE IN THE LIMIT

Theorem 7.1 ($n(\cdot) = 1$ for small $\sigma^2 \log(\sigma^{-1}) \int |A|^4$). *Assume $\Phi \in C^\infty(\bar{B}_r^2(z), \mathcal{M}^m)$ is a conformal immersion, critical for (4.1) on $B_r^2(z)$ and satisfying*

- $\ell^{-1}(\Phi(z + r\cdot) - p) \in \mathcal{R}_{K_0, \delta_0}^\Pi$ for some $\sqrt{\sigma/\varepsilon_0''} < \ell < 1$ and $p \in \mathcal{M}^m$,
- $\frac{1}{2} \int_{B_r^2(z)} |\nabla\Phi|^2 \leq E_0\ell^2$,
- $\int_{\Phi^{-1}(B_\ell^q(p))} d\text{vol}_{g_\Phi} \leq V\pi\ell^2$ and $\int_{\Phi^{-1}(B_{\eta_0\ell}^q(p))} d\text{vol}_{g_\Phi} \leq V\pi(\eta_0\ell)^2$,

- $\sigma^2 \log(\sigma^{-1}) \int_{B_s^2} |A|^4 d\text{vol}_{g_\Phi} \leq \frac{\varepsilon_0''}{E_0} \int_{B_s^2} d\text{vol}_{g_\Phi}$ for all $0 < s \leq r$.

Then, if σ and ℓ are small enough (independently of each other), we have

$$n(\Phi - p, B_{s_0^2 r}^2(z), B_{\eta_0 \ell}^\Pi) = 1.$$

Proof of Theorem 7.1. Let $r_0 := r$, $p_0 := p$, $\ell_0 := \ell$, $\tau_0 := \sigma \ell_0^{-2}$ and $\Pi_0 := \Pi$. Note that

$$\Psi_0 := \ell^{-1}(\Phi - p) = \ell_0^{-1}(\Phi - p_0)$$

is critical for (4.3), with $\tau := \tau_0 \leq \varepsilon_0''$. Thus Lemma 6.3 applies to Ψ_0 (if ℓ is small enough), giving a new radius $\varepsilon_0' r_0 < r_1 < s_0 r_0$, a new point $p' \in \mathcal{M}_{p, \ell}^m$, a new scale $\varepsilon_0' < \ell' < \frac{1}{2}$ and a new 2-plane Π' . Setting $r_1 := r'$, $p_1 := p_0 + \ell_0 p'$, $\ell_1 := \ell' \ell_0$, $\tau_1 := \sigma \ell_1^{-2}$, $\Pi_1 := \Pi'$ and recalling (6.10), the map

$$\Psi_1 := (\ell')^{-1}(\Psi_0 - p') = \ell_1^{-1}(\Phi - p_1)$$

still satisfies the hypotheses of Lemma 6.3, with the parameters $r_1, \tau_1, p_1, \ell_1, \Pi_1$, provided that $\tau_1 \leq \varepsilon_0'$: indeed, note that (assuming $\tau_1 \leq \varepsilon_0' < 1$)

$$\begin{aligned} \tau_1^2 \log(\tau_1^{-1}) \int_{B_{r_1}^2(z)} |A_{\Psi_1}|^4 d\text{vol}_{g_{\Psi_1}} &\leq \tau_1^2 \log(\sigma^{-1}) \int_{B_{r_1}^2(z)} |A_{\Psi_1}|^4 d\text{vol}_{g_{\Psi_1}} \\ &= \ell_1^{-2} \sigma^2 \log(\sigma^{-1}) \int_{B_{r_1}^2(z)} |A_\Phi|^4 d\text{vol}_{g_\Phi} \leq \frac{\varepsilon_0'' \ell_1^{-2}}{E_0} \int_{B_{r_1}^2(z)} d\text{vol}_{g_\Phi} = \frac{\varepsilon_0''}{2E_0} \int_{B_{r_1}^2(z)} |\nabla \Psi_1|^2 \leq \varepsilon_0''. \end{aligned}$$

Hence, we can iterate and define $r_j, \tau_j, p_j, \ell_j, \Pi_j$, for $j = 0, 1, \dots$, up to a maximum index $k \geq 1$ such that $\tau_j \leq \varepsilon_0'' \leq \varepsilon_0'$ for $1 \leq j < k$ and $\tau_k > \varepsilon_0''$: such k exists since $\tau_j = \ell_j^{-2} \sigma \geq 4^j \tau_0$. With the same computation as above, this implies

$$(7.1) \quad \int_{B_{r_k}^2(z)} |A|^4 d\text{vol}_{g_{\Psi_k}} \leq \frac{\varepsilon_0''}{\tau_k^2 \log(\sigma^{-1})} \leq \frac{1}{\varepsilon_0'' \log(\sigma^{-1})}.$$

