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Singular Bundles with Bounded L^2-curvatures

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Abstract

The goal of this dissertation consists in producing a suitable framework for the calculus of variations associated with the Yang-Mills functional beyond the critical dimension four. We focus primarily on principal $SU(2)$-bundles over the five dimensional unit ball B^5. In this case, the usual direct variational method together with the classical compactness results by K. Uhlenbeck do not yield the existence of minimizers for the Yang-Mills functional. As in the theory of harmonic maps from B^3 into S^2, we may however expect that minimizers do exist in the class of "singular bundles", defined as the closure of weak curvatures with a finite number of topological point singularities. The class of "singular bundles" then consists of possibly almost everywhere singular principal $SU(2)$-bundles over B^5 with bounded L^2-curvatures.

To determine the closure of weak curvatures with isolated singularities, we first investigate the simpler Abelian case given by principal $U(1)$-bundles over B^3. With the help of a cubic decomposition of B^3, we show that the closure consists of generalized curvatures with integer-valued integral over all spheres in B^3. The proof greatly relies on the gauge invariance of the curvature in an Abelian gauge theory.

The results obtained in the Abelian case enable us to study the closure of curvatures with point singularities in the non-Abelian setting consisting of principal $SU(2)$-bundles over B^5 and to define a suitable class of "singular bundles". This class given by the trace of the tensor product of generalized curvatures incorporates the change in curvature under gauge transformations in non-Abelian gauge theories.

The second part of the dissertation is devoted to one-dimensional currents associated with the aforementioned weak singular curvatures. These currents enable us to investigate whether the singular set of singular curvatures can be connected. In particular, we show that to the currents associated with the curvatures of "singular bundles" there exists a minimal one-dimensional rectifiable current – known as a minimal connection – which completes the initial current to one without boundary.

Zusammenfassung

Die vorliegende Dissertation hat zur Aufgabe, einen geeigneten Rahmen für die Variationsrechnung mit dem Yang-Mills Funktional in Dimensionen größer als die kritische Dimension vier zu schaffen. Exemplarisch werden dazu Prinzipalbündel mit Strukturgruppe $SU(2)$ über dem fünfdimensionalen Einheitsball B^5 betrachtet, für welche die direkte Methode der Variationsrechnung zusammen mit den klassischen Kompaktheitsresultaten von K. Uhlenbeck keine Existenz von Minimierenden des Yang-Mills Funktionals liefert. Die Analogie zur Theorie der harmonischen Abbildungen von B^3 nach S^2 legt es nahe, die Existenz von Minimierenden in der Klasse von "singulären Bündeln" bestimmt durch den Abschluß von schwachen Krümmungen mit einer endlichen Anzahl von topologischen Punktsingularitäten zu erwarten. Die Klasse von "singulären Bündeln" besteht dann aus möglicherweise fast überall singulären $SU(2)$-Prinzipalbündeln über B^5 mit beschränkten L^2-Krümmungen.

Zur Bestimmung des Ablschlusses von schwachen Krümmungen mit isolierten Singularitäten soll zunächst der vereinfachte Abelsche Fall von $U(1)$-Prinzipalbündeln über B^3 untersucht werden. Mittels einer kubischen Zerlegung von B^3 läßt sich dann ein Resultat für den Abschluß beweisen. Genauer besteht dieser Abschluß aus verallgemeinerten Krümmungen mit ganzzahligem Integral über alle Sphären in B^3. Die für Abelsche Eichtheorin charakteristische geometrische Eigenschaft der Eichinvarianz von Krümmungen unter Eichtransformationen ist für den Beweis von großer Bedeutung.

Aus den Erkenntnissen des Abelschen Falls lassen sich dann sowohl der schwache Abschluss von Krümmungen mit Punktsingularitäten für den nicht Abelschen Fall von $SU(2)$-Prinzipalbündeln über B^5 als auch eine geeignete Klasse von "singulären Bündeln" ableiten. Diese Klasse gegeben über die Spur des Tensorprodukts von verallgemeinerten Krümmungen berücksichtigt die Krümmungsänderungen bei Umeichung nicht Abelscher Eichtheorien.

In einem zweiten Teil der Dissertation werden zu den vorher eingeführten schwachen singulären Krümmungen des Abelschen und nicht Abelschen Falls eindimensionale Ströme definiert, anhand welcher sich die Singularitätenmenge der Krümmungen auf ihre Verbindbarkeit untersuchen lässt. Insbesondere wird gezeigt, dass für die Ströme assoziiert zu den Krümmungen von "singulären Bündeln" ein eindimensionaler minimaler rektifizierbarer Strom – der sogenannte minimale Zusammenhang – existiert, welcher den Ausgangsstrom zu einem randlosen Strom vervollständigt.

Résumé

Le but de cette thèse est de donner un cadre convenable pour le calcul variationnel de la fonctionnelle de Yang-Mills en dimension supérieure à la dimension critique quatre. Nous concentrons notre étude sur les fibrés principaux avec groupe de structure $SU(2)$ sur la boule-unité B^5 en dimension cinq, pour lesquels la méthode variationnelle directe combinée aux résultats classiques de compacité de K. Uhlenbeck ne fournit pas l'existence de minimiseurs pour la fonctionnelle de Yang-Mills. Pareillement au cas des applications harmoniques sur B^3 à valeur dans S^2, on peut cependant s'attendre à ce qu'il existe des minimiseurs dans la classe des "fibrés singuliers" déterminée par la fermeture des courbures faibles avec un nombre fini de points singuliers topologiques. La classe des "fibrés singuliers" contient alors des fibrés principaux $SU(2)$ sur B^5 peut-être singuliers presque partout avec des L^2-courbures bornées.

Afin de déterminer la fermeture des courbures faibles avec des singularitées isolées, nous étudions d'abord le cas Abélian, plus simple, des fibrés principaux avec groupe de structure $U(1)$ sur la boule-unité B^3. A l'aide d'une décomposition cubique, nous démontrons que la fermeture est constituée de courbures généralisées avec une intégrale entière sur toutes les sphères dans B^3. La preuve repose sur l'invariance des courbures sous transformations de jauge, charactèristique à la théorie Abélienne.

Les résultats obtenus dans le cas Abélien nous permettent d'étudier la fermeture des courbures avec des singularitées isolées dans le cas non-Abélien consistant en un fibré principal $SU(2)$ sur B^5 et de définir une classe convenable de "fibrés singuliers". Cette classe, qui fait intervenir des traces du produit tensoriel de courbures généralisées, prend en compte le comportement des courbures sous transformations de jauge quand le groupe de structure est non-Abélien.

Dans la seconde partie de la thèse, nous définissons des courants un-dimensionels associés aux courbures singulières dans les cas Abélien et non-Abélien. Ces courants permettent d'étudier l'ensemble singulier des courbures sur sa connectivité. Nous démontrons en particulier qu'aux courants associés aux courbures des "fibrés singuliers" il correspond un courant un-dimensionel rectifiable – appelé connection minimale – qui complète le courant initial en un courant sans bord.

Contents

Chapter 1

Introduction

The goal of this dissertation consists in producing a suitable framework for the calculus of variations associated with the Yang-Mills functional beyond the critical dimension four. In particular, we investigate the framework for which we expect to obtain existence of minimizers. The aim of this introduction is not only to present the main results in this direction, but also to motivate the used strategy. We refer at several places to the next chapter which is devoted to additional background material.

Calculus of Variations

Let $\pi : P \longrightarrow M$ be a principal G-bundle over a compact n-dimensional Riemannian manifold M with Riemannian metric g. The structure group G of P is assumed to be a compact Lie group with Lie algebra $\mathfrak{g}$. We denote by $\mathcal{A}(P)$ the space of connections on P. For any connection A let $F(A) \in \Omega^2(M, \mathrm{Ad}(P))$ denote the curvature of A. The Yang-Mills functional is then defined as

$$YM(A) = \int_M |F(A)|^2 dvol_g \,,$$

where $dvol_g$ denotes the canonical volume form on M. The norm of $F(A)$ is induced by the Killing form on $\mathfrak{g}$ and the Riemannian metric on M. Details can be found in Section 2.1.

As a first step, we look for a smooth connection which minimizes the Yang-Mills functional YM among all smooth connections in $\mathcal{A}_\infty(P)$. In other words, we ask if the following infimum is attained:

$$m(P) = \inf_{A \in \mathcal{A}_\infty(P)} \int_M |F(A)|^2 \, dvol_g \,. \tag{1.1}$$

The most natural approach in order to show the existence of a minimizer consists in applying the direct method in the calculus of variations. We take a minimizing sequence, i.e., a sequence $(A_k)_{k\in\mathbb{N}}$ of smooth connections in $\mathcal{A}_\infty(P)$ such that $YM(A_k) \longrightarrow m(P)$ for $k \to \infty$, which we hope will converge to a minimizer in $\mathcal{A}_\infty(P)$. The curvatures $F(A_k)$ associated to the minimizing sequence obviously have a uniform L^2-bound. This bound, however, gives no compactness result for the minimizing sequence in the C^∞-topology.

Hence, the direct method in the calculus of variations applied to $\mathcal{A}_\infty(P)$ does not imply the existence of a minimizer in this class.

It is instructive to get familiar with the problem of existence of minimizers for the Yang-Mills functional in the conformal four-dimensional case first. Results on the existence of minimizers in four dimensions are based on the so-called Coulomb or Uhlenbeck gauge introduced by K. Uhlenbeck, [46]. Let $A \in \mathcal{A}_{W^{1,2}}(P)$ be a $W^{1,2}$-Sobolev connection on a smooth principal G-bundle $P \in \mathcal{P}^G_\infty(M)$ over a four-dimensional compact Riemannian manifold M (see Section 2.2 for the notation) and let $U \subset M$ open be trivializing for P. Then, under the assumption of small L^2-energy for the curvature $F(A)$ of A, we can find a local gauge transformation $\sigma_{Coulomb} \in W^{2,2}(U, G)$ such that the Coulomb gauge $A_{Coulomb} = \sigma_{Coulomb}(A)$ satisfies $d_* A_{Coulomb} = 0$ and $\|A_{Coulomb}\|_{W^{1,2}(U)} \le C_{Uh} \|F(A)\|_{L^2(U)}$. For more details the reader should read Section 2.4. As already announced in [46], the local Coulomb gauge transformations $\sigma_{Coulomb}$ for smooth connections are also smooth. – In this thesis, we will rewrite this statement in a slightly different way and establish the regularity result using Hodge decomposition and Lorentz space techniques.

Applying the direct method in the calculus of variations, we take a minimizing sequence of connections for the Yang-Mills functional YM in four dimensions. Since the estimate $\|A_{Coulomb}\|_{W^{1,2}(U)} \le C_{Uh} \|F(A)\|_{L^2(U)}$ implies the "coercivity" of YM, the Coulomb gauge is the key point in order to get weak compactness for the minimizing sequence. Thus the direct method yields a limiting object which can be characterized as "$W^{1,2}$-Sobolev connection on some $W^{2,2}$-Sobolev bundle". Using the regularity results for Yang-Mills connections (see K. Uhlenbeck [46], [47] and the explanations in Section 2.4) we then conclude the existence of a minimizer in the class $\mathcal{A}_{W^{1,2}}(P)$. As a consequence, the "singular bundles" $\mathcal{A}_{W^{1,2}}(P)$ – consisting of Sobolev connections on smooth bundles – give a suitable framework for the calculus of variations in four dimensions.

In contrast to the previous four-dimensional case, finding a notion of "singular bundles" for the Yang-Mills functional in dimensions greater than four is much more involved and is one of the main aims of this thesis. As a model case, from now on we consider principal $SU(2)$-bundles over the five-dimensional unit ball B^5. Note that the triviality of P implies that any smooth connection $A \in \mathcal{A}_\infty(P)$ is a globally well-defined element in $\Omega^1_\infty(B^5, \mathfrak{su}(2))$ and furthermore, that $F(A) = dA + A \wedge A$ globally on B^5.

As in the four-dimensional case, the most natural way to enlarge $\mathcal{A}_\infty(P)$ for obtaining existence of minimizers is to consider the class $\mathcal{A}_{W^{1,2}}(P)$ of $W^{1,2}$-Sobolev connections on some trivial smooth principal $SU(2)$-bundle $P \in \mathcal{P}^{SU(2)}_\infty(B^5)$ (again see Section 2.2 for notation). Note that for the L^2-curvature we have $F(A) = dA + A \wedge A$ globally on B^5. Although the Yang-Mills functional YM is still well-defined for $\mathcal{A}_{W^{1,2}}(P)$ the infimum is not attained in this class. This is because the crucial theorem by K. Uhlenbeck for the existence of a Coulomb gauge does not apply to L^2-curvatures in five dimensions and hence the Yang-Mills functional is not "coercive". Therefore the class $\mathcal{A}_{W^{1,2}}(P)$ does not give the right framework for the calculus of variations for the Yang-Mills in five dimensions and we have to look for a suitable replacement.

Harmonic Maps

In order to get a better understanding of our strategy to obtain a setting in which minimizers for the Yang-Mills functional in B^5 exist, we briefly review some results for harmonic maps (see Section 2.3 for more details). For maps $u : B^2 \longrightarrow S^2$ there exists a minimizer for the Dirichlet functional which is also smooth. This conformal two-dimensional case for harmonic maps corresponds to the four-dimensional case for the Yang-Mills problem. Recall that in both cases we obtain existence and smoothness of minimizers. Since we are interested in the calculus of variations for the Yang-Mills functional in five dimensions, we have to investigate the three-dimensional harmonic map analogue. Let $u : B^3 \longrightarrow S^2$ and the Dirichlet energy functional $E(u)$ given by

$$E(u) = \int_{B^3} |du|^2 d^3x \,.$$

It is well-known that the infimum of E is attained in the Sobolev space $W^{1,2}(B^3, S^2)$. From H. Brezis, [2] and R. Schoen, K. Uhlenbeck, [41] we then deduce that the minimizer u_0 is smooth except at a finite number of points $a_1, \dots, a_N \in B^3$ with Brouwer degree given by

$$\deg(u_0, a_i) = \frac{1}{4\pi} \int_{\partial B^3_r(a_i)} \iota^*_{\partial B^3_r(a_i)} (u_0^* \omega_{S^2}) \neq 0 \,.$$

In other words, minimizers for the Dirichlet functional in three dimensions have isolated topological singularities and hence belongs to $R^{1,2}(B^3, S^2)$, the space of $W^{1,2}$-Sobolev maps being smooth except at a finite number of points with topological degrees.

In fact, even more is true: A density result by F. Bethuel [8], [9] says that the space $R^{1,2}(B^3, S^2)$ is dense in $W^{1,2}(B^3, S^2)$. Therefore the strong $W^{1,2}$-closure of $R^{1,2}(B^3, S^2)$ equals the Sobolev space $W^{1,2}(B^3, S^2)$. As a consequence the closure of $R^{1,2}(B^3, S^2)$ gives a suitable class for the existence of minimizers for the Dirichlet functional in three dimensions. It is precisely this scenario we want to mimic for defining "singular bundles" below.