If σ and ℓ are small enough, Lemma 6.6 applies to the map $\Psi_k := \ell_k^{-1}(\Psi - p_k)$, on the ball $B_{r_k}^2(z)$: indeed, note that $\ell_k \leq \ell$ and $\int_{B_{r_k}^2(z)} |A|^4 d\text{vol}_{g_{\Psi_k}}$ can be assumed arbitrarily small (by taking σ and ℓ small enough), by virtue of (7.1). This, together with Lemma A.1, gives

$$n(\Psi_k, B_{s_0^2 r_k}^2(z), B_{\eta_0}^{\Pi_k}) = 1.$$

Also, Lemma 6.5 applies for all $j = 0, \dots, k-1$, giving

$$\begin{aligned} n(\Phi - p, B_{s_0^2 r}^2(z), B_{\eta_0 \ell}^\Pi) &= n(\Psi_0, B_{s_0^2 r_0}^2(z), B_{\eta_0}^{\Pi_0}) \\ &= n(\Psi_1, B_{s_0^2 r_1}^2(z), B_{\eta_0}^{\Pi_1}) \\ &= \dots \\ &= n(\Psi_k, B_{s_0^2 r_k}^2(z), B_{\eta_0}^{\Pi_k}) \\ &= 1. \end{aligned} \quad \blacksquare$$

As in Section 4, assume now that $\Phi_k : \Sigma \rightarrow \mathcal{M}^m$ is a sequence of critical points for

$$(7.2) \quad \int_{\Sigma} d\text{vol}_{g_{\Phi_k}} + \sigma_k^2 \int_{\Sigma} (1 + |A|^2)^2 d\text{vol}_{g_{\Phi_k}}$$

with controlled area, namely

$$\lambda \leq \int_{\Sigma} d\text{vol}_{g_{\Phi_k}} \leq \Lambda,$$

and with

$$\sigma_k \rightarrow 0, \quad \sigma_k^2 \log(\sigma_k^{-1}) \int_{\Sigma} (1 + |A|^2)^2 d\text{vol}_{g_{\Phi_k}} \rightarrow 0.$$

By the main result of [13], up to subsequences the varifolds \mathbf{v}_k induced by Φ_k converge to a parametrized stationary varifold (see e.g. [12, Definition 2.2] and [12, Remark 2.3] for two equivalent definitions of this notion).

As explained in [13], there could be bubbling points, and also the conformal structures induced by Φ_k could degenerate (in the space of conformal structures up to diffeomorphisms). The latter cannot happen if Σ is a sphere, while it can yield a cylinder \mathbb{C}/\mathbb{Z} (or, equivalently, $\mathbb{C} \setminus \{0\} = \widehat{\mathbb{C}} \setminus \{0, \infty\}$) in the limit when Σ is a torus. While both of these statements are easy consequences of the uniformization theorem for Riemann surfaces, if the genus is at least two then the general picture of degenerating conformal structures is more complicated; we refer the reader to [4, Section IV.5] for its description.⁸

In the remainder of the paper, we will assume for simplicity that there is no bubbling and no degeneration of the conformal structure. Note that the arguments will apply also to the general case, working on appropriate domains different from Σ .⁹

Up to precomposing Φ_k with suitable diffeomorphisms of Σ , we can thus assume that there exist metrics g_k of constant curvature (1, 0 or -1 , depending on the genus of Σ) such that $\Phi_k : (\Sigma, g_k) \rightarrow \mathcal{M}^m$ is conformal, and such that g_k converges smoothly to a limiting Riemannian metric g_{∞} . The limiting varifold \mathbf{v}_{∞} is a parametrized stationary varifold, of the form $(\Sigma_{\infty}, \Theta_{\infty}, N_{\infty})$, where $\Theta_{\infty} : \Sigma_{\infty} \rightarrow \mathcal{M}^m$ is a smooth branched minimal immersion. Also, calling $\Phi_{\infty} \in W^{1,2}(\Sigma, \mathcal{M}^m)$ the weak limit of Φ_k (up to subsequences), $\Theta_{\infty} = \Phi_{\infty} \circ \varphi_{\infty}^{-1}$ for some quasiconformal homeomorphism $\varphi_{\infty} : \Sigma \rightarrow \Sigma_{\infty}$, with respect to g_{∞} . Here Σ_{∞} is a Riemann surface homeomorphic (by means of φ_{∞}) to Σ : see the proof of [13, Lemma III.13]. In particular, Φ_{∞} is continuous.

⁸Given a sequence of closed hyperbolic surfaces, we can assume that they have been decomposed into marked pairs of pants, with a constant combinatorial configuration (see e.g. [4, Section IV.3]). We can then cut them open along the marked geodesics whose length converges to zero and apply [4, Proposition IV.5.1] to the resulting surfaces with boundary.

⁹Namely, if the conformal structure does not degenerate, on the complement of suitable small disks centered at the bubbling points, or (as in a standard bubble tree analysis) on such small disks (suitably rescaled to bigger and bigger disks converging to \mathbb{C}), again working far from children bubbles; if the conformal structure degenerates, one also has to discard small tubular neighborhoods of collapsing geodesics (see also the previous footnote).