Bundles with Defects

We begin by defining the Yang-Mills analogue of $R^{1,2}(B^3, S^2)$, in other words of harmonic maps with defects. In five dimensions, we expect for the minimum of the Yang-Mills functional – as in the case of harmonic maps from B^3 into S^2 – to have isolated topological singularities. A point $a \in B^5$ represents a topological singularity if

$$\frac{1}{8\pi^2} \int_{\partial B^5_r(a)} \iota^*_{\partial B^5_r(a)} \mathrm{Tr}(F \wedge F) \neq 0 \,, \tag{1.2}$$

for small enough radii $r > 0$. In words, the second Chern number $c_2(\iota^*_{\partial B^5_r(a)} P)$ of the restricted principal $SU(2)$-bundle $\iota^*_{\partial B^5_r(a)} P$ over $\partial B^5_r(a)$ does not vanish and is hence non-trivial. As a consequence, there is no global representation $F = dA + A \wedge A$ for some $A \in \Omega^1_{W^{1,2}}(B^5, \mathfrak{su}(2))$. Note that this would be required for any connection in the class $\mathcal{A}_{W^{1,2}}(P)$ introduced before which is yet another reason why the latter is not the right class to work with.

In order to define weak connections with topological singularities first consider smooth boundary data given by a smooth principal $SU(2)$-bundle q over S^4 with connection $b \in \mathcal{A}_\infty(q)$. Then there exists an extension to a smooth principal $SU(2)$-bundle over B^5 with smooth connection $A \in \mathcal{A}_\infty(P)$ such that $\iota^*_{S^4} A = b$ if and only if $c_2(q) = 0$ (see

T. Isobe [25], Lemma 3.4). A vanishing second Chern number on the boundary hence gives no topological obstruction for smoothness in the interior. This is similar to the existence of a smooth extension for maps from B^3 into S^2 with vanishing Brouwer degree on the boundary as described in Section 2.3. From the above observations, we conclude that the space

$$\mathcal{A}_{\infty,b}(\bar{B}^5) = \left\{ \tilde{A} \in \mathcal{A}_\infty(P) \;:\; P \in \mathcal{P}_\infty^{SU(2)}(B^5) \text{ and } \iota_{S^4}^* \tilde{A} = b \right\}, \tag{1.3}$$

is not empty. For the class of connections allowing topological singularities, we recall that there is no global interpretation of a connection as $\mathfrak{su}(2)$-valued one-form. With smoothness except at a finite number of points $a_1, \dots, a_N \in B^5$, we then define

$$\begin{aligned} \mathcal{A}_{R,b}(B^5) = \Big\{ A \in \mathcal{A}_\infty(P) \;:\; & P \in \mathcal{P}_\infty^{SU(2)}\big(B^5 \setminus \{a_1, \dots, a_N\}\big), \\ & \|F(A)\|_{L^2(B^5)} < \infty \text{ and } \iota_{S^4}^* A = b \Big\}. \end{aligned} \tag{1.4}$$

Note that in this class the singular set of the bundles itself can also vary.

Under the assumption that the second Chern number vanishes on the boundary, the existing smooth extension is however not necessarily minimizing for the Yang-Mills energy. Indeed, it is shown in T. Kessel and T. Rivière, [27] that we can find a smooth principal $SU(2)$-bundle q over S^4 with connection $b \in \mathcal{A}_\infty(q)$ and $c_2(q) = 0$ such that

$$\inf_{A \in \mathcal{A}_{R,b}(B^5)} \int_{B^5} \big|F(A)\big|^2 \, d^5x < \inf_{\tilde{A} \in \mathcal{A}_{\infty,b}(\bar{B}^5)} \int_{B^5} \big|F(\tilde{A})\big|^2 \, d^5x. \tag{1.5}$$

Such a gap phenomenon which we also encounter in the context of harmonic maps (see Section 2.3) confirms the necessity of working with singular bundles.

Analogously to $\mathcal{A}_\infty(\bar{B}^5)$ we define the following class of smooth $\mathfrak{su}(2)$-valued two-forms on B^5:

$$\begin{aligned} \mathcal{F}_\infty(\bar{B}^5) \;=\; \Big\{ & F \in \Omega^2_{L^2}\big(\bar{B}^5, \mathfrak{su}(2)\big) \;:\; \exists\; A \in \Omega^1_\infty\big(\bar{B}^5, \mathfrak{su}(2)\big) \\ & \text{st. } F \overset{\sigma}{=} dA + A \wedge A \text{ and } d\,\mathrm{Tr}(F \wedge F) = 0 \Big\}. \end{aligned} \tag{1.6}$$

By $F \overset{\sigma}{=} dA + A \wedge A$ we mean equality up to gauge transformations σ. More precisely, there exists a gauge transformation $\sigma : B^5 \longrightarrow SU(2)$ such that $\sigma^{-1} F \sigma = dA + A \wedge A$ for $A \in \Omega^1_\infty\big(\bar{B}^5, \mathfrak{su}(2)\big)$. Elements in $\mathcal{F}_\infty(\bar{B}^5)$ can be interpreted as curvatures of connections in $\mathcal{A}_\infty(\bar{B}^5)$. Moreover, analogously to $\mathcal{A}_R(B^5)$ we define

$$\begin{aligned} \mathcal{F}_R(B^5) = \Big\{ F \;=\; & \{F_i\}_{i \in I} \in \Omega^2_{L^2}\big(B^5, \mathfrak{su}(2)\big) \;:\; \exists\; a_1, \dots, a_N \in B^5 \text{ st.} \\ & F \overset{\sigma,\, loc.}{=} dA + A \wedge A \text{ locally smooth on } B^5 \setminus \{a_1, \dots, a_N\}, \\ & d\,\mathrm{Tr}(F \wedge F) = 8\pi^2 \left(\sum_{i=1}^N d_i \delta_{a_i} \right) dx^1 \wedge \ldots \wedge dx^5 \text{ in } \mathcal{D}' \Big\}. \end{aligned} \tag{1.7}$$

For weak curvatures in $\mathcal{F}_R(B^5)$ there exist hence locally on $B^5 \setminus \{a_1, \dots, a_N\}$ a gauge transformation σ and a smooth $\mathfrak{su}(2)$-valued one-form A such that $\sigma^{-1} F \sigma = dA + A \wedge A$ is locally smooth. Connections in $\mathcal{A}_R(B^5)$ give rise to curvatures belonging to $\mathcal{F}_R(B^5)$. Note that the integers $d_1, \ldots, d_N \in \mathbb{Z}$ in (1.7) denote the second Chern numbers $c_2(\iota^*_{\partial B^5_r(a_i)} P)$ around the finite number of topological singularities $a_1, \ldots, a_N \in B^5$. Roughly speaking we have that $d\,\mathrm{Tr}(F \wedge F)$ detects the topological singularities of the bundle. The condition $d\,\mathrm{Tr}(F \wedge F) = 0$ in the sense of distributions is necessary and sufficient for the approximability in some L^2-sense of $\mathcal{F}_R(B^5)$ by smooth curvatures in $\mathcal{F}_\infty(\bar{B}^5)$ (see T. Isobe, [25]). This is very similar to the case of harmonic maps $u : B^3 \longrightarrow S^2$. In fact, a map $u \in R^{1,2}(B^3, S^2)$ has an approximation by smooth maps in $C^\infty(\bar{B}^3, S^2)$ if and only if $dD(u) = d(u^* \omega_{S^2}) = 0$ in the sense of distributions. This characterization of the smooth approximability goes back to F. Bethuel, [7] and the reader should again read Section 2.3 for more details.

Abelian Singular Bundles

The analogue to $R^{1,2}(B^3, S^2)$ in the case of the Yang-Mills functional is the space $\mathcal{F}_R(B^5)$ of singular connections with isolated point singularities. Motivated by the fact that for harmonic maps the minimizer belongs to the closure of $R^{1,2}(B^3, S^2)$, it is natural to expect a minimizer for the Yang-Mills functional to be in the closure of $\mathcal{F}_R(B^5)$. However, so far this closure is not known. In other words, the analogue to Sobolev maps $W^{1,2}(B^3, S^2)$ consisting of "singular bundles" which can be singular everywhere has not been determined yet. Hence, in order to get a framework for the existence of minimizers for YM, we have in a first step to give a precise definition of "singular bundles" or of the closure of $\mathcal{F}_R(B^5)$.

First we consider a simplified version of this problem. Let P be a principal $U(1)$-bundle over B^3. Owing to the Abelian Lie group $U(1)$, we call this Abelian gauge theory the "Abelian case". Note that in the Abelian case the curvature $F(A)$ of a connection $A \in \mathcal{A}(P)$ has the representation $F(A) = dA$ and is hence a linear function of the connection. A topological singularity $a \in B^3$ in the Abelian case has non-vanishing first Chern number

$$\frac{1}{2\pi} \int_{\partial B^3_r(a)} \iota^*_{\partial B^3_r(a)} F \,, \tag{1.8}$$

for radii $r > 0$ small enough. In analogy to $\mathcal{F}_R(B^5)$, we define the following class of globally defined real-valued two-forms on B^3:

$$\begin{aligned} \mathcal{F}_R(B^3) = \Big\{ F \in \Omega^2_{L^1}(B^3) \;\; : \;\; & \exists \; a_1, \ldots, a_N \in B^3 \;\text{ st. }\; F \in \Omega^2_\infty \left(B^3 \setminus \bigcup_{i=1}^N a_i \right) , \\ & F \overset{\text{loc.}}{=} dA \;\text{ on }\; B^3 \setminus \{a_1, \ldots, a_N\} \,, \\ & dF = 2\pi \left(\sum_{i=1}^N d_i \delta_{a_i} \right) dx^1 \wedge dx^2 \wedge dx^3 \;\text{ in }\; \mathcal{D}' \Big\} \,. \end{aligned} \tag{1.9}$$

In the Abelian case the topological singularities are detected by dF. Though the underlying principal $U(1)$-bundle for the class $\mathcal{F}_R(B^3)$ may have topological singularities and is thus not trivial, the curvatures are globally well-defined two-forms on B^3 (see Section 2.1). This gauge invariance of the curvature makes the Abelian case much simpler to handle. –

Note that curvatures in the Abelian case can be seen as a kind of natural generalization of the "D-field" $u^*\omega_{S^2}$ used for the study of harmonic maps $u : B^3 \longrightarrow S^2$. For more details on the link between curvatures and maps we refer to T. Kessel and T. Rivière, [27] (see also the article by M.S. Narasimhan and S. Ramanan, [34]).

As a consequence of the simplified geometry in the Abelian case, it makes sense to define the closure of $\mathcal{F}_R(B^3)$ as the global L^1-closure. The first main result of this dissertation describes completely the closure of $\mathcal{F}_R(B^3)$.

Theorem 1.1. *Assume that there exists a sequence $(F_k)_{k\in\mathbb{N}}$ in $\mathcal{F}_R(B^3)$ such that $F_k \overset{k\to\infty}{\longrightarrow}$ F in L^1 for some $F \in \Omega^2_{L^1}(B^3)$. Then, we have that F belongs to*

$$\begin{aligned} \mathcal{F}_{\mathbb{Z}}(B^3) \;&=\; \Big\{ F \in \Omega^2_{L^1}(B^3) \,:\, \frac{1}{2\pi}\int_{\partial B^3_r(x)} \iota^*_{\partial B^3_r(x)} F \in \mathbb{Z}\,, \\ &\qquad\qquad \textit{for } \forall x \in B^3 \;\textit{ and }\; \textit{a.e } 0 < r \le \textit{dist}(x, \partial B^3) \Big\}\,. \end{aligned} \tag{1.10}$$

Conversely, assuming that $F \in \mathcal{F}_{\mathbb{Z}}(B^3)$, we can find a sequence $(F_k)_{k\in\mathbb{N}}$ in $\mathcal{F}_R(B^3)$ such that

$$F_k \longrightarrow F \quad \textit{in } L^1 \qquad (k \to \infty)\,.$$

This then leads to the definition of "singular bundles" in the Abelian case or, more precisely, to the definition of everywhere singular principal $U(1)$-bundles over B^3 with bounded L^1-curvatures.

Definition 1.1 (L^1-curvature of a singular $U(1)$-bundle). *An L^1-curvature of a singular principal $U(1)$-bundle over B^3 is a measurable real-valued two-form F on B^3 such that the following holds:*

(i) The two-form F has bounded L^1-norm, i.e.,

$$\int_{B^3} |F|\, d^3x < +\infty\,. \tag{1.11}$$

(ii) For every $x \in B^3$ and for almost every $0 < r < \textit{dist}(x, \partial B^3)$ we have

$$\frac{1}{2\pi}\int_{\partial B^3_r(x)} \iota^*_{\partial B^3_r(x)} F \in \mathbb{Z}\,. \tag{1.12}$$

The proof of Theorem 1.1 is based on a decomposition of B^3 into little cubes of side length $\varepsilon > 0$. This is a technique we also encounter in Section 2.3 for the density of $R^{1,2}(B^3, S^2)$ in $W^{1,2}(B^3, S^2)$. The main steps of the proof can then be summarized as follows:

a) *Choice of the cubic decomposition*: We take the average of the given $F \in \mathcal{F}_{\mathbb{Z}}(B^3)$ and choose the grid in such a way that difference of F and its average $\bar{F}$ vanishes on the boundary in the limit $\varepsilon \to 0$.

b) *Good and bad cubes*: The first Chern number on the boundary of the good cubes vanishes. The bad cubes are the remaining ones whose volume vanishes in the limit $\varepsilon \to 0$.

c) *Smoothing on the boundary*: On the two-dimensional boundary of the cubic decomposition we are in the critical case and can hence approximate F by smooth curvatures applying the density result in Corollary 1.7 below.

d) *Gauge fixing*: On the boundary of the good cubes we pass to a "linear Coulomb gauge". It is important to mention that since we are dealing with a Abelian gauge theory, the curvature remains unchanged under gauge transformations.

e) *Extensions*: On the good cubes we take a harmonic and on the bad cubes a radial extension in order to define the approximating sequence in $\mathcal{F}_R(B^3)$. The bad cubes give rise to the topological singularities.

f) *Passing to the limit*: On the good cubes, the approximating sequence tends for $\varepsilon \to 0$ to the constant $\bar{F}$. This is a consequence of the choice d) for the gauge and of a) saying that F is close to the constant $\bar{F}$ on the boundary of the cubic decomposition. So the fact of being closed to a constant can be transfered from the boundary to the interior of the good cubes, if we choose the correct gauge for the harmonic extension. Once we have shown that the approximating sequence converges to $\bar{F}$, we apply Lebesgue's theorem in order to conclude the convergence to F as desired.

   On the bad cubes, using the properties of radial extensions, there exists a uniform bound for the approximating sequence. Thus, we can apply dominated convergence implying together with the vanishing volume that the approximating sequence vanishes in the limit $\varepsilon \to 0$ on the bad cubes.