In local conformal coordinates for (Σ, g_∞) , as in (4.5) we have

$$(7.3) \quad \text{vol}_{g_{\Phi_k}} \xrightarrow{*} N_\infty |\partial_1 \Phi_\infty \wedge \partial_2 \Phi_\infty| \mathcal{L}^2 \geq \frac{1}{2} |\nabla \Phi_\infty|^2 \mathcal{L}^2.$$

By the regularity result of [12], which was already exploited in Section 6, N_∞ is locally a.e. constant and thus a.e. constant (being Σ_∞ connected).

Setting $\nu_k := \text{vol}_{g_{\Phi_k}}$ and $\nu_\infty := N_\infty |\partial_1 \Phi_\infty \wedge \partial_2 \Phi_\infty| \mathcal{L}^2$ (in local conformal coordinates for Σ), by (7.3) we have $\nu_k \xrightarrow{*} \nu_\infty$. We can find a conformal disk $U \subset (\Sigma, g_\infty)$, which we identify with $B_1^2 \subset \mathbb{C}$ and fix in the sequel, such that $\nu_\infty(B_{1/2}^2) > 0$.

Definition 7.2. We denote by ν the constant value of N_∞ . Also, we call T the set of bad points $z \in B_1^2$ which are not Lebesgue for $\nabla \Phi_\infty$, or such that $\nabla \Phi_\infty(z)$ does not have full rank, or such that

$$(7.4) \quad \max_{|x|=1} |\langle \nabla \Phi_\infty(z), x \rangle| > 2\nu \min_{|x|=1} |\langle \nabla \Phi_\infty(z), x \rangle|.$$

We have $\mathcal{L}^2(T) = 0$, since $\nabla \Phi_\infty$ has full rank a.e. by conformality (hence the same holds for Φ_∞ by the chain rule¹⁰) and since (7.4) implies $\nu |\partial_1 \Phi_\infty \wedge \partial_2 \Phi_\infty|(z) < \frac{1}{2} |\nabla \Phi_\infty|^2(z)$ (as it can be immediately verified using a singular value decomposition for $\nabla \Phi_\infty(z)$).

Definition 7.3. We now specify $K'' := 2\nu$ and we set $E'' := \pi\nu((K'')^2 + 1)$. Finally, we choose $V > 0$ such that $V = \lfloor V \rfloor + \frac{1}{2}$ and

$$(7.5) \quad \|\mathbf{v}_\infty\|(\overline{B}_\ell^q(p)) < V\pi\ell^2$$

for all $\ell > 0$ and all $p \in \mathcal{M}^m$. Such V exists by the monotonicity formula satisfied by the stationary varifold \mathbf{v}_∞ . Note that now also the constants K_0, E_0, s_0, η_0 , as well as $\varepsilon_0, \delta_0, \varepsilon'_0$ and ε''_0 , are determined.

Theorem 7.4 (multiplicity one). *We have $N_\infty = 1$ a.e., or equivalently $\nu = 1$.*

Proof of Theorem 7.4. Let \mathcal{B}_k be the Borel set of points $z \in B_{1/2}^2$ such that

$$\sigma_k^2 \log(\sigma_k^{-1}) \int_{B_r^2(z)} |A|^4 d\text{vol}_{g_{\Phi_k}} \geq \frac{\varepsilon_0''}{E_0} \int_{B_r^2(z)} d\text{vol}_{g_{\Phi_k}}$$

for some radius $0 < r < \frac{1}{2}$. By Besicovitch's covering lemma, we get a collection of points $z_i \in \mathcal{B}_k$ and radii $0 < r_i < \frac{1}{2}$ such that

$$\sigma_k^2 \log(\sigma_k^{-1}) \int_{B_{r_i}^2(z_i)} |A|^4 d\text{vol}_{g_{\Phi_k}} \geq \frac{\varepsilon_0''}{E_0} \int_{B_{r_i}^2(z_i)} d\text{vol}_{g_{\Phi_k}}, \quad \mathbf{1}_{\mathcal{B}_k} \leq \sum_i \mathbf{1}_{B_{r_i}^2(z_i)} \leq \mathfrak{N},$$

for some universal constant \mathfrak{N} . Thus we get

$$\begin{aligned} \nu_k(\mathcal{B}_k) &\leq \sum_i \text{vol}_{g_{\Phi_k}}(B_{r_i}^2(z_i)) \leq \frac{E_0}{\varepsilon_0''} \sigma_k^2 \log(\sigma_k^{-1}) \sum_i \int_{B_{r_i}^2(z_i)} |A|^4 d\text{vol}_{g_{\Phi_k}} \\ &\leq \frac{E_0 \mathfrak{N}}{\varepsilon_0''} \sigma_k^2 \log(\sigma_k^{-1}) \int_\Sigma |A|^4 d\text{vol}_{g_{\Phi_k}} \rightarrow 0. \end{aligned}$$

¹⁰See e.g. [5, Lemma 4.12] and [8, Lemma III.6.4].

Up to subsequences, we can assume that $\overline{B_{1/2}^2 \setminus \mathcal{B}_k}$ converges in the Hausdorff topology to some compact set $S \subseteq \overline{B_{1/2}^2}$. We remark that $\nu_\infty(S) > 0$: indeed, for any compact neighborhood F of S in B_1^2 , we have $B_{1/2}^2 \setminus \mathcal{B}_k \subseteq F$ eventually and so

$$\nu_\infty(F) \geq \limsup_{k \rightarrow \infty} \nu_k(F) \geq \limsup_{k \rightarrow \infty} (\nu_k(B_{1/2}^2) - \nu_k(\mathcal{B}_k)) = \limsup_{k \rightarrow \infty} \nu_k(B_{1/2}^2) \geq \nu_\infty(B_{1/2}^2) > 0.$$

It follows from (7.3) that $\mathcal{L}^2(S) > 0$.

We now show that $N_\infty = 1$ a.e. on $S \setminus T$, which has positive Lebesgue measure. This will show that $\nu = 1$, as desired. Fix any $z \in S \setminus T$ and take a sequence $z_k \in B_{1/2}^2 \setminus \mathcal{B}_k$ with $z_k \rightarrow z$. Locally we can find conformal reparametrizations $\tilde{\Phi}_k$ of $\Phi_k(z_k + \cdot)$, by means of diffeomorphisms converging smoothly to the identity on a small neighborhood of 0.¹¹ By weak convergence $\tilde{\Phi}_k \rightharpoonup \Phi_\infty(z + \cdot)$ in $W^{1,2}$, for a.e. radius $r > 0$ we have

$$(7.6) \quad \tilde{\Phi}_k(r \cdot) \rightarrow \Phi_\infty(z + r \cdot) \quad \text{in } C^0(\partial B_1^2 \cup \partial B_{s_0}^2 \cup \partial B_{s_0^2}^2)$$

up to further subsequences.¹² Using [12, Lemma A.4] and the fact that $z \notin T$, we can assume that r satisfies

$$(7.7) \quad |\Phi_\infty(z + rx) - \Phi_\infty(z) - \langle \nabla \Phi_\infty(z), rx \rangle| < \delta_0 \ell \quad \text{for } x \in \partial B_1^2 \cup \partial B_{s_0}^2 \cup \partial B_{s_0^2}^2,$$

$$(7.8) \quad \frac{1}{2} \int_{B_r^2(z)} |\nabla \Phi_\infty|^2 < (\pi r^2) |\nabla \Phi_\infty(z)|^2 \leq \ell^2 \pi ((K'')^2 + 1),$$

with $\ell := r \min_{|x|=1} |\langle \nabla \Phi_\infty(z), x \rangle|$. Note that (7.7) will guarantee (below) that the first assumption in Theorem 7.1 holds for $\tilde{\Phi}_k$. Setting $p := \Phi_\infty(z)$, note that (7.5) gives

$$\|\mathbf{v}_k\| (B_\ell^q(p)) < V\pi\ell^2, \quad \|\mathbf{v}_k\| (B_{\eta_0\ell}^q(p)) < V\pi(\eta_0\ell)^2$$

eventually, which trivially implies

$$(7.9) \quad \int_{\tilde{\Phi}_k^{-1}(B_\ell^q(p))} d\text{vol}_{g_{\tilde{\Phi}_k}} < V\pi\ell^2, \quad \int_{\tilde{\Phi}_k^{-1}(B_{\eta_0\ell}^q(p))} d\text{vol}_{g_{\tilde{\Phi}_k}} < V\pi(\eta_0\ell)^2.$$

Also, (7.3) and (7.8) give

$$\lim_{k \rightarrow \infty} \frac{1}{2} \int_{B_r^2(z)} |\nabla \tilde{\Phi}_k|^2 = \lim_{k \rightarrow \infty} \nu_k(B_r^2(z)) \leq \frac{\nu}{2} \int_{B_r^2(z)} |\nabla \Phi_\infty|^2 < E''\ell^2.$$

Thanks to the fact that $z_k \notin \mathcal{B}_k$ and the above inequalities, eventually $\tilde{\Phi}_k$ satisfies the hypotheses of Theorem 7.1 on the ball B_r^2 , provided that r (and thus ℓ) is chosen small enough. Setting $\Psi_k := \ell^{-1}(\tilde{\Phi}_k - p)$, we infer that

$$(7.10) \quad n(\Psi_k, B_{s_0^2 r}^2, B_{\eta_0}^\Pi) = 1,$$

where Π is the 2-plane spanned by $\nabla \Phi_\infty(z)$.

¹¹For instance, one can isometrically identify a neighborhood of z_k in (Σ, g_k) with a neighborhood of z in (Σ, g_∞) , by means of the exponential map.

¹²This can be obtained by applying [12, Lemma A.5] to the weakly converging \mathbb{R}^{3q} -valued maps

$$(\tilde{\Phi}_k, \tilde{\Phi}_k(s_0 \cdot), \tilde{\Phi}_k(s_0^2 \cdot)) \rightharpoonup (\Phi_\infty(z + \cdot), \Phi_\infty(z + s_0 \cdot), \Phi_\infty(z + s_0^2 \cdot)).$$

Since r can be chosen arbitrarily small (possibly changing the subsequence guaranteeing (7.6)), the argument used in the proof of [13, Lemma III.10] shows that $N_\infty(z) = 1$.