Non-Abelian Singular Bundles

We now come back to the initial non-Abelian case of principal $SU(2)$-bundles over B^5 and try to adapt Theorem 1.1 to this case. A direct translation of $\mathcal{F}_{\mathbb{Z}}(B^3)$ to the non-Abelian case leads to

$$\begin{aligned}\mathcal{F}_{\mathbb{Z}}(B^5) = \Big\{F \in \Omega^2_{L^2}\big(B^5, \mathfrak{su}(2)\big) \ :\ & \iota^*_{\partial B^5_r(x)} F \ \text{ is a curvature form on } \partial B^5_r(x) \\ & \text{and } \frac{1}{8\pi^2}\int_{\partial B^5_r(x)} \iota^*_{\partial B^5_r(x)} \operatorname{Tr}(F\wedge F) \in \mathbb{Z}\,, \\ & \text{for } \forall x \in B^5 \text{ and a.e } 0 < r \le \operatorname{dist}(x, \partial B^5)\Big\}\,. \end{aligned} \tag{1.13}$$

On the other hand, we can avoid in the non-Abelian case the problem of only locally defined curvatures – and also the problem of smooth curvatures up to gauge transformations (see (1.7)) – by considering the gauge invariant quantity $\operatorname{Tr}(F\wedge F)$ being a globally well-defined real-valued integrable four-form on B^5. Moreover, for $F \in \mathcal{F}_R(B^5)$ we have that $\operatorname{Tr}(F\wedge F)$ is globally smooth on B^5 except at a finite of points. This leads us to the next main theorem.

Theorem 1.2. *Let $F \in \mathcal{F}_{\mathbb{Z}}(B^5)$. Then there exists a sequence $(F_k)_{k\in\mathbb{N}}$ in $\mathcal{F}_R(B^5)$ such that the following weak convergence holds:*

$$\int_{B^5} \mathrm{Tr}(F_k \wedge F_k) \wedge \omega \stackrel{k\to\infty}{\longrightarrow} \int_{B^5} \mathrm{Tr}(F \wedge F) \wedge \omega \,, \qquad \forall \omega \in \Omega^1_{C^0}(B^5)\,. \tag{1.14}$$

As a direct consequence, we obtain that

$$d\,\mathrm{Tr}(F_k \wedge F_k) \longrightarrow d\,\mathrm{Tr}(F \wedge F) \quad \text{in } \mathcal{D}' \qquad (k \to \infty)\,. \tag{1.15}$$

This shows that the topology of the principal bundles given by $d\,\mathrm{Tr}(F \wedge F)$ is approximatively preserved. Though the topology is normally not preserved under weak convergence – see Section 2.3 for the weak convergence of harmonic maps with topology $d(u^*\omega_{S^2})$ – we still refer to Theorem 1.2 as the weak density result. Note that the strategy for the proof of this result is almost a copy of the main steps for Theorem 1.1. Only step d) – passing to the Coulomb gauge on the boundary of the cubic decomposition – is not needed.

Theorem 1.2 is a first step for the determination of the closure of $\mathcal{F}_R(B^5)$. However, this result is still unsatisfactory, since we are rather interested in some strong closure than in the weak closure of $\mathcal{F}_R(B^5)$ in order to define the notion of non-Abelian "singular bundle". For the strong closure, it turns out that – as in the Abelian case – we have to pass to the Coulomb gauge on the boundary of the good cubes in the cubic decomposition whose existence in four-dimensions follows from K. Uhlenbeck's theorem (see [46]). Since the curvature is not gauge invariant under the Coulomb gauge transformations, we would obtain a kind of L^2-convergence "up to gauge transformations". Thus, we cannot work with curvatures and the standard L^2-topology in order to obtain the strong closure of $\mathcal{F}_R(B^5)$. The problem of gauge invariance in the weak density result was solved by considering the wedge product $\mathrm{Tr}(F \wedge F)$. We are however not able to characterize uniquely with this wedge product the curvature. Passing to the tensor product $\mathrm{Tr}(F \otimes F)$, we finally end up with an object which fulfills simultaneously the requirements of gauge invariance and characterization of the curvature – modulo the adjoint action of $SU(2)$.

Lemma 1.3. *Let F and G be two elements in $\Omega^2_{L^2}(S^4, \mathfrak{su}(2))$. Then there exists a map $\sigma \in L^\infty(S^4, SU(2))$ such that $F = \sigma^{-1} G \sigma$ if and only if*

$$\mathrm{Tr}(F \otimes F) = \mathrm{Tr}(G \otimes G)\,. \tag{1.16}$$

Note that for $\Omega = \mathrm{Tr}(F \otimes F) : B^5 \longrightarrow \bigwedge^2(TB^5) \otimes \bigwedge^2(TB^5)$ we observe that

$$\|F\|_{L^2(B^5)} \leq \|\Omega\|_{L^1(B^5)} = \|\mathrm{Tr}(F \otimes F)\|_{L^1(B^5)}\,. \tag{1.17}$$

As outcome of the above considerations we suggest a definition for "singular bundles" in the non-Abelian case – or, more precisely – for everywhere singular principal $SU(2)$-bundles over B^5 with bounded L^2-curvature.

Definition 1.2 (L^2-curvature of a singular $SU(2)$-bundle). *An L^2-curvature of a singular principal $SU(2)$-bundle over B^5 is defined as an element Ω in $L^1\big(B^5, \bigwedge^2(TB^5) \otimes \bigwedge^2(TB^5)\big)$ with the following properties:*

(i) There exists $F \in \Omega^2_{L^2}(B^5, \mathfrak{su}(2))$ such that $\Omega = \mathrm{Tr}(F \otimes F)$.

(ii) For every $x \in B^5$ and a.e. $0 < r \leq dist(x, \partial B^5)$ there exists a curvature f of a $W^{1,2}$-connection on a smooth principal $SU(2)$-bundle over $\partial B^5_r(x)$ such that $\iota^*_{\partial B^5_r(x)}\Omega = \mathrm{Tr}(f \otimes f)$.

The class of L^2-curvatures of singular principal $SU(2)$-bundles over B^5 is denoted by $\mathcal{F}_{\otimes}(B^5)$.

We mention that a $\mathfrak{su}(2)$-valued two-form F as in Definition 1.2 for an L^2-curvature for a singular principal $SU(2)$-bundle belongs in particular to $\mathcal{F}_{\mathbb{Z}}(B^5)$, since condition (ii) implies that

$$\frac{1}{8\pi^2}\int_{\partial B^5_r(x)} \iota^*_{\partial B^5_r(x)}\mathrm{Tr}(F \wedge F) \in \mathbb{Z}\,,$$

for every $x \in B^5$ and almost every $0 < r \leq \mathrm{dist}(x, \partial B^5)$. For the strong closure of $\mathcal{F}_R(B^5)$ we then have to solve the following open problem:

Open Problem 1. *Let $\Omega \in \mathcal{F}_{\otimes}(B^5)$. Does there exist a sequence $(F_k)_{k\in\mathbb{N}}$ in $\mathcal{F}_R(B^5)$ such that*

$$\mathrm{Tr}(F_k \otimes F_k) \longrightarrow \Omega \qquad \textit{in } L^1\ ? \tag{1.18}$$

Remark 1.1. a) Note that in (1.18) we consider on the space of singular principal $SU(2)$-bundles with bounded L^2-curvature a metric given by

$$\delta(F_1, F_2) = \int_{B^5} \big|\mathrm{Tr}(F_1 \otimes F_1) - \mathrm{Tr}(F_2 \otimes F_2)\big|\, d^5x\,. \tag{1.19}$$

b) Denoting the alternated Ω by $\hat{\Omega} \in L^1\big(\bigwedge^4(TB^5)\big) = \Omega^4_{L^1}(B^5)$, the weak density result (1.14) can be rewritten as

$$\int_{B^5} \mathrm{Tr}(F_k \wedge F_k) \wedge \omega \overset{k\to\infty}{\longrightarrow} \int_{B^5} \hat{\Omega} \wedge \omega\,, \qquad \forall \omega \in \Omega^1_{C^0}(B^5)\,. \tag{1.20}$$

Hence it can be interpreted as consequence of the strong convergence (1.18).

There is another alternative definition for non-Abelian "singular bundles" given in T. Kessel and T. Rivière, [27].

Definition 1.3 (L^2-curvature of a singular $SU(2)$-bundle). *A representative of an L^2-curvature of a singular principal $SU(2)$-bundle over B^5 is a $\mathfrak{su}(2)$-valued measurable two-form F on B^5 such that the following holds:*

(i) The two-form F has bounded L^2-norm, i.e.,

$$\int_{B^5} |F|^2\, d^5x < +\infty\,. \tag{1.21}$$

(ii) For every $x \in B^5$ and a.e. $0 < r \leq dist(x, \partial B^5)$ there exists a curvature f of a $W^{1,2}$-connection on a smooth principal $SU(2)$-bundle over $\partial B^5_r(x)$ such that $\iota^*_{\partial B^5_r(x)}F = f$.

An L^2-curvature of a singular principal $SU(2)$-bundle over B^5 is an equivalence class $[F]$ in the space of two-forms $F \in \Omega_{L^2}(B^5, \mathfrak{su}(2))$ satisfying (i) and (ii) for the equivalence relation given by the adjoint action of measurable maps $\sigma \in L^\infty(B^5, SU(2))$.

As a consequence of Lemma 1.3, we obtain that Definition 1.2 and 1.3 are equivalent. Together with Definition 1.3 we can adjust the topology on the space of L^2-curvatures of singular principal $SU(2)$-bundles using the metric given by (see T. Kessel and T. Rivière, [27])

$$d([F_1],[F_2]) = \inf_{\sigma \in L^\infty(B^5, SU(2))} \left(\int_{B^5} |F_1 - \sigma^{-1} F_2 \sigma|^2 \, d^5x \right)^{1/2} . \tag{1.22}$$

We will show that the two metrics d and δ induce the same topology (see Proposition 1.5 below).

Next let φ be a smooth curvature for some smooth trivial principal $SU(2)$-bundle q over S^4. For this boundary data we denote by $\mathcal{F}_{R,\varphi}(B^5)$ the space of curvatures in $\mathcal{F}_R(B^5)$ whose restriction to ∂B^5 is gauge equivalent to φ. Moreover, we denote by $\mathcal{F}_{\otimes,\varphi}(B^5)$ the closure of $\mathcal{F}_{R,\varphi}(B^5)$ for the topology induced by the metric δ (see (1.19)). As final outcome of our investigations on a suitable framework for the existence of minimizers for the Yang-Mills functional in five dimensions, we arrive at the following open problem:

Open Problem 2. *Is the infimum of the Yang-Mills functional in five dimensions*

$$\inf_{F \in \mathcal{F}_{\otimes,\varphi}(B^5)} \int_{B^5} |F|^2 \, d^5x \tag{1.23}$$

attained in $\mathcal{F}_{\otimes,\varphi}(B^5)$?

In a next step, the question of regularity has to be answered. It is well-known that for maps from B^3 into S^2 there is a partial regularity result of R. Schoen and K. Uhlenbeck, [41] saying that minimizers – which belong a priori only to $W^{1,2}(B^3, S^2)$ – are smooth except at a finite number of points. It is an open problem if such partial regularity result also holds for minimizers in $\mathcal{F}_{\otimes,\varphi}(B^5)$.

Open Problem 3. *Do we have some partial regularity result for minimizers in $\mathcal{F}_{\otimes,\varphi}(B^5)$? In particular, do they belong to $\mathcal{F}_{R,\varphi}(B^5)$?*

Three Different Metrics in Four Dimensions

For a better understanding of our results on non-Abelian "singular bundles" – and in particular of Open Problem 1 – we have to take a closer look on $W^{1,2}$-connections with small curvatures on some trivial principal $SU(2)$-bundle over S^4. First, we will show the following proposition saying that Coulomb gauges with curvatures in the small L^2-energy regime are unique up to a constant gauge transformation σ_0:

Proposition 1.4. *Let $A \in \mathcal{A}_{W^{1,2}}(S^4)$ with $\|F(A)\|_{L^2(S^4)} \le \varepsilon < \delta_{Uh}$ and let $\sigma_{Coulomb} \in W^{2,2}(S^4, G)$ denote the gauge transformation such that $A_{Coulomb} = \sigma_{Coulomb}(A)$ satisfies*

$$d_* A_{Coulomb} = 0 , \qquad \|A_{Coulomb}\|_{W^{1,2}(S^4)} \le C_{Uh} \, \|F(A)\|_{L^2(S^4)} . \tag{1.24}$$

Moreover, assume that there exists another gauge transformation $\sigma'_{Coulomb} \in W^{2,2}(S^4, G)$ *such that* $B_{Coulomb} = \sigma'_{Coulomb}(A)$ *satisfies*

$$d_* B_{Coulomb} = 0\,, \qquad \|B_{Coulomb}\|_{W^{1,2}(S^4)} \leq C_{Uh}\, \|F(A)\|_{L^2(S^4)}\,. \tag{1.25}$$

Then, for sufficiently small $\varepsilon > 0$*, the Coulomb gauge is unique up to a constant gauge transformation* σ_0*, i.e., we have that* $\sigma_0^{-1} A_{Coulomb} \sigma_0 = B_{Coulomb}$.

The next proposition plays an important role for the analysis of the non-Abelian case.

Proposition 1.5. *Let* $A, B \in \mathcal{A}_{W^{1,2}}(S^4)$ *with* $\|F(A)\|_{L^2(S^4)}, \|F(B)\|_{L^2(S^4)} \leq \varepsilon < \delta_{Uh}$ *and denote by* $A_{Coulomb}, B_{Coulomb}$ *the corresponding Coulomb gauges. Moreover, let* d*,* δ *and* γ *be three different metrics given by*

(i)

$$d\big(F(A), F(B)\big) = \inf_{\sigma \in L^\infty(S^4, SU(2))} \left(\int_{S^4} \left| F(A) - \sigma^{-1} F(B) \sigma \right|^2 d^4x \right)^{1/2}, \tag{1.26}$$

(ii)

$$\delta\big(F(A), F(B)\big) = \int_{S^4} \left| \mathrm{Tr}\big(F(A) \otimes F(A)\big) - \mathrm{Tr}\big(F(B) \otimes F(B)\big) \right| d^4x\,, \tag{1.27}$$

(iii)

$$\gamma\big(F(A), F(B)\big) = \inf_{\sigma_0 \in SU(2)} \|A_{Coulomb} - \sigma_0^{-1} B_{Coulomb} \sigma_0\|_{W^{1,2}(S^4)}\,. \tag{1.28}$$

Then we have that d *and* δ *induce the same topology, whereas* d *and* γ *do not induce the same topology.*

Remark 1.2. a) The definition of the third metric γ is motivated by Proposition 1.4.

b) The fact that the metrics d and δ do not induce the same topology than γ is the main difficulty in order to solve Open Problem 1.

c) Lemma 1.3 which was already used for the definition of L^2-curvatures of singular $SU(2)$-bundles is also crucial for the proof of the first statement in the previous Proposition 1.5.