Alternatively, (7.10) gives

$$\left| \frac{\|\Pi_* \mathbf{v}'_k\| (B_{\eta_0}^\Pi)}{\pi \eta_0^2} - 1 \right| < \frac{1}{8},$$

where the varifold \mathbf{v}'_k is induced by $\Psi_k|_{B_{s_0^2 r}^2}$ and converges to the varifold \mathbf{v}'_∞ induced by $(\varphi_\infty(B_{s_0^2 r}^2(z)), \ell^{-1}(\Theta_\infty - p), \nu)$. Assuming without loss of generality that $\nabla \Theta_\infty(\varphi_\infty(z)) \neq 0$, $\Pi \circ \Theta_\infty$ is a diffeomorphism from $\varphi_\infty(B_{s_0^2 r}^2(z))$ onto its image (for r small enough). Hence, at a.e. point of Π the varifold $\Pi_* \mathbf{v}'_\infty$ has density either 0 or ν . Since $\Pi \circ \Phi_\infty(B_{s_0^2 r}^2(z))$ is a superset of $B_{\eta_0 \ell}^\Pi(\Pi(p))$ (by Lemma A.1), it follows that

$$\frac{\|\Pi_* \mathbf{v}'_\infty\| (B_{\eta_0}^\Pi)}{\pi \eta_0^2} = \nu.$$

The convergence $\frac{\|\Pi_* \mathbf{v}'_k\| (B_{\eta_0}^\Pi)}{\pi \eta_0^2} \rightarrow \frac{\|\Pi_* \mathbf{v}'_\infty\| (B_{\eta_0}^\Pi)}{\pi \eta_0^2}$ thus gives $|\nu - 1| \leq \frac{1}{8}$, and again we conclude that $\nu = 1$. \blacksquare

APPENDIX.

Lemma A.1 (big image under a boundary constraint). *Assume that $F \in C^0(\overline{B}_1^2, \mathbb{R}^2)$ satisfies*

$$(A.1) \quad |F(x) - \varphi(x)| \leq \delta \quad \text{for all } x \in \partial B_1^2$$

for some $0 < \delta < 1$ and some homeomorphism $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, with $\varphi(0) = 0$ and $\min_{|x|=1} |\varphi(x)| \geq 1$. Then

$$(A.2) \quad F(B_1^2) \supseteq B_{1-\delta}^2.$$

Proof of Lemma A.1. It suffices to show that, for a fixed $y \in B_{1-\delta}^2$, the closed curve $\Gamma' := F|_{\partial B_1^2}$ is not contractible in $\mathbb{R}^2 \setminus \{y\}$: once this is done, if we had $y \notin F(B_1^2)$, i.e. $y \notin F(\overline{B}_1^2)$, then F would provide a homotopy from Γ' to the constant curve $F(0)$ in $\mathbb{R}^2 \setminus \{y\}$, yielding a contradiction.

Letting $\Gamma := \varphi|_{\partial B_1^2}$ and $\gamma := \Gamma' - \Gamma$, we have $|\gamma(x)| \leq \delta$ for all $x \in \partial B_1^2$. Hence, Γ is homotopic to Γ' in $\mathbb{R}^2 \setminus B_{1-\delta}^2 \subseteq \mathbb{R}^2 \setminus \{y\}$ by means of the homotopy

$$\Gamma + t\gamma, \quad 0 \leq t \leq 1.$$

So we are left to show that Γ is not contractible in $\mathbb{R}^2 \setminus \{y\}$, i.e. that $\Gamma - y$ is not contractible in $\mathbb{R}^2 \setminus \{0\}$. The curve $\Gamma - y$ is homotopic to Γ in $\mathbb{R}^2 \setminus \{0\}$, by means of the homotopy

$$\Gamma - ty, \quad 0 \leq t \leq 1,$$

which avoids the origin since $|y| < 1$. Finally, Γ is not contractible in $\mathbb{R}^2 \setminus \{0\}$, since φ (once restricted to a homeomorphism of $\mathbb{R}^2 \setminus \{0\}$) induces an automorphism of $\pi_1(\mathbb{R}^2 \setminus \{0\})$

sending the class of the generator $\text{id}_{\partial B_1^2}$ to the class of Γ . Hence, $\Gamma - y$ is not contractible in $\mathbb{R}^2 \setminus \{0\}$, too, as desired. \blacksquare

Lemma A.2 (elliptic $W^{2,4}$ -estimate). *For a function $\Psi \in C^\infty(\overline{B_1})$ and a $0 < \tau < 1$ we have*

$$\|\Psi\|_{W^{2,4}(B_\tau^2)} \leq C(\tau)(\|\Delta\Psi\|_{L^4(B_1^2)} + \|\nabla\Psi\|_{L^2(B_1^2)} + \|\Psi\|_{L^2(B_1^2)}).$$

Proof of Lemma A.2. Given two radii $0 < r < s \leq 1$, let us choose a cut-off function $\rho \in C_c^\infty(B_s^2)$ with $\rho = 1$ on B_r^2 . Since $\rho\Psi \in C_c^\infty(\mathbb{R}^2)$, standard Calderón–Zygmund estimates give

$$(A.3) \quad \begin{aligned} \|\nabla^2\Psi\|_{L^p(B_r^2)} &\leq \|\nabla^2(\rho\Psi)\|_{L^p(\mathbb{R}^2)} \leq C(p) \|\Delta(\rho\Psi)\|_{L^p(\mathbb{R}^2)} \\ &\leq C(p, r, s)(\|\Delta\Psi\|_{L^p(B_s^2)} + \|\nabla\Psi\|_{L^p(B_s^2)} + \|\Psi\|_{L^p(B_s^2)}) \end{aligned}$$

for all $1 < p < \infty$. Setting $t := \frac{1+\tau}{2}$ and applying (A.3) with $p := 2$, $r := t$ and $s := 1$ we get

$$\|\nabla^2\Psi\|_{L^2(B_t^2)} \leq C(\tau)(\|\Delta\Psi\|_{L^2(B_1^2)} + \|\nabla\Psi\|_{L^2(B_1^2)} + \|\Psi\|_{L^2(B_1^2)}),$$

hence $\|\Psi\|_{W^{2,2}(B_t^2)}$ is bounded by the desired quantity. Using Sobolev's embedding $W^{2,2}(B_t^2) \hookrightarrow W^{1,4}(B_t^2)$ and (A.3) with $p := 4$, $r := \tau$ and $s := t$, we obtain

$$\begin{aligned} \|\Psi\|_{W^{2,4}(B_\tau^2)} &\leq C(\tau)(\|\Delta\Psi\|_{L^4(B_t^2)} + \|\Psi\|_{W^{2,2}(B_t^2)}) \\ &\leq C(\tau)(\|\Delta\Psi\|_{L^4(B_1^2)} + \|\nabla\Psi\|_{L^2(B_1^2)} + \|\Psi\|_{L^2(B_1^2)}). \end{aligned} \quad \blacksquare$$

Lemma A.3 (compactness of normal solutions to Beltrami equation). *Given a sequence $\psi_k : \mathbb{C} \rightarrow \mathbb{C}$ of K -quasiconformal homeomorphisms with the normalization conditions*

$$\psi_k(0) = 0, \quad \psi_k(1) = 1,$$

there exists a K -quasiconformal homeomorphism $\psi_\infty : \mathbb{C} \rightarrow \mathbb{C}$ satisfying the same normalization condition and such that, up to subsequences, $\psi_k \rightarrow \psi_\infty$ and $\psi_k^{-1} \rightarrow \psi_\infty^{-1}$ in $C_{loc}^0(\mathbb{C})$.

Proof of Lemma A.3. Let $\mu_k \in \mathcal{E}_K$ be defined by $\partial_{\bar{z}}\psi_k = \mu_k\partial_z\psi_k$.¹³ Existence and uniqueness of a K -quasiconformal homeomorphism satisfying this equation and the normalization conditions is shown in [5, Theorem 4.30].

Given $M > 0$, we consider the set $\mathcal{E}_K^M := \{\mu \in \mathcal{E}_K : \mu = 0 \text{ a.e. on } \mathbb{C} \setminus B_M^2\}$. If F^μ denotes the normal solution to the equation $\partial_{\bar{z}}F^\mu = \mu\partial_zF^\mu$ (in the sense of [5, Theorem 4.24]), then F^μ satisfies estimates (4.21) and (4.24) in [5]. Applying them with the points 0 and 1, we infer that also the map $f^\mu := F^\mu(1)^{-1}F^\mu$ satisfies estimates of the form

$$(A.4) \quad |f^\mu(z_1) - f^\mu(z_2)| \leq C|z_1 - z_2|^\alpha + C|z_1 - z_2|,$$

¹³Actually, the coefficient μ_k is uniquely determined a.e., as $\partial_z\psi_k \neq 0$ a.e. (this follows from $\text{id}_{\mathbb{C}} = \psi_k^{-1} \circ \psi_k$ and the chain rule [8, Lemma III.6.4], together with $|\partial_{\bar{z}}\psi_k| \leq |\partial_z\psi_k|$).

$$(A.5) \quad |z_1 - z_2| \leq C |f^\mu(z_1) - f^\mu(z_2)|^\alpha + C |f^\mu(z_1) - f^\mu(z_2)|,$$

with C and $0 < \alpha < 1$ depending only on K and M . Given a sequence of homeomorphisms $f_k : \mathbb{C} \rightarrow \mathbb{C}$ satisfying these estimates, Ascoli–Arzelà theorem applies to f_k and f_k^{-1} and so we can extract a subsequence (not relabeled) such that

$$f_k \rightarrow f_\infty, \quad f_k^{-1} \rightarrow \tilde{f}_\infty \quad \text{in } C_{loc}^0(\mathbb{C}).$$

From $f_k^{-1} \circ f_k = f_k \circ f_k^{-1} = \text{id}_{\mathbb{C}}$ we get $\tilde{f}_\infty \circ f_\infty = f_\infty \circ \tilde{f}_\infty = \text{id}_{\mathbb{C}}$ and thus $f_\infty : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism, with $\tilde{f}_\infty = f_\infty^{-1}$. Also, since $f_k(z), f_k^{-1}(z) \rightarrow \infty$ uniformly as $z \rightarrow \infty$, we deduce that the canonical extensions $\hat{f}_k : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ converge uniformly to \hat{f}_∞ and that the same holds for \hat{f}_k^{-1} .