Density Results in Critical Dimensions

In the proofs of the above theorems we made use of some density results in critical dimensions for gauge field theories which we now describe. Let P be a principal G-bundle over an n-dimensional compact Riemannian manifold M. Starting with L^p-bounded curvatures, we get from the geometry that the corresponding Sobolev connections and Sobolev bundles have to be at least in $\mathcal{A}_{W^{1,p}}(P)$ and $\mathcal{P}^G_{W^{2,p}}(P)$ respectively (see Section 2.2). It is then natural to ask if there exists a smooth approximation for $W^{2,p}$-Sobolev bundles. The non-linear constraint for the smooth approximation comes from the cocycle condition $g_{ij} g_{jl} = g_{il}$, where the transition functions $g = \{g_{ij}\}_{i,j \in I}$ belong to $W^{2,p}(U_i \cap U_j, G)$ with $U = \{U_i\}_{i \in I}$ an open covering of M. This non-linear constraint causes no problems in the case $p > n/2$ due to the Sobolev embedding $W^{2,p} \hookrightarrow C^0$. It turns out that the density result still holds in the critical case $p = n/2$. A similar situation occurs in the context of maps between manifolds, where the smooth approximability can also be extended to critical dimensions in the sense of Sobolev embeddings as shown by R. Schoen and K. Uhlenbeck [39], [40].

The smooth approximation for connections can be shown with the help of a simple formula taking into consideration the non-linear constraint in terms of the compatibility condition for connections.

Proposition 1.6. *Let $p \geq n/2$ and $A = \{A_i\}_{i\in I}$ a $W^{1,p}$-Sobolev connection on $P_{U,g} \in \mathcal{P}^G_\infty(M)$. Then, there exists a sequence $(\tilde{A}^k)_{k\in\mathbb{N}}$ of smooth connections on $P_{U,g}$ such that*

$$\lim_{k\to\infty} \|\tilde{A}^k_i - A_i\|_{W^{1,p}(U_i)} = 0\,, \tag{1.29}$$

for every $i \in I$.

As a direct consequence, we obtain the smooth approximability of curvatures.

Corollary 1.7. *Let $A = \{A_i\}_{i\in I} \in \mathcal{A}_{W^{1,p}}(P_{U,g})$ and let $F(A)$ be its curvature. Assume also that there exists a sequence $(\tilde{A}^k)_{k\in\mathbb{N}}$ of smooth connections such that*

$$\lim_{k\to\infty} \|\tilde{A}^k_i - A_i\|_{W^{1,p}(U_i)} = 0\,, \tag{1.30}$$

for every $i \in I$. Then, for the sequence of smooth curvatures $\big(F(\tilde{A}^k)\big)_{k\in\mathbb{N}}$ associated to $\tilde{A}^k$, we have that

$$\lim_{k\to\infty} \|F_i(\tilde{A}^k) - F_i(A)\|_{L^p(U_i)} = 0\,, \tag{1.31}$$

for every $i \in I$.

Connecting Singularities

In a second part of the present dissertation, we want to solve the problem of connecting the singularities of the previously defined various analytic classes of weak curvatures for the Abelian and non-Abelian case. In other words, we answer the question of existence of a minimal connection. The notion of minimal connection first appeared in H. Brezis, J.-M. Coron and E. H. Lieb, [2] for harmonic maps from B^3 into S^2 and was then extended to the context of Cartesian currents by M. Giaquinta, G. Modica and J. Souček, [17]. The reader should read the preliminaries to Section 6 for more details.

We start again with the Abelian case. For $F \in \Omega^2_{L^1}(B^3)$, we define the one-dimensional current $T_F \in \mathcal{D}_1(B^3)$ given by

$$T_F(\omega) = \frac{1}{2\pi}\int_{B^3} F \wedge \omega\,, \qquad \forall \omega \in \mathcal{D}^1(B^3)\,. \tag{1.32}$$

In the case of $\mathcal{F}_\infty(B^3) = \{F \in \Omega^2_{L^1}(B^3) \;:\; dF = 0\}$ representing smooth curvatures on B^3 and $\mathcal{F}_R(B^3)$ the boundary of T_F – which we will denote by S_F – reads as $S_F = 0$

and $S_F = -\sum_{i=1}^N d_i\, [\![a_i]\!]$, respectively. Then we introduce the following class of curvature currents on B^3:

$$\begin{aligned}\mathrm{Curv}(B^3) = \Big\{T \in \mathcal{D}_1(B^3) \quad : \quad & \exists\, F_T \in \Omega^2_{L^1}(B^3) \ \text{ and } \exists\, L_T \in \mathcal{R}_1(B^3) \\ & \text{s.t. } T = T_{F_T} + L_T\,,\ \ \partial T = 0 \ \text{ on } \ B^3\Big\}\,,\end{aligned} \tag{1.33}$$

where $\mathcal{R}^1(B^3)$ denotes one-dimensional integer multiplicity (i.m.) rectifiable currents on B^3. It is not difficult to check that the currents T_F for F belonging to $\mathcal{F}_\infty(B^3)$ and $\mathcal{F}_R(B^3)$ are elements of $\mathrm{Curv}(B^3)$.

Consider now an open set $\tilde{B}^3 \supset\supset B^3$ and smooth boundary data φ on $\tilde{B}^3 \setminus B^3$ with vanishing integral on ∂B^3. Then, we define $F \in \mathcal{F}_{R,\varphi}(\tilde{B}^3)$ as element in $\mathcal{F}_R(B^3)$ extended by φ on $\tilde{B}^3 \setminus B^3$ and its minimal connection L_F as one-dimensional i.m. rectifiable current supported in $\bar{B}^3$ of minimal mass whose boundary satisfies $\partial L_F = -S_F$. Moreover, we denote the mass $M(L_F)$ of L_F by $L(F)$ and refer to it as the length of a minimal connection. Following closely the approach of M. Giaquinta, G. Modica and J. Souček, [17] we use some results on minimal currents in order to obtain the formula

$$L(F) = \frac{1}{2\pi} \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} \left\{ \int_{B^3} F \wedge d\xi - \int_{\partial B^3} \iota^*_{\partial B^3}(F\,\xi) \right\}. \tag{1.34}$$

The right-hand side can be seen as definition for the length of a minimal connection for elements in the strong L^1-closure of $\mathcal{F}_{R,\varphi}(\tilde{B}^3)$. Owing to the strong density Theorem 1.1, we denote the L^1-closure of $\mathcal{F}_{R,\varphi}(\tilde{B}^3)$ by $\mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3)$ and we will show that $\mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3) \cap \mathcal{F}_{\mathbb{Z},\eta}(\tilde{B}^3) \neq \emptyset$ with η some other smooth boundary data implies $\varphi = \eta$ (see Proposition 6.7). The next main theorem of the present dissertation says that the singularities of elements in $\mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3)$ can also be connected by an i.m. rectifiable current.

Theorem 1.8. *Let $F \in \mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3)$. Then, there exists $L_T \in \mathcal{R}_1(\tilde{B}^3)$ with $spt\, L_T \subset \bar{B}^3$ such that $\partial L_T = -S_F$. In other words, we have*

$$T = T_F + L_T \in Curv_\varphi(\tilde{B}^3)\,.$$

Moreover, the one-dimensional i.m. rectifiable current L_T can be chosen in such a way that

$$M(L_T) = L(F)\,,$$

where $L(F)$ denotes the length of a minimal connection for F.

Now, we transfer the previous results to the non-Abelian case. To $F \in \Omega^2_{L^2}\big(B^5, \mathfrak{su}(2)\big)$ we associate the four-form $\hat{\Omega} = \mathrm{Tr}(F \wedge F)$ belonging to $\Omega^4_{L^1}(B^5)$ and define then the one-dimensional current $T_{\hat{\Omega}} \in \mathcal{D}_1(B^5)$ given by

$$T_{\hat{\Omega}}(\omega) = \frac{1}{8\pi^2}\int_{B^5} \hat{\Omega} \wedge \omega = \frac{1}{8\pi^2}\int_{B^5} \mathrm{Tr}(F \wedge F) \wedge \omega\,, \qquad \forall \omega \in \mathcal{D}^1(B^5)\,. \tag{1.35}$$

As in the Abelian case, it is not difficult to check that for $\mathcal{F}_\infty(\bar{B}^5)$ and $\mathcal{F}_R(B^5)$ the current $T_{\hat{\Omega}}$ belongs to the following class of curvature currents on B^5:

$$\mathrm{Curv}(B^5) = \Big\{ T \in \mathcal{D}_1(B^5) \quad : \quad \exists\, \hat{\Omega}_T \in \Omega^4_{L^1}(B^5) \ \text{ and } \ \exists\, L_T \in \mathcal{R}_1(B^5) \\ \text{s.t. } \ T = T_{\hat{\Omega}_T} + L_T\,, \ \ \partial T = 0 \ \text{ on } \ B^5 \Big\}\,. \tag{1.36}$$

The length of a minimal connection for $F \in \mathcal{F}_{R,\varphi}(B^5)$ has the formula

$$\begin{aligned} L(F) &= \frac{1}{8\pi^2} \sup_{\substack{\xi : B^5 \to \mathbb{R} \\ \|d\xi\|^{co}_{L^\infty} \le 1}} \left\{ \int_{B^5} \hat{\Omega} \wedge d\xi - \int_{\partial B^5} \iota^*_{\partial B^5}(\hat{\Omega}\,\xi) \right\} \\ &= \frac{1}{8\pi^2} \sup_{\substack{\xi : B^5 \to \mathbb{R}^3 \\ \|d\xi\|^{co}_{L^\infty} \le 1}} \left\{ \int_{B^5} \mathrm{Tr}(F \wedge F) \wedge d\xi - \int_{\partial B^5} \iota^*_{\partial B^5}\big(\mathrm{Tr}(F \wedge F)\xi\big) \right\}\,, \end{aligned} \tag{1.37}$$

which also gives a definition for the length of a minimal connection $L(\Omega)$ of elements Ω in $\mathcal{F}_{\otimes,\varphi}(B^5)$ with appropriate boundary conditions φ such that the integral of $\mathrm{Tr}(\varphi \wedge \varphi)$ on ∂B^3 vanishes. Note that as in the Abelian case we have $\mathcal{F}_{\otimes,\varphi}(B^5) \cap \mathcal{F}_{\otimes,\eta}(B^5) \neq \emptyset$ with η some other smooth boundary data implies $\varphi = \eta$. The last formula (1.37) for $\mathcal{F}_{R,\varphi}(B^5)$ already appeared in T. Isobe, [25] as a natural generalization of the length $L(u)$ of a minimal connection for $u \in R^{1,2}(B^3, S^2)$ without using the language of minimal currents. Proceeding as in the Abelian case, we can show that singularities of $\mathcal{F}_{\otimes,\varphi}(B^5)$ can also be connected by an i.m. rectifiable current.

Theorem 1.9. *Let $\Omega \in \mathcal{F}_{\otimes,\varphi}(\tilde{B}^5)$ with $\hat{\Omega}$ the associated alternated four-form. Then, there exists $L_T \in \mathcal{R}_1(\tilde{B}^5)$ with $spt\, L_T \subset \bar{B}^5$ such that $\partial L_T = -S_{\hat{\Omega}}$. In other words, we have*

$$T = T_{\hat{\Omega}} + L_T \in Curv_\varphi(\tilde{B}^5)\,.$$

Moreover, the one-dimensional i.m. rectifiable current L_T can be chosen in such a way that

$$M(L_T) = L(\Omega)\,,$$

where $L(\Omega)$ denotes the length of a minimal connection for Ω.

Next, note that the problem of minimizing the Yang-Mills functional in five dimensions with fixed singularities was solved by T. Isobe, [25]. More precisely, let q be a smooth $SU(2)$-bundle over S^4 with vanishing second Chern number $c_2(q) = 0$ and curvature φ. Then, the minimal Yang-Mills energy in the class

$$\mathcal{E} = \Big\{ F \in \mathcal{F}_{R,\varphi}(B^5) \ : \ d\,\mathrm{Tr}(F \wedge F) = 8\pi^2 \left(\sum_{i=1}^N d_i \delta_{a_i} \right) dx^1 \wedge \ldots \wedge dx^5 \ \text{ in } \ \mathcal{D}' \Big\}$$

with prescribed singularities is given by $8\pi^2\, L(F)$, where $L(F)$ denotes the minimal connection of F. Following the strategy of F. Bethuel, H. Brezis and J.-M. Coron, [6] for harmonic maps $u : B^3 \longrightarrow S^2$ explained in Section 2.3, we then introduce the relaxed Yang-Mills functional.

Open Problem 4. *Is the infimum of the relaxed Yang-Mills functional given by*

$$YM_{rel}(F) = \int_{B^5} |F|^2\, d^5x + 8\pi^2\, L(\Omega)\,, \tag{1.38}$$

or, equivalently,

$$YM_{rel}(F) = \int_{B^5} |F|^2\, d^5x + \min\ 8\pi^2\, M(L_T)\,, \tag{1.39}$$

attained in the class $\mathcal{F}_{\otimes,\varphi}(B^5)$*?*

Chapter 2

Background Material

2.1 Concepts from Differential Geometry

We present the basic concepts from differential geometry on which the theory of gauge fields is based. For simplicity, in this section we assume that all objects are smooth. Several aspects are explained in more detail including references to the literature in Appendices B and C.

Let $\pi : P \longrightarrow M$ be a principal G-bundle over a compact n-dimensional Riemannian manifold M. The structure group G of P is assumed to be a compact Lie group with Lie algebra $\mathfrak{g}$. Given an open covering $U = \{U_i\}_{i\in I}$ of M and trivializations (π, φ_i) : $\pi^{-1}(U_i) \longrightarrow U_i \times G$ for the principal G-bundle P, we can define transition functions $g_{ij} : U_i \cap U_j \longrightarrow G$ satisfying the cocycle condition $g_{ij}g_{jl} = g_{il}$ on $U_i \cap U_j \cap U_l \neq \emptyset$. The open covering U and the family of transition functions $g = \{g_{ij}\}_{i,j\in I}$ uniquely determine the principal G-bundle P. In the following, we will hence sometimes use the notation $P_{U,g}$ for P being an element in the space $\mathcal{P}^G(M)$ of all principal G-bundle over M.

We associate the adjoint bundle $P \times_{Ad} \mathfrak{g}$ to P – denoted by $\mathrm{Ad}(P)$ – with $Ad : G \times \mathfrak{g} \longrightarrow \mathfrak{g}$ the adjoint action of the Lie group G on the Lie algebra $\mathfrak{g}$. Differential s-forms on M with values in the adjoint bundle $\mathrm{Ad}(P)$ are sections of $\bigwedge^s(TM) \otimes \mathrm{Ad}(P)$ for which we will use the notation $\Omega^s(M, \mathrm{Ad}(P))$. The space of connections on P is then defined by

$$\mathcal{A}(P) = \left\{\omega = \omega_0 + A \ : \ A \in \Omega^1(M, \mathrm{Ad}(P))\right\}, \tag{2.1}$$

where ω_0 is a fixed reference connection. Given a trivialization, we can also think of elements in $\mathcal{A}(P)$ as family $A = \{A_i\}_{i\in I}$ of $\mathfrak{g}$-valued one-forms A_i on U_i satisfying the compatibility condition

$$A_j = g_{ij}^{-1}\, A_i\, g_{ij} + g_{ij}^{-1} dg_{ij} \qquad \text{on } \ U_i \cap U_j\,. \tag{2.2}$$

Hence we can write

$$\mathcal{A}(P) = \left\{A = \{A_i\}_{i\in I} \ : \ A_i \in \Omega^1(U_i, \mathfrak{g}) \ \text{ s.t. } (2.2) \text{ holds}\right\}.$$

A connection on P induces a covariant derivative ∇_A on sections of the associated vector bundle $\mathrm{Ad}(P)$. Together with the Levi-Cività connection on M this covariant derivative extends to a covariant derivative ∇_A on differential forms taking their values in $\mathrm{Ad}(P)$. The exterior covariant derivative D_A on $\Omega^s(M, \mathrm{Ad}(P))$ is then the completely anti-symmetric part of ∇_A (see Appendix C for details).

The exterior covariant derivative of a connection in $\mathcal{A}(P)$ then defines its curvature F being a two-form on M with values in $\mathrm{Ad}(P)$. The curvature can also be interpreted as family $F = \{F_i\}_{i \in I}$ of $\mathfrak{g}$-valued two-forms F_i on U_i satisfying the compatibility condition

$$F_j = g_{ij}^{-1}\, F_i\, g_{ij} \qquad \text{on } U_i \cap U_j\,. \tag{2.3}$$

Moreover, we have the local representation

$$F_i = dA_i + A_i \wedge A_i \qquad \text{on } U_i\,. \tag{2.4}$$

The group of gauge transformation $\mathcal{G}(P)$ consists of bundle automorphisms on P. Gauge transformations can be identified with sections of the associated automorphism bundle $P \times_c G$ – denoted by $\mathrm{Aut}(P)$– with $c : G \times G \longrightarrow G$ the conjugate action of the Lie group G on itself. In a trivialization $\sigma \in \mathcal{G}(P)$ is represented by local gauge transformations $\sigma_i : U_i \longrightarrow G$ satisfying $g_{ij} = \sigma_i^{-1} g_{ij} \sigma_j$ on $U_i \cap U_j$. This shows that the transition functions remain unchanged under gauge transformations. Locally, a gauge transformation acts on a connection $A \in \mathcal{A}(P)$ by

$$\sigma_i(A) = \sigma_i^{-1} A_i \sigma_i + \sigma_i^{-1} d\sigma_i\,, \tag{2.5}$$

and on its curvature by

$$F_i\big(\sigma(A)\big) = \sigma_i^{-1} F_i(A) \sigma_i\,. \tag{2.6}$$

Also note that a single local gauge transformation can be seen as a change of trivialization.

Two principal G-bundles $P, Q \in \mathcal{P}^G(M)$ are said to be equivalent or isomorphic to each other if there exists a bundle isomorphism $h : P \longrightarrow Q$, i.e., a G-equivariant map inducing the identity on M. In terms of transition functions this translates to the existence of maps $\rho_i : U_i \longrightarrow G$ such that

$$h_{ij} = \rho_i\, g_{ij}\, \rho_j^{-1} \qquad \text{on } U_i \cap U_j\,, \tag{2.7}$$

where $\{g_{ij}\}_{i,j \in I}$ and $\{h_{ij}\}_{i,i \in I}$ are transition functions of P and Q respectively over the same covering $\{U_i\}_{i \in I}$ of M. Note that different trivializations of the same principal G-bundle satisfy (2.7). For the following two cases the equivalence classes are characterized by integers:

a) Principal $U(1)$-bundles P over the unit two-sphere $S^2 \subset \mathbb{R}^3$ with curvature F are classified by the integer

$$c_1(P) = \frac{1}{2\pi} \int_{S^2} F\,, \tag{2.8}$$

called first Chern number of P. Note that the integrand is gauge invariant and belongs to the first Chern class.

b) Principal $SU(2)$-bundles P over the unit four-sphere $S^4 \subset \mathbb{R}^5$ with curvature F are classified by the integer

$$c_2(P) = \frac{1}{8\pi^2} \int_{S^4} \mathrm{Tr}(F \wedge F)\,, \tag{2.9}$$

called second Chern number of P. Note that the integrand is again gauge invariant but belongs to the second Chern class.

For these results and a very readable introduction to characteristic classes we refer to G.L. Naber [32], Chapter 6, and in particular to Theorem 6.4.2 therein.

On $\Omega^s(M, \mathrm{Ad}(P))$, there exists an adjoint action invariant pointwise scalar product $\langle\cdot,\cdot\rangle$ induced by the Riemannian metric g on M and the Killing form on the Lie algebra $\mathfrak{g}$. Using this scalar product, we then define the Yang-Mills energy functional $YM : \mathcal{A}(P) \longrightarrow \mathbb{R}$ by

$$YM(A) = \int_M |F(A)|^2 \, dvol_g \,, \tag{2.10}$$

where $dvol_g$ is the volume form on M. Note that the Yang-Mills functional is gauge invariant, i.e., $YM\big(\sigma(A)\big) = YM(A)$ for $\sigma \in \mathcal{G}(P)$. The critical points of the Yang-Mills functional are called Yang-Mills connections. They satisfy a partial differential equation given by the Euler-Lagrange equation

$$D_A^* F(A) = 0 \,,$$

where D_A^* denotes the formally adjoint operator to the exterior covariant derivative D_A with respect to the scalar product mentioned above.

2.2 Weak Formulation of Gauge Field Theories

In this section, we describe the weak Sobolev formulation for gauge field theories. From previous considerations, we already know that various geometrical objects occur in the theory of gauge fields. Therefore care must be taken when defining their corresponding Sobolev analogue. In particular, the Sobolev space of bundle-valued differential forms has to be well defined. The reader is referred to Appendix C and references therein for more details.

First, we introduce the notion of Sobolev principal bundles. For this purpose, from now on we assume that the compact Lie group G is a matrix Lie group and we can therefore regard G as subset of $\mathbb{R}^{l\times l} = \mathbb{R}^m$. As in the case of maps between manifolds (see Section 2.3 below), the Sobolev space of maps $g : U \subset M \longrightarrow G$ is given by

$$W^{k,p}(U,G) = \left\{ g \in W^{k,p}(U,\mathbb{R}^m) \;:\; g(x) \in G \text{ for a.e. } x \in U \right\} . \tag{2.11}$$

For $kp > \dim M = n$ the Sobolev embedding says that elements in $W^{k,p}(U,G)$ are even continuous. Hence we easily obtain that the pointwise multiplication and inversion are well-defined and continuous in $W^{k,p}(U,G)$. Making use of the compactness of G, it can be shown that this remains true for $1 \le kp \le n$.

Lemma 2.1. *Let $1 \le p \le \infty$. Then the pointwise multiplication and inversion are well-defined and continuous in the Sobolev space $W^{k,p}(U,G)$.*

Generalized transition functions in the Sobolev space (2.11) lead to Sobolev principal bundles. Note that the next definition only makes sense because of Lemma 2.1.

Definition 2.1. *Let M be an n-dimensional Riemannian manifold with open covering $U = \{U_i\}_{i\in I}$ and G a matrix Lie group. Moreover, let $g = \{g_{ij}\}_{i,j\in I}$ be a family of G-valued functions each defined on $U_i \cap U_j \neq \emptyset$ such that the following holds:*

(i) For every $i, j \in I$, we have that $g_{ij} \in W^{k,p}(U_i \cap U_j, G)$.

(ii) For a.e. $p \in U_i \cap U_j \cap U_k \neq \emptyset$, the cocycle condition $g_{ij}(p)g_{jk}(p) = g_{ik}(p)$ is satisfied.

This open covering U and family of functions g then define a $W^{k,p}$-Sobolev principal G-bundle over M. It is denoted by $P_{U,g}$ and we write $\mathcal{P}^G_{W^{k,p}}(M)$ for the set of all $W^{k,p}$-Sobolev principal G-bundles over M.

Next, we explain how to define Sobolev gauge transformations. As sections of $\mathrm{Aut}(P)$, not only are they locally maps with values in the Lie group G, but also characterized by gluing conditions. For Sobolev gauge transformations we therefore have on the one hand to keep the global characteristics and on the other hand to require the associated local gauge transformations to be Sobolev maps. More precisely, we define

$$\begin{aligned} \mathcal{G}_{W^{k,p}}(P) = \big\{\sigma \in \mathcal{G}(P) \quad : \quad & \sigma_i \in W^{k,p}(U_i, \mathbb{R}^m) \\ & \text{and } \ \sigma(x) \in G \ \text{ for a.e. } \ x \in M\big\} . \end{aligned} \tag{2.12}$$

Note that a single local Sobolev gauge transformation can simply be seen as element of $W^{k,p}(U, G)$ for some $U \subset M$.

With the help of the covariant derivative ∇_A and the scalar product $\langle \cdot, \cdot \rangle$ introduced in Section 2.1, we can define the Sobolev spaces $\Omega^s_{W^{k,p}}(M, \mathrm{Ad(P)})$ whose elements α are Ad(P)-valued s-forms with finite Sobolev norm given by

$$\|\alpha\|_{W^{k,p}} = \left(\sum_{i=0}^{k} \int_M |\nabla^i_A \alpha|^p \, dvol_g\right)^{1/p} = \left(\sum_{i=0}^{k} \|\nabla^i_A \alpha\|^p_{L^p}\right)^{1/p} .$$

Here ∇^i_A denotes the i-th covariant derivative. Together with (2.1) we can then define

$$\mathcal{A}_{W^{k,p}}(P) = \big\{\omega = \omega_0 + A \ : \ A \in \Omega^1_{W^{k,p}}(M, \mathrm{Ad}(P))\big\} , \tag{2.13}$$

the Sobolev space of connections on P. From now on we will only consider L^p-curvatures, i.e. the case $k = 1$, since (2.4) indicates that the connection has one more derivative than the curvature. From (2.2) we then get that the relevant Sobolev principal bundles have two derivatives and hence belongs to $\mathcal{P}^G_{W^{2,p}}(M)$. Due to (2.5) gauge transformations are of the same $W^{2,p}$-Sobolev type.

The interpretation of a connection as a family of locally defined $\mathfrak{g}$-valued one-forms satisfying the compatibility condition (2.2) leads to another equivalent definition of Sobolev spaces of connections.

Definition 2.2. *Let $P_{U,g} \in \mathcal{P}^G_{W^{2,p}}(M)$. A $W^{1,p}$-Sobolev connection on $P_{U,g}$ is then defined as a family $A = \{A_i\}_{i \in I}$ of $\mathfrak{g}$-valued one-forms on U_i such that the following holds:*

(i) For every $i \in I$, we have that $A_i \in W^{1,p} \cap L^{2p}(U_i, T^*U_i \otimes \mathfrak{g})$.

(ii) The compatibility condition $A_j = g_{ij}^{-1} \, A_i \, g_{ij} + g_{ij}^{-1} dg_{ij}$ is satisfied a.e. on $U_i \cap U_j$.

The space of $W^{1,p}$-Sobolev connections on $P_{U,g}$ is denoted by $\mathcal{A}_{W^{1,p}}(P_{U,g})$.

Note that the Riemannian metric and the Levi-Civitá connection on M are used in order to define the Sobolev space $W^{1,p}(U_i, T^*U_i \otimes \mathfrak{g}) = \Omega^1_{W^{1,p}}(U_i, \mathfrak{g})$. The integrability condition $A_i \in W^{1,p} \cap L^{2p}(U_i, T^*U_i \otimes \mathfrak{g})$ in the previous definition together with Hölder's inequality ensures the L^p-boundedness of the curvature

$$\begin{aligned} \|F_i\|_{L^p} &\leq \|dA_i\|_{L^p} + \|A_i \wedge A_i\|_{L^p} \\ &\leq \|dA_i\|_{L^p} + \|A_i\|^2_{L^{2p}} \leq \|A_i\|_{W^{1,p}} + \|A_i\|^2_{L^{2p}} . \end{aligned} \tag{2.14}$$

Moreover, if we assume that $p \geq n/2$ then the Sobolev embedding $W^{1,p} \hookrightarrow L^{2p}$ implies that the first condition in Definition 2.2 can be relaxed to $A_i \in W^{1,p}(U_i, T^*U_i \otimes \mathfrak{g})$, since we then have

$$\|F_i\|_{L^p} \leq \|A_i\|_{W^{1,p}} + \|A_i\|^2_{L^{2p}} \leq \|A_i\|_{W^{1,p}} + C\,\|A_i\|^2_{W^{1,p}} . \tag{2.15}$$

Estimate (2.14) gives a sense to the next definition.

Definition 2.3. *Let $A = \{A_i\}_{i \in I} \in \mathcal{A}_{W^{1,p}}(P_{U,g})$. The L^p-Sobolev curvature associated to A is the family $F(A) = \{F_i(A)\}_{i \in I}$ of $\mathfrak{g}$-valued two-forms on U_i defined by*

$$F_i(A) = dA_i + A_i \wedge A_i \qquad \text{on } U_i .$$

Note that as a direct consequence we have $F_j(A) = g_{ij}^{-1}\, F_i(A)\, g_{ij}$ almost everywhere on $U_i \cap U_j$. In Definition 2.3 we encountered a first formulation of weak curvatures.

2.3 Review of Harmonic Map Theory

Our strategy for producing a suitable framework for the calculus of variations associated with the Yang-Mills functional beyond the critical dimension four, is strongly based on the one used for similar problems occurring in harmonic map theory. – Harmonic maps $u : \mathcal{M} \longrightarrow \mathcal{N}$ between two Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$ have been a vast field of research in mathematics for the last decades. Analytic methods used for harmonic maps can in part be transfered to gauge field theories. Although many of the results summarized in this section hold in a more general context, we shall only be concerned with those aspects of harmonic map theory which can be used as motivation for solving our initial problem. We will hence focus our attention on harmonic maps $u : B^3 \longrightarrow S^2$. For a better understanding of the strategies used in this thesis some ideas of proofs are also added. – A detailed overview of results in harmonic map theory and the techniques involved in proving them can be found in H. Brezis, [4], F. Hélein and J.C. Wood, [23].

We consider maps $u : B^n \longrightarrow S^m$, where B^n is the unit ball in $\mathbb{R}^n$ and S^m the unit sphere in $\mathbb{R}^{m+1}$. Both are equipped with the metric induced by the standard Euclidean metric. The Dirichlet energy functional is defined by

$$E(u) = \int_{B^n} |du|^2\, d^n x ,$$

with the usual norm on vector-valued one-forms. The critical points of the Dirichlet functional are called harmonic maps. They satisfy a system of non-linear partial differential equations given by the Euler-Lagrange equation

$$\Delta u + u|du|^2 = 0 , \tag{2.16}$$

where Δ denotes the Laplace operator.

We will see that for given smooth boundary data $\varphi : \partial B^n \longrightarrow S^m$, the Sobolev space

$$W^{1,2}_{\varphi}(B^n, S^m) = \big\{ u \in W^{1,2}(B^n, \mathbb{R}^{m+1}) \ : \ u = \varphi \text{ on } \partial B^n\,, \\ u(x) \in S^m \text{ for a.e. } x \in B^n \big\}$$

allows the application of the Dirichlet principle also called the direct method in the calculus of variations.