We now closely examine the proof of [5, Theorem 4.30]: let $\tilde{\mu}_k \in \mathcal{E}_K^1$ be given by equation (4.25) in [5], with $\mu_k \mathbf{1}_{\mathbb{C} \setminus B_1^2}$ in place of μ , and

$$g_k : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}, \quad g_k(z) := \widehat{f^{\tilde{\mu}_k}}(z^{-1})^{-1}.$$

This map corresponds to the map f^{μ_1} in the aforementioned proof (with μ_k in place of μ). The lower bound (A.5), applied with $f^{\tilde{\mu}_k}$ and $z_1 := f^{\tilde{\mu}_k}(z^{-1}), z_2 := 0$, shows that $|g_k(z)|$ is bounded above by some M' , for all k and all $z \in \overline{B_1^2}$. Hence, defining $\mu_{k,2}$ as in equation (4.27) in [5] (with μ_k in place of μ), we get $\mu_{k,2} \in \mathcal{E}_K^{M'}$. Calling $h_k : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ the associated quasiconformal homeomorphism, normalized so that $h_k(0) = 0$ and $h_k(1) = 1$, by the above argument (with $M := M'$) we obtain the uniform convergence

$$g_k \rightarrow g_\infty, \quad g_k^{-1} \rightarrow g_\infty^{-1}, \quad h_k \rightarrow h_\infty, \quad h_k^{-1} \rightarrow h_\infty^{-1}$$

up to subsequences, for suitable homeomorphisms g_∞ and h_∞ of the Riemann sphere $\widehat{\mathbb{C}}$. Setting $\psi_\infty := h_\infty \circ g_\infty|_{\mathbb{C}}$ and observing that $\psi_k = h_k \circ g_k|_{\mathbb{C}}$, we get the desired convergence $\psi_k \rightarrow \psi_\infty$ and $\psi_k^{-1} \rightarrow \psi_\infty^{-1}$ in $C_{loc}^0(\mathbb{C})$.

Finally, we show that ψ_∞ is a K -quasiconformal homeomorphism. Given an open rectangle $R \subset \subset \mathbb{C}$, [5, Lemma 4.12] gives

$$\mathcal{L}^2(\psi_k(R)) = \int_R (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2) \geq \int_R (1 - k^2) |\partial_z \psi_k|^2 \geq (1 - k^2) k^{-2} \int_R |\partial_{\bar{z}} \psi_k|^2,$$

where $k := \frac{K-1}{K+1}$. Since $\mathcal{L}^2(\psi_k(R)) \rightarrow \mathcal{L}^2(\psi_\infty(R))$, we deduce that ψ_k is bounded in $W^{1,2}(R)$, thus ψ_∞ is the limit of ψ_k in the weak $W_{loc}^{1,2}(\mathbb{C})$ -topology. Given $\rho, \psi^1, \psi^2 \in C_c^\infty(\mathbb{C})$, integration by parts shows that

$$(A.6) \quad \int_{\mathbb{C}} \rho (\partial_1 \psi^1 \partial_2 \psi^2 - \partial_2 \psi^1 \partial_1 \psi^2) = - \int_{\mathbb{C}} (\partial_1 \rho \psi^1 \partial_2 \psi^2 - \partial_2 \rho \psi^1 \partial_1 \psi^2).$$

Writing $\psi_k = \varphi_k^1 + i\psi_k^2$, a standard density argument shows that (A.6) still holds with ψ^1, ψ^2 replaced by ψ_k^1, ψ_k^2 , for $k \in \mathbb{N} \cup \{\infty\}$. Hence, observing that $|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2 = (\partial_1 \psi_k^1 \partial_2 \psi_k^2 - \partial_2 \psi_k^1 \partial_1 \psi_k^2)$, we get

$$(A.7) \quad \int_{\mathbb{C}} \rho (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2) \rightarrow \int_{\mathbb{C}} \rho (|\partial_z \psi_\infty|^2 - |\partial_{\bar{z}} \psi_\infty|^2).$$

Defining the positive measures $\nu_k := (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2) \mathcal{L}^2$, up to further subsequences we can assume that $\nu_k \xrightarrow{*} \nu_\infty$ as Radon measures. For any rectangle R such that $\nu_\infty(\partial R) = 0$, approximating $\mathbf{1}_R$ from above and below with smooth functions and applying (A.7) we get

$$\int_R (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2) \rightarrow \int_R (|\partial_z \psi_\infty|^2 - |\partial_{\bar{z}} \psi_\infty|^2).$$