Before proceeding in this direction, we want to focus our attention on some density results for Sobolev maps. For the linear problem of weakly harmonic functions, the well-known approximability of Sobolev functions by smooth functions using standard mollification techniques plays a crucial role. In the present non-linear context, it hence seems natural to study whether such an approximation result still holds. It turns out that this depends on the dimension n of the domain B^n. For $n = 1$ the density of smooth maps $C^{\infty}(\bar{B}^1, S^m)$ in $W^{1,2}(B^1, S^m)$ for the strong $W^{1,2}$-topology follows directly from the Sobolev embedding theorem. As shown in R. Schoen and K. Uhlenbeck [39], [40] the density of smooth maps can also be extended to the critical case $n = 2$. However, in the case of $n > 2$ the radial projection map $u_{\odot}(x) = x/|x|$ gives an example for a map in $W^{1,2}(B^3, S^2)$ which can not be strongly approximated by smooth maps $C^{\infty}(\bar{B}^3, S^2)$. We will come back later to the smooth approximability of $W^{1,2}(B^3, S^2)$.

Using the direct method in the calculus of variations, we now look for a minimizer of the Dirichlet functional. Equivalently, we ask whether the following infimum is attained:

$$m = \inf_{u \in W^{1,2}_{\varphi}(B^n, S^m)} \int_{B^n} |du|^2 \, d^n x\,. \tag{2.17}$$

Two-dimensional Case

We first treat the case $n = 2$ with maps from B^2 into S^2. Note that in the sense of Sobolev embeddings we are in the critical dimension and that because of the previously mentioned density result, in (2.17) we can also take the infimum over $C^{\infty}_{\varphi}(\bar{B}^2, S^2)$. Consider a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ in $W^{1,2}_{\varphi}(B^2, S^2)$ such that $E(u_k) \longrightarrow m$ for $k \to \infty$. Weak compactness then gives a limiting $W^{1,2}$-Sobolev map u_{∞}. It is not difficult to check that u_{∞} even belongs to $W^{1,2}_{\varphi}(B^2, S^2)$. Using the properties of weakly convergent sequences and the lower semicontinuity of the L^2-norm, we deduce that u_{∞} is a minimizer, i.e., $m = E(u_{\infty})$. For an indepth discussion of the direct method in the calculus of variations we refer to Chapter I of the textbook by M. Struwe, [44]. As minimizer u_{∞} satisfies the harmonic map equation (2.16) in a weak sense. The classical regularity result of C.B. Morrey (see M. Giaquinta, G. Modica and J. Souček [17], Section 3.2.2 for an exposition of the original proof) for weakly harmonic maps in the critical dimension then implies the smoothness of u_{∞}. Thus problem (2.17) has a smooth solution in two dimensions.

When topology comes into play in the sense that we want to find a harmonic representant in a prescribed homotopy class the situation becomes more involved. First recall that in the case of smooth maps from S^2 into S^2, we have that $f, g \in C^{\infty}(S^2, S^2)$ are homotopy equivalent $f \sim g$ or in the same homotopy class if and only if $\deg(f) = \deg(g)$.

Here $\deg(f)$ denotes the Brouwer degree of f given by

$$\deg(f) = \frac{1}{4\pi} \int_{S^2} f^* \omega_{S^2} \,, \tag{2.18}$$

where $f^*\omega_{S^2}$ denotes the pull-back by f of the canonical volume form ω_{S^2} on S^2. The degree of a Sobolev map in $W^{1,2}(S^2, S^2)$ can then be defined by the degree of $W^{1,2}$-close smooth maps which exists because of the strong density of smooth maps as explained before. Using a gluing technique this leads to a definition of the homotopy class for maps in $W^{1,2}_{\varphi}(B^2, S^2)$. One makes the following important observation: The weakly convergent minimizing subsequence for (2.17) does not preserve the homotopy class. Thus the infimum in the class of maps in $W^{1,2}_{\varphi}(B^2, S^2)$ with prescribed homotopy class is not achieved. This failure of compactness can be illustrated with the so-called "bubbling of spheres" as discovered by J. Sacks and K. Uhlenbeck, [37].

Three-dimensional Case

We now consider the three dimensional case with maps from B^3 into S^2. Recall that the radial projection $u_{\odot}$ is an example of a map belonging to $W^{1,2}(B^3, S^2)$ which can not be approximated by smooth maps (see R. Schoen and K. Uhlenbeck, [40]). The following theorem says that the singularities of $u_{\odot}$ are somehow generic:

Theorem 2.2 ([8],[9],[10]). *Let*

$$R^{1,2}(B^3, S^2) = \Big\{ u \in W^{1,2}(B^3, S^2) \;\;:\;\; \exists\; a_1, \ldots, a_N \in B^3 \;\; \text{such that} \;\; u \in C^{\infty}\Big(B^3 \setminus \bigcup_{i=1}^{N} a_i, S^2 \Big) \Big\}$$

be the subset of $W^{1,2}$-Sobolev maps which are smooth except at a finite number of points. Then, for every $u \in W^{1,2}(B^3, S^2)$, there exists a sequence $(u_k)_{k \in \mathbb{N}}$ in $R^{1,2}(B^3, S^2)$ such that

$$u_k \longrightarrow u \qquad \text{in } W^{1,2} \,. \tag{2.19}$$

For this density result we refer to F. Bethuel and X. Zeng [10], Theorem 4, F. Bethuel [8], Theorem 2 and especially to F. Bethuel [9], Proposition 1. The idea of the proof is to divide B^3 into small cubes of side length $\varepsilon > 0$. In this cubic decomposition one identifies the "good" cubes as those with small Dirichlet energy and the "bad" cubes as the remaining ones with large energy. In order to construct an approximating sequence in $R^{1,2}(B^3, S^2)$, one chooses a harmonic extension of the given map in $W^{1,2}(B^3, S^2)$ on the good cubes and extend it radially on the bad cubes. Considering the good and bad cubes separately one can show that in the limit $\varepsilon \to 0$ this sequence converges to the given map. In fact, for the convergence on the good cubes one uses Poincaré's inequality since the L^2-norm on the boundary of the cubes can be controlled by a particular choice of the cubic decomposition. By a simple counting argument the volume of the bad cubes tends to zero in the limit $\varepsilon \to 0$ and hence dominated convergence applies.

The question of approximability of $W^{1,2}(B^3, S^2)$ thus relies on approximating $R^{1,2}(B^3, S^2)$ by smooth maps. We associate to $u \in R^{1,2}(B^3, S^2)$ the so-called "D-field" defined by the integrable one-form $D(u) = u^*\omega_{S^2}$ on B^3 whose exterior derivative reads in the distributional sense as

$$dD(u) = \left(4\pi \sum_{i=1}^{N} d_i\, \delta_{a_i}\right) dx^1 \wedge dx^2 \wedge dx^3 \quad \text{in } \mathcal{D}' . \tag{2.20}$$

Here δ_{a_i} denotes the Dirac measure at $a_i \in B^3$ and the integers $d_i = \deg(u, a_i)$ denote the Brouwer degree of u restricted to any small sphere $S^2_r(a_i)$ centered at a_i. Roughly speaking $dD(u)$ detects the locations and the degrees of the topological singularities of $u \in R^{1,2}(B^3, S^2)$. Using (2.20), there is a simple proof of the non-approximability of the radial projection $u_\odot$. For an argument by contradiction assume that $(v_k)_{k\in\mathbb{N}}$ in $C^\infty(\bar{B}^3, S^2)$ converges in $W^{1,2}$ to $u_\odot$. It follows that $D(v_k) \longrightarrow D(u_\odot)$ in the L^1-norm and hence $dD(v_k) \rightharpoonup dD(u_\odot)$ weakly in the sense of distributions. This is not possible, since $dD(v_k) = 0$ for all k and $dD(u_\odot) = 4\pi\delta_0\, dx^1 \wedge dx^2 \wedge dx^3$. This observation gives one direction in Theorem 1 of F. Bethuel, [7] characterizing the smooth approximability of $W^{1,2}$-maps.

Theorem 2.3 ([7]). *Let $u \in W^{1,2}(B^3, S^2)$. Then there exists a sequence $(v_k)_{k\in\mathbb{N}}$ in $C^\infty(\bar{B}^3, S^2)$ such that $v_k \longrightarrow u$ in $W^{1,2}$ if and only if*

$$dD(u) = d(u^*\omega_{S^2}) = 0 . \tag{2.21}$$

The main steps for the other direction in the theorem are the following: In a first step, due to Theorem 2.2 one can take an approximating sequence in $R^{1,2}(B^3, S^2)$ for the given $W^{1,2}$-Sobolev map with $dD(u) = 0$. In a second step, one considers a so-called "dipole" defined by a map $w : U \longrightarrow S^2$ on some open domain U of B^3 such that $w \in C^\infty\big(U \setminus \{a_+, a_-\}, S^2\big)$ with $\deg(w, a_+) = 1$ and $\deg(w, a_-) = -1$ for the pair of singularities $a_+, a_- \in U$. Modification of w inside a small tubular neighborhood of the line segment $[a_+, a_-]$ joining the two singularities in order to obtain a smooth map, induces a bounded extra cost in $W^{1,2}$-norm. More precisely, the bound is given by the length $8\pi|a_+ - a_-|$ of the dipole. In a third step, the singularities of the $R^{1,2}$-maps in the approximating sequence are splitted into several dipoles. Each of these can be removed by the modification just described without loosing control of the $W^{1,2}$-norm. The control is given explicitly by summing the length of the dipoles. This quantity will later be called "length of a minimal connection". Further explanations are given in Chapter 6 below. The length $L(u)$ of a minimal connection for the given $u \in W^{1,2}(B^3, S^2)$ vanishes, since $dD(u) = 0$ by assumption. Finally, one concludes the proof using the continuity of the length of minimal connections and the fact that for smooth maps the length of a minimal connection is zero.

In contrast to the strong convergence, as a by-product of the previous theorem, one obtains that there is no restriction on the *weak* approximability of $W^{1,2}(B^3, S^2)$ by smooth maps (see F. Bethuel [7], Theorem 3 and F. Bethuel, H. Brezis and J.-M. Coron [6], Theorem 2).

Theorem 2.4 ([6],[7]). *Let $u : B^3 \longrightarrow S^2$ be any $W^{1,2}$-Sobolev map. Then there exists a sequence $(v_k)_{k\in\mathbb{N}}$ in $C^\infty(\bar{B}^3, S^2)$ such that*

$$v_k \rightharpoonup u \qquad \textit{weakly in } W^{1,2} .$$

Now we come back to the calculus of variations for the Dirichlet functional in dimension $n = 3$. First of all, it is important to mention that the direct method in the calculus of variations gives – as in the two-dimensional case – the existence of a minimizer in the class $W^{1,2}_\varphi(B^3, S^2)$. However, this does not yield a smooth solution of (2.17) in the three-dimensional case, because we only have a partial regularity result for minimizing weakly harmonic maps (see R. Schoen and K. Uhlenbeck, [41]).

Next, we consider a simpler problem than (2.17) in the sense that the minimizer can only have a finite number of singularities. More precisely, we want to determine the minimal Dirichlet energy m in the class

$$\mathcal{E}_0 = \left\{u \in R^{1,2}_\varphi(B^3, S^2) \;:\; \deg(u, a_i) = d_i \text{ and } \varphi(x) = \text{const}\right\}, \tag{2.22}$$

where the locations $a_1, \dots, a_N \in B^3$ and the degrees $d_1, \dots, d_N \in \mathbb{Z}$ of the topological singularities are prescribed. It turns out that $m = 8\pi\, L$, where L is the length of a minimal connection for the prescribed topological singularities in $\mathcal{E}_0$. For a proof of this result we refer to H. Brezis, J.-M. Coron and E. H. Lieb, [2]. Note that the assumption of a constant map φ on the boundary implies that $\deg(u|_{\partial B^3}) = \deg(\varphi) = 0$ or equivalently that $\sum_{i=1}^N d_i = 0$. Under this assumption the length of a minimal connection $L(u)$ for $u \in R^{1,2}_\varphi(B^3, S^2)$ can be calculated by the formula (see Section 6.1 below)

$$L(u) = \frac{1}{4\pi} \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|_{L^\infty} \le 1}} \left\{ \int_{B^3} D(u) \wedge d\xi - \int_{\partial B^3} \iota^*_{\partial B^3}(D(u)\,\xi) \right\}. \tag{2.23}$$

The right-hand side of the last equation makes sense for arbitrary maps $u \in W^{1,2}_\varphi(B^3, S^2)$ and hence can be seen as extension of $L(u)$ on $W^{1,2}_\varphi(B^3, S^2)$ with $\deg(\varphi) = 0$.

Returning to the problem (2.17) there are cases for which the minimizer of the Dirichlet functional can be calculated explicitly (see H. Brezis, J.-M. Coron and E. H. Lieb [2], Theorem 7.1).

Theorem 2.5 ([2]). *The unique minimizer of the Dirichlet functional in the class*

$$\mathcal{E}_1 = \left\{u \in W^{1,2}_\varphi(B^3, S^2) \;:\; \varphi(x) = x\right\}$$

is given by the radial projection $u_\odot = x/|x|$.

A direct computation yields $E(u_\odot) = 8\pi$. Next one shows that $E(u) \ge 8\pi$ for every $u \in W^{1,2}_\varphi(B^3, S^2)$ with $\varphi(x) = x$. Using the regularity result of R. Schoen and K. Uhlenbeck, [41] for minimizing weakly harmonic maps saying that such maps are smooth except at a finite number of points, it suffices to test $E(u) \ge 8\pi$ in the class $R^{1,2}_\varphi(B^3, S^2)$. Alternatively, we can also use the density result in Theorem 2.2. Different approaches for the rest of the proof including uniqueness are presented in H. Brezis [3], Section II.1. More generally, using the partial regularity result of R. Schoen and K. Uhlenbeck, [41] the following characterization of minimizers holds (see H. Brezis [3], Theorem 6):

Theorem 2.6 ([3]). *Let $u_0 : B^3 \longrightarrow S^2$ be a minimizer for the Dirichlet functional $E(u)$ with any boundary data $\varphi : \partial B^3 \longrightarrow S^2$. Then all singularities of u_0 have non-vanishing degree. More precisely, assuming that $u_0 \in C^\infty\big(B^3 \setminus \{a_1, \dots, a_N\}\big)$, we have that*

$$deg(u_0, a_i) = \frac{1}{4\pi} \int_{\partial B^3_r(a_i)} \iota^*_{\partial B^3_r(a_i)}(u_0^* \omega_{S^2}) \neq 0\,. \tag{2.24}$$

What happens if we now assume smooth boundary data $\varphi : \partial B^3 \longrightarrow S^2$ satisfying $\deg(\varphi) = 0$? Do we then get the existence of a smooth solution to problem (2.17) in three dimensions? Vanishing degree means that we do not impose topological obstructions to the regularity, since φ can be extended smoothly inside B^3. However, such smooth extensions are not minimizing and in fact, introducing singularities can lower the energy. More precisely, it can be shown that there exist smooth boundary data $\varphi : \partial B^3 \longrightarrow S^2$ with $\deg(\varphi) = 0$ such that

$$\min_{u\in W^{1,2}_\varphi(B^3,S^2)} \int_{B^3} |du|^2\, d^3x < \inf_{v\in C^\infty_\varphi(\bar{B}^3,S^2)} \int_{B^3} |dv|^2\, d^3x\,. \tag{2.25}$$

This is the so-called gap phenomenon first observed by R. Hardt and F.-H. Lin, [20] being in agreement with the density result in Theorem 2.3.

Exploiting the fact that $C^\infty_\varphi(\bar{B}^3, S^2)$ is weakly dense in $W^{1,2}_\varphi(B^3, S^2)$, we define the relaxed energy E_{rel} on $W^{1,2}_\varphi(B^3, S^2)$ by

$$\begin{aligned} E_{rel}(u) = \inf\Big\{ \liminf_{k\to\infty} \int_{B^3} |dv_k|^2\, d^3x \quad : \quad & (v_k)_{k\in\mathbb{N}} \text{ in } C^\infty_\varphi(\bar{B}^3, S^2) \text{ with} \\ & v_k \rightharpoonup u \quad \text{weakly in } W^{1,2} \Big\}\,. \end{aligned} \tag{2.26}$$

It is shown in F. Bethuel, H. Brezis and J.-M. Coron, [6] that for the relaxed energy in the case of $\deg(\varphi) = 0$ we have

$$E_{rel}(u) = E(u) + 8\pi L(u)\,, \tag{2.27}$$

where $L(u)$ is the length of a minimal connection associated to $u \in W^{1,2}_\varphi(B^3, S^2)$ given by the formula (2.23). From (2.27) we deduce the following properties for the relaxed energy:

a) We have that $\inf_{u\in W^{1,2}_\varphi(B^3,S^2)} E_{rel}(u) = \inf_{v\in C^\infty_\varphi(\bar{B}^3,S^2)} E(v)$. Note that due to the lower semicontinuity of E_{rel} on $W^{1,2}(B^3, S^2)$ the minimum on the left-hand side of the last equation is achieved.

b) For every $u \in W^{1,2}_\varphi(B^3, S^2)$ we obviously have $E_{rel}(u) \geq E(u)$. Equality holds if $u \in C^\infty_\varphi(\bar{B}^3, S^2)$, since then $L(u) = 0$.

We return to the existence of a smooth harmonic extension to a given smooth map $\varphi : \partial B^3 \longrightarrow S^2$ with $\deg(\varphi) = 0$. One strategy for solving this problem might be to minimize E_{rel} over $W^{1,2}_\varphi(B^3, S^2)$. Then the energy gap in (2.25) coming from introducing singularities is canceled by adding an extra cost in energy given by the length of a minimal connection. However, this is only a heuristic argument, since at the moment there are no regularity results for minimizers of the relaxed energy known to us.

2.4 Weak Compactness and Coulomb Gauge

This section is a first encounter with the analytic and topological aspects of gauge fields. In particular, we discuss classical results for the calculus of variations associated with the Yang-Mills functional in four dimensions. The important question of weak compactness was answered by K. Uhlenbeck by choosing a good gauge – called Coulomb or Uhlenbeck gauge – which makes the Yang-Mills functional "coercive". Note that the method of passing to the Coulomb gauge is very common for gauge theories and will also play a crucial role in this thesis.

The fundamental compactness result by K. Uhlenbeck [46], Lemma 3.5 and Theorem 3.6 is a key point for understanding the analytic and topological aspects of gauge fields. An entire book was written on this subject by K. Wehrheim, [49].

Theorem 2.7 (Weak Compactness, [46]). *Let $P \in \mathcal{P}^G_\infty(M)$ with M an n-dimensional compact Riemannian manifold and $(A^k)_{k\in\mathbb{N}}$ a sequence of smooth connections in $\mathcal{A}_\infty(P)$. Moreover, we assume that $\|F(A^k)\|_{L^p}$ is uniformly bounded for $p > n/2$. Then, we can choose an open covering $\{U_i\}_{i\in I}$ of M and trivializations of P – or, in other words, local smooth gauge transformations $\sigma^k_i : U_i \longrightarrow G$ – such that for the transition functions $g^k_{ij} : U_i \cap U_j \longrightarrow G$ and the gauge transformed connections $\sigma^k_i(A^k) = (\sigma^k_i)^{-1}A^k_i\sigma^k_i + (\sigma^k_i)^{-1}d\sigma^k_i$ the following holds:*

(*i*) *There exists a family $g^\infty = \{g^\infty_{i,j}\}_{i,j\in I}$ of functions in $W^{2,p}(U_i \cap U_j, G)$ satisfying the cocycle condition such that for a subsequence*

$$g^k_{ij} \rightharpoonup g^\infty_{ij} \qquad \textit{weakly in } W^{2,p}(U_i \cap U_j, G)\,. \tag{2.28}$$

(*ii*) *There exists $A^\infty = \{A^\infty_i\}_{i\in I}$ a $W^{1,p}$-Sobolev connection on the $W^{2,p}$-Sobolev bundle P_{U,g^∞} such that for a subsequence*

$$\sigma^k_i(A^k_i) \rightharpoonup A^\infty_i \qquad \textit{weakly in } W^{1,p}(U_i, T^*U_i \otimes \mathfrak{g})\,. \tag{2.29}$$

For the proof one chooses a finite open covering $\{U_i\}_{i\in I}$ in such a way that on every U_i the L^p-norm of the curvatures $F(A^k)$ is small independently of k. This is possible because of the uniform bound on the curvatures and since $p > n/2$ by assumption. Then one locally passes to a particular gauge called *Coulomb gauge* or also *Uhlenbeck gauge* following K. Uhlenbeck [46], Theorem 2.1. For a motivation of such "gauge fixing" we refer to S.K. Donaldson and P.B. Kronheimer [11], Chapter 2.3.

Theorem 2.8 (Coulomb Gauge, [46]). *Let $P \in \mathcal{P}^G_\infty(M)$ and a connection $A \in \mathcal{A}_{W^{1,p}}(P)$ for $p \geq n/2$. Then, there exist constants $0 < \delta_{Uh} \ll 1$ and C_{Uh} such that if the $L^{n/2}$-norm of the curvature F in a trivializing open set $U \subset M$ of P is bounded by δ_{Uh}, i.e., $\|F\|_{L^{n/2}(U)} \leq \delta_{Uh}$, we can find a local gauge transformation $\sigma_{Coulomb} \in W^{2,p}(U, G)$ making the connection A locally gauge equivalent to the so-called Coulomb gauge $A_{Coulomb} = \sigma_{Coulomb}(A)$ which satisfies*

$$d_* A_{Coulomb} = 0\,, \tag{2.30}$$

and

$$\|A_{Coulomb}\|_{W^{1,p}(U)} \leq C_{Uh}\,\|F\|_{L^p(U)}\,. \tag{2.31}$$

Remark 2.1. The Coulomb gauge estimate (2.31) has to be compared with (2.15).

In the Coulomb gauge one obtains a $W^{1,p}$-bound on the connections from the bound on the curvatures. Using weak compactness, one hence gets a limiting connection and

(2.29). Exploiting (2.2), one deduces a $W^{2,p}$-bound on the transition functions from the bound on the connections. The weak convergence (2.28) is again a consequence of weak compactness. – These are the main steps in the proof of Theorem 2.7.

So far we can only characterize the limiting objects from the compactness result as $W^{1,p}$-Sobolev connection on some $W^{2,p}$-Sobolev bundle. The compact Sobolev embedding $W^{2,p} \hookrightarrow C^0$ which holds due to the assumption $p > n/2$ implies that the weak convergence in (2.28) can be replaced by strong convergence in the C^0-norm. From Proposition 3.2 in [46], we then deduce the existence of continuous maps $\rho_i^k : V_i \longrightarrow G$ such that

$$g_{ij}^k = \rho_i^k \, g_{ij}^\infty \, (\rho_j^k)^{-1} \qquad \text{on } U_i \cap U_j \,,$$

for k sufficiently large, where $\{V_i\}_{i\in I}$ denotes a refinement of the covering $\{U_i\}_{i\in I}$ of M. It follows from (2.7) that the initial bundle P and the limiting bundle P_{U,g^∞} – which in the following we will simply denote by P^∞ – are topologically equivalent.

If in dimension $n = 3$ we consider in dimension $n = 3$ a sequence of $W^{1,2}$-Sobolev connections with uniformly bounded Yang-Mills L^2-energy then Theorem 2.7 applies. In dimension $n = 4$ the Yang-Mills energy is given by the critical exponent $p = n/2$ and hence weak compactness fails. However, as shown in S. Sedlacek [42], Theorem 3.1, there exists a singular set S consisting of a finite number of points $a_1, \ldots, a_N$ such that weak compactness still holds on the smaller set $M \setminus S$. This follows from the fact that one can find an open covering $\{U_i\}_{i\in I}$ on which one can locally pass into Coulomb gauge only for $M \setminus S$.

Another result on the limiting behavior of sequences of connections (see T. Rivière [36], Theorem IV.1) says that under strong convergence of connections in the $W^{1,n/2}$-norm and the additional assumption of connections locally being in Coulomb gauge, the topology of the underlying principal G-bundle remains unchanged.

Existence of Minimizers

With the help of the previous weak compactness results, we now want to solve the problem of existence of minimizers arising in the calculus of variations for the Yang-Mills functional. Given a smooth principal G-bundle P over M, we ask if the following infimum is attained:

$$m(P) = \inf_{\tilde{A} \in \mathcal{A}_\infty(P)} \int_M |F(\tilde{A})|^2 \, dvol_g \,. \tag{2.32}$$

First, we treat the case $n = 3$. Applying the weak compactness Theorem 2.7 to a minimizing sequence $(\tilde{A}^k)_{k\in\mathbb{N}}$ in $\mathcal{A}_\infty(P)$ with $YM(\tilde{A}^k) \longrightarrow m(P)$ for $k \to \infty$, we deduce the existence of a limiting $W^{2,2}$-Sobolev principal G-bundle P^∞ with preserved topology and a connection $A^\infty \in \mathcal{A}_{W^{1,2}}(P^\infty)$. From the weak convergence (2.29), we obtain that $F\big(\sigma^k(\tilde{A}^k)\big) \rightharpoonup F(A^\infty)$ weakly in L^2, where a standard estimate similar to (2.14) is used. Note that the global gauge transformations (σ^k) are obtained by gluing the local gauge transformations in Theorem 2.7 together in a suitable way (see K. Uhlenbeck [46], Theorem 3.6). Hence the weak lower semicontinuity of the L^2-norm and gauge invariance of the Yang-Mills energy imply that

$$\begin{aligned} YM(A^\infty) = \int_M |F(A^\infty)|^2 \, dvol_g \;&\leq\; \liminf_{k\to\infty} \int_M \big|F\big(\sigma^k(\tilde{A}^k)\big)\big|^2 \, dvol_g \\ &=\; \liminf_{k\to\infty} \int_M |F(\tilde{A}^k)|^2 \, dvol_g = m(P) \,. \end{aligned}$$

Moreover, the $W^{1,2}$-Sobolev connection A^∞ is a weak solution of the Yang-Mills equation

$$D_A^* F(A) = 0 \qquad \text{in } \mathcal{D}' ,$$

meaning that $\langle F(A), D_A\phi\rangle = 0$ for every $\phi \in \Omega^1_\infty(M, \mathrm{Ad(P)})$. The smoothness of the weak Yang-Mills connection A^∞ can then be deduced, since the Coulomb gauge condition (2.30) is also preserved under weak limits. This elliptic regularity result can be found in K. Uhlenbeck [46], Corollary 1.4. Thus A^∞ satisfying $YM(A^\infty) = m(P)$ is a solution of problem (2.32) in three dimensions.

We now focus on the four-dimensional case investigated in detail by S. Sedlacek, [42]. The minimizing sequence of connections for the Yang-Mills functional converges weakly – as seen before – on M except for the singular set S containing a finite number of points. The limiting $W^{1,2}$-Sobolev connection A^∞ on some $W^{2,2}$-Sobolev principal G-bundle P^∞ over $M \setminus S$ is shown to be a weak Yang-Mills connection on $M \setminus S$. Applying the previous elliptic regularity result, which still holds in the critical case $p = n/2$, we obtain that A^∞ is smooth. The theorem on the removability of singularities in Yang-Mills connections (see K. Uhlenbeck [47], Theorem 4.1) then implies that A^∞ and P^∞ extend to a smooth Yang-Mills connection $\bar{A}$ on an extended smooth principal G-bundle $\bar{P}$ over M. It is important to notice that the initial smooth principal G-bundle P and $\bar{P}$ are in general not topologically equivalent or in other words isomorphic to each other. This is similar to the two-dimensional case for harmonic maps where the homotopy class is not preserved under weak convergence (see Section 2.3). For a more detailed discussion including the "bubbling phenomenon" the reader is refered to B. Lawson [29], Section V. However, as shown in S. Sedlacek [42], Theorem 5.5, a certain topological invariant is preserved under weak convergence. In the case of $G = U(m)$ this invariant coincides with the first Chern class of P. Thus, we obtain the existence of a minimizer for (2.32) in the case $n = 4$ if we allow the initial principal G-bundle P to vary within a class of principal G-bundles determined by this given fixed topological invariant. – Previous ideas led A. Marini, [30] to a solution of the Dirichlet and Neumann boundary value problem for Yang-Mills connections using again the direct method in the calculus of variations.

In order to avoid the difficulty of a possibly new limiting principal G-bundle in four dimensions, in the special case of $G = SU(2)$ it is possible to minimize the Yang-Mills functional in a class of connections on generalized principal G-bundles. These are roughly speaking obtained as weak limit of their second Chern class (see T. Isobe, [25]). Recall that in four dimensions principal $SU(2)$-bundles are isomorphic if and only if their second Chern number coincides.

Chapter 3

Analysis of $W^{1,2}$-connections in Four Dimensions

3.1 Density Results in Critical Dimensions

Let M be a n-dimensional compact Riemannian manifold and G a compact Lie group with Lie algebra $\mathfrak{g}$. A smooth principal G-bundle P over M is then defined in terms of a bundle atlas as described in Definition B.1. The bundle atlas then leads to smooth transition functions $g_{ij} : U_i \cap U_j \longrightarrow G$ where $U = \{U_i\}_{i \in I}$ denotes an open covering of M (see (B.2)). Transition functions satisfy the cocycle condition

$$g_{ij} g_{jk} = g_{ik} \,, \qquad \text{on } U_i \cap U_j \cap U_k \,. \tag{3.1}$$

From Proposition B.1 we know that it is conversely sufficient to construct a smooth principal G-bundle in $\mathcal{P}^G_\infty(M)$ if transition functions $g = \{g_{ij}\}_{i,j \in I}$ and an open covering U are given. This characterization of smooth principal bundles by their transition functions turns out to be suitable in order define weak Sobolev bundles. We just require a family of maps $g = \{g_{ij}\}_{i,j \in I}$ with $g_{ij} : U_i \cap U_j \longrightarrow G$ to be in some Sobolev space and to satisfy the cocycle condition (3.1). From these generalized transition functions we then get weak Sobolev principal G-bundles $\mathcal{P}^G_{W^{2,p}}(M)$ of Definition 2.1.