By monotonicity of both sides, this actually holds for every rectangle R . On the other hand, by lower semicontinuity of the L^2 -norm,

$$\begin{aligned} \int_R (1 - k^2) |\partial_z \psi_\infty|^2 &\leq \liminf_{k \rightarrow \infty} \int_R (1 - k^2) |\partial_z \psi_k|^2 \leq \lim_{k \rightarrow \infty} \int_R (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2) \\ &= \int_R (|\partial_z \psi_\infty|^2 - |\partial_{\bar{z}} \psi_\infty|^2). \end{aligned}$$

Since R is arbitrary, we get $|\partial_{\bar{z}} \psi_\infty| \leq k |\partial_z \psi_\infty|$ a.e., as desired. \blacksquare

Corollary A.4 (compactness of \mathcal{D}_K). *Given a sequence $\varphi_k \in \mathcal{D}_K$, there exists $\varphi_\infty \in \mathcal{D}_K$ such that, up to subsequences, $\varphi_k \rightarrow \varphi_\infty$ and $\varphi_k^{-1} \rightarrow \varphi_\infty^{-1}$ in $C_{loc}^0(\mathbb{C})$.*

Proof of Corollary A.4. Let $\mu_k \in \mathcal{E}_K$ be defined by $\partial_{\bar{z}} \varphi_k = \mu_k \partial_z \varphi_k$ for all k and let $\psi_k : \mathbb{C} \rightarrow \mathbb{C}$ be the unique K -quasiconformal homeomorphism satisfying the same differential equation, as well as $\psi_k(0) = 0$, $\psi_k(1) = 1$ (see [5, Theorem 4.30]).

By Lemma A.3, up to subsequences there exists a K -quasiconformal homeomorphism ψ_∞ such that $\psi_k \rightarrow \psi_\infty$ and $\psi_k^{-1} \rightarrow \psi_\infty^{-1}$ in $C_{loc}^0(\mathbb{C})$.

By the chain rule (see [8, Lemma III.6.4]), the map $\psi_k \circ \varphi_k^{-1} : \mathbb{C} \rightarrow \mathbb{C}$ is a biholomorphism and fixes the origin, so it equals the multiplication by a nonzero complex number λ_k , i.e. $\psi_k = \lambda_k \varphi_k$. On the other hand,

$$|\lambda_k| = \min_{x \in \partial B_1^2} |\psi_k(x)| \rightarrow \min_{x \in \partial B_1^2} |\psi_\infty(x)| \in (0, \infty).$$

Hence, up to further subsequences we can suppose that $\lambda_k \rightarrow \lambda_\infty \in \mathbb{C} \setminus \{0\}$. The statement follows with $\varphi_\infty := \lambda_\infty^{-1} \psi_\infty$. \blacksquare

Remark A.5. In general, given $\varphi_k \in \mathcal{D}_K$ (for $k \in \mathbb{N} \cup \{\infty\}$) with $\varphi_k \rightarrow \varphi_\infty$ and $\varphi_k^{-1} \rightarrow \varphi_\infty^{-1}$ locally uniformly, it is not true that the corresponding Beltrami coefficients satisfy $\mu_k \xrightarrow{*} \mu_\infty$ in $L^\infty(\mathbb{C})$. For instance, let $\mu_0(z) := \frac{1}{2}$ if $\Re(z) \in \bigcup_{n \in \mathbb{Z}} [n, n + \frac{1}{2})$ and $\mu_0(z) := -\frac{1}{2}$ otherwise. Then the bi-Lipschitz homeomorphism $\psi_0 : \mathbb{C} \rightarrow \mathbb{C}$ given by

$$\psi_0(x + iy) := \begin{cases} n + \frac{9}{5}(x - n) + \frac{3}{5}iy = n + \frac{6}{5}(z - n) + \frac{3}{5}(\bar{z} - n) & n \leq x \leq n + \frac{1}{2} \\ n + \frac{4}{5} + \frac{x-n}{5} + \frac{3}{5}iy = n + \frac{4}{5} + \frac{2}{5}(z - n) - \frac{1}{5}(\bar{z} - n) & n + \frac{1}{2} \leq x \leq n + 1 \end{cases}$$

satisfies $\partial_{\bar{z}} \psi_0 = \mu_0 \partial_z \psi_0$, with the normalization $\psi_0(0) = 0$ and $\psi_0(1) = 1$. So $\mu_k := \mu_0(2^k \cdot)$ and $\psi_k := 2^{-k} \psi_0(2^k \cdot)$ satisfy $\partial_{\bar{z}} \psi_k = \mu_k \partial_z \psi_k$ with the same normalization. Moreover, they converge uniformly to $\psi_\infty(x + iy) = x + \frac{3}{5}iy = \frac{4}{5}z + \frac{1}{5}\bar{z}$, together with their inverses. The homeomorphism ψ_∞ satisfies $\partial_{\bar{z}} \psi_\infty = \mu_\infty \partial_z \psi_\infty$ with $\mu_\infty := \frac{1}{4}$, but $\mu_k \xrightarrow{*} 0$. Dividing each ψ_k by $\min_{|z|=1} |\psi_k(z)|$, we obtain a counterexample in the class \mathcal{D}_3 .

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