The aim is now to find some smooth approximation of $P_{U,g} \in \mathcal{P}^G_{W^{2,p}}(M)$ by smooth principal G-bundles in $\mathcal{P}^G_\infty(M)$. The difficulty comes from the fact we can not only take the mollification of the transition functions g of $P_{U,g}$, but also have to ensure that the mollification of g satisfies the cocycle condition (3.1). Only in this case the mollification defines smooth approximating principal bundles. The non-linear constraint (3.1) is easy to handle under the hypothesis $p > n/2$, since $W^{2,p}$-Sobolev maps embedds continuously in the space of continous maps (see for example R. Schoen and K. Uhlenbeck, [39]). Though the smooth approximability becomes much more involved, it still holds in the critical case $p = n/2$.

Proposition 3.1. *Let $P_{U,g} \in \mathcal{P}^G_{W^{2,p}}(M)$ with $p \geq n/2$. Then, there exists a sequence $(\tilde{g}^k_{ij})_{k \in \mathbb{N}}$ of smooth transition functions defining smooth principal G-bundles $P_{U,\tilde{g}^k}$, for every $k \in \mathbb{N}$, such that*

$$\lim_{k \to \infty} \|\tilde{g}^k_{ij} - g_{ij}\|_{W^{2,p}(U_i \cap U_j)} = 0 \,, \tag{3.2}$$

for every $i, j \in I$.

We now sketch how to deal with the non-linear constraint in the critical case $p = n/2$. – Let $\varphi_i \in W^{2,p}(U_i, G)$ be such that $g_{ij} = \varphi_i \varphi_j^{-1}$ on $U_i \cap U_j$ (see (B.2)). Then we define

$$\tilde{\varphi}_i^k = \eta^k \star \varphi_i \in C^\infty(U_i, G)\,, \tag{3.3}$$

where $(\eta^k)_{k\in\mathbb{N}}$ denotes a mollifying sequence. We already know that (see R. Schoen and K. Uhlenbeck, [39])

$$\lim_{k\to\infty} \|\tilde{\varphi}_i^k - \varphi_i\|_{W^{2,p}(U_i)} = 0\,. \tag{3.4}$$

Now, let $(\tilde{g}_{ij}^k)_{k\in\mathbb{N}}$ be the sequence of smooth maps defined by $\tilde{g}_{ij}^k = \tilde{\varphi}_i^k(\tilde{\varphi}_j^k)^{-1}$ on $U_i \cap U_j$. Using Lemma 2.1 and (3.4), we observe that

$$\begin{aligned}
\|\tilde{g}_{ij}^k - g_{ij}\|_{W^{2,p}(U_i\cap U_j)} &= \|\tilde{\varphi}_i^k(\tilde{\varphi}_j^k)^{-1} - \varphi_i\varphi_j^{-1}\|_{W^{2,p}(U_i\cap U_j)} \\
&\le \big\|\big(\tilde{\varphi}_i^k - \varphi_i\big)(\tilde{\varphi}_j^k)^{-1}\big\|_{W^{2,p}(U_i\cap U_j)} \\
&\quad + \big\|\varphi_i\big((\tilde{\varphi}_j^k)^{-1} - \varphi_j^{-1}\big)\big\|_{W^{2,p}(U_i\cap U_j)} \\
&\le C\,\|\tilde{\varphi}_i^k - \varphi_i\|_{W^{2,p}(U_i\cap U_j)} + \|(\tilde{\varphi}_j^k)^{-1} - \varphi_j^{-1}\|_{W^{2,p}(U_i\cap U_j)}\,,
\end{aligned} \tag{3.5}$$

and hence the convergence in the limit $k \to \infty$. It remains to show that $\tilde{g}^k = \{\tilde{g}_{ij}^k\}_{i,j\in I}$ for every $k \in \mathbb{N}$ satisfy the cocylce condition (3.1). A straightforward computation yields

$$\tilde{g}_{ij}^k \tilde{g}_{jl}^k = \big(\tilde{\varphi}_i^k(\tilde{\varphi}_j^k)^{-1}\big)\big(\tilde{\varphi}_j^k(\tilde{\varphi}_l^k)^{-1}\big) = \tilde{\varphi}_i^k(\tilde{\varphi}_l^k)^{-1} = \tilde{g}_{il}^k\,. \tag{3.6}$$

Combining (3.5) with (3.6) we end up with an approximating sequence of smooth transition functions.

In a next step, we investigate what happens with the topology of the bundles by an approximating procedure. Consider $P_{U,g} \in \mathcal{P}^G_{W^{2,p}}(M)$ with a smooth approximation $P_{U,\tilde{g}^k}$ such that

$$\lim_{k\to\infty} \|\tilde{g}_{ij}^k - g_{ij}\|_{W^{2,p}(U_i\cap U_j)} = 0\,, \tag{3.7}$$

for every $i, j \in I$. In the case of $p > n/2$, we deduce from (3.7) that the transition functions g and $\tilde{g}^k$ are also C^0-close. Thus, thanks to K. Uhlenbeck [46], Proposition 3.2, there exists continuous maps $\rho_i^k : V_i \longrightarrow G$ such that

$$\tilde{g}_{ij}^k = \rho_i^k\, g_{ij}\, (\rho_j^k)^{-1} \qquad \text{on } U_i \cap U_j\,, \tag{3.8}$$

for k sufficiently large, where $\{V_i\}_{i\in I}$ denotes a refinement of the covering $\{U_i\}_{i\in I}$ of M. Hence, using the characterization (2.7) for equivalent bundles, the approximating procedure leave the topology unchanged for $p > n/2$.

The Definition 2.2 of $W^{1,p}$-Sobolev connections on $P_{U,g} \in \mathcal{P}^G_{W^{2,p}}(M)$ is based on a geometrical result given in Proposition B.4. For $p \ge n/2$ recall that $A = \{A_i\}_{i\in I}$ belongs to $\mathcal{A}_{W^{1,p}}(P_{U,g})$ if each A_i is a $W^{1,p}$-Sobolev $\mathfrak{g}$-valued one-form on U_i and if the compatibility condition

$$A_j = g_{ij}^{-1}\, A_i\, g_{ij} + g_{ij}^{-1} dg_{ij} \qquad \text{on } U_i \cap U_j\,. \tag{3.9}$$

holds.

For the existence of a smooth approximation for $A \in \mathcal{A}_{W^{1,p}}(P_{U,g})$ in the case $p \ge n/2$, we can assume the smoothness of the underlying bundle $P_{U,g}$. This is a consequence of Proposition 3.1. The non-linear constraint (3.9) to be satisfied by the smooth approximation can then be handled by a simple formula (see (3.13) below). We give the following proof for the smooth approximability of connections in Proposition 1.6:

Proof. Let $A = \{A_i\}_{i\in I} \in \mathcal{A}_{W^{1,p}}(P_{U,g})$ be a connection on a smooth principal G-bundle $P_{U,g}$ over M. We define

$$A_i^k = \eta^k \star A_i \in C^\infty(U_i, T^*U_i \otimes \mathfrak{g})\,, \tag{3.10}$$

for every $i \in I$, where $(\eta^k)_{k\in\mathbb{N}}$ is a mollifying sequence. The following convergence is a standard result:

$$\lim_{k\to\infty} \|A_i^k - A_i\|_{W^{1,p}(U_i)} = 0\,. \tag{3.11}$$

Next, we denote by $\{\theta_i\}_{i\in I}$ the partition of unity subordinate to the open covering $U = \{U_i\}_{i\in I}$. It is well-known that by definition

$$\sum_{i\in I} \theta_i \equiv 1 \qquad \text{on } M\,. \tag{3.12}$$

For every $i \in I$ and $k \in \mathbb{N}$, we then define a smooth $\mathfrak{g}$-valued one-form $\tilde{A}_i^k$ on U_i by

$$\tilde{A}_i^k = \theta_i A_i^k + \sum_{j\neq i\in I} \theta_j \big(g_{ji}^{-1} A_j^k g_{ji} + g_{ji}^{-1} dg_{ji}\big)\,, \tag{3.13}$$

or, with $g_{ii} \equiv e$ equivalently,

$$\tilde{A}_i^k = \sum_{j\in I} \theta_j \big(g_{ji}^{-1} A_j^k g_{ji} + g_{ji}^{-1} dg_{ji}\big) \qquad \text{on } U_i\,. \tag{3.14}$$

Setting $\theta_j \equiv 0$ on $U_i \setminus U_i \cap U_j$, for every $j \in I$, we get that $\tilde{A}_i^k$ is well-defined on U_i. Note also that the smoothness of $\tilde{A}_i^k$ follows directly from (3.10) and the smoothness of the transition functions $g_{ij} \in C^\infty(U_i \cap U_j, G)$ for $P_{U,g}$.

In order for $\tilde{A}^k = \{\tilde{A}_i^k\}_{i\in I}$ to be an element in $\mathcal{A}_\infty(P_{U,g})$, we have to show that the compatibility condition (3.9) holds. – For this purpose, we compute on $U_l \cap U_i$

$$\begin{aligned} g_{il}^{-1} \tilde{A}_i^k g_{il} + g_{il}^{-1} dg_{il} &= g_{li} \tilde{A}_i^k g_{il} + g_{li} dg_{il} \\ &\overset{(3.14)}{=} g_{li} \left(\sum_{j\in I} \theta_j \big(g_{ij} A_j^k g_{ji} + g_{ij} dg_{ji}\big) \right) g_{il} + g_{li} dg_{il} \\ &= \sum_{j\in I} \theta_j \big(g_{li}\ g_{ij} A_j^k g_{ji}\ g_{il} + g_{li}\ g_{ij} dg_{ji}\ g_{il}\big) + g_{li} dg_{il} \\ &= \sum_{j\in I} \theta_j \big(g_{lj} A_j^k g_{jl}\big) + \sum_{j\in I} \theta_j \big(g_{lj} dg_{ji}\ g_{il}\big) + g_{li} dg_{il}\,, \end{aligned} \tag{3.15}$$

where the cocylce condition (see (3.1))

$$g_{li} g_{ij} = g_{lj}\,,$$

with $j \in I$ such that $U_l \cap U_i \cap U_j \neq \emptyset$, is used several times. The inverse of the last equation has the following differential:

$$dg_{ji} g_{il} + g_{ji} dg_{il} = dg_{jl}\,.$$

Multiplication by g_{lj} then leads to

$$g_{lj}dg_{ji}g_{il} = g_{lj}dg_{jl} - g_{li}dg_{il}\,,$$

where the cocycle condition (3.1) is used again. After inserting this into the second term on the right-hand side of (3.15), we arrive at

$$\begin{aligned} g_{il}^{-1}\tilde{A}_i^k g_{il} + g_{il}^{-1}dg_{il} &= \sum_{j\in I}\theta_j\big(g_{lj}A_j^k g_{jl}\big) \\ &\quad + \sum_{j\in I}\theta_j\big(g_{lj}dg_{jl} - g_{li}dg_{il}\big) + g_{li}dg_{il} \\ &= \sum_{j\in I}\theta_j\big(g_{lj}A_j^k g_{jl}\big) \\ &\quad + \sum_{j\in I}\theta_j\big(g_{lj}dg_{jl}\big) - \left(\sum_{j\in I}\theta_j\right) g_{li}dg_{il} + g_{li}dg_{il} \\ &= \sum_{j\in I}\theta_j\big(g_{lj}A_j^k g_{jl}\big) + \sum_{j\in I}\theta_j\big(g_{lj}dg_{jl}\big)\,, \end{aligned}$$

the last equality being a consequence of (3.12). Together with the defining equation (3.14), we thus end up with

$$\tilde{A}_l^k = g_{il}^{-1}\tilde{A}_i^k g_{il} + g_{il}^{-1}dg_{il} \qquad \text{on } U_l\cap U_i\,, \tag{3.16}$$

which is exactly the compatibility condition (3.9).

In a next step, we establish the convergence

$$\lim_{k\to\infty}\|\tilde{A}_i^k - A_i\|_{W^{1,p}(U_i)} = 0\,, \tag{3.17}$$

for the sequence $(\tilde{A}^k)_{k\in\mathbb{N}}$ defined by (3.13). – For this purpose, we first observe that

$$\lim_{k\to\infty}\|g_{ji}^{-1}A_j^k g_{ji} + g_{ji}^{-1}dg_{ji} - A_i\|_{W^{1,p}(U_i\cap U_j)} = 0\,. \tag{3.18}$$

In order to see this, we compute

$$\begin{aligned} &\|g_{ji}^{-1}A_j^k g_{ji} + g_{ji}^{-1}dg_{ji} - A_i\|_{W^{1,p}(U_i\cap U_j)} \\ =\ &\|g_{ji}^{-1}A_j^k g_{ji} + g_{ji}^{-1}dg_{ji} - (g_{ji}^{-1}A_j g_{ji} + g_{ji}^{-1}dg_{ji})\|_{W^{1,p}(U_i\cap U_j)} \\ =\ &\|g_{ji}^{-1}A_j^k g_{ji} - g_{ji}^{-1}A_j g_{ji}\|_{W^{1,p}(U_i\cap U_j)} \\ =\ &\|g_{ji}^{-1}(A_j^k - A_j)g_{ji}\|_{W^{1,p}(U_i\cap U_j)} = \|A_j^k - A_j\|_{W^{1,p}(U_i\cap U_j)}\,, \end{aligned}$$

where we used the compatibility condition (3.9) for A and that the scalar product on the Lie algebra $\mathfrak{g}$ is invariant by the adjoint action of the Lie group G. The convergence (3.11) then yields (3.18).

Now, we can estimate

$$\begin{aligned}\|\tilde{A}_i^k - A_i\|_{W^{1,p}(U_i)} &= \Big\|\tilde{A}_i^k - \sum_{j\in I}\theta_j A_i\Big\|_{W^{1,p}(U_i)} \\ &\le \|\theta_i(A_i^k - A_i)\|_{W^{1,p}(U_i)} \\ &\quad + \sum_{j\neq i\in I}\big\|\theta_j\big(g_{ji}^{-1}A_j^k g_{ji} + g_{ji}^{-1}dg_{ji} - A_i\big)\big\|_{W^{1,p}(U_i\cap U_j)} \\ &\le \|A_i^k - A_i\|_{W^{1,p}(U_i)} \\ &\quad + \sum_{j\neq i\in I}\|g_{ji}^{-1}A_j^k g_{ji} + g_{ji}^{-1}dg_{ji} - A_i\|_{W^{1,p}(U_i\cap U_j)}\,.\end{aligned}$$

Together with (3.11) and (3.18) the desired convergence (3.17) then follows. □

We are interested in transfering the previous smooth approximation of connections to the corresponding weak curvatures, since they will be the main concern of the next chapters. So we assume $(\tilde{A}^k)_{k\in\mathbb{N}}$ to be a smooth approximation for $A \in \mathcal{A}_{W^{1,p}}(M)$, i.e.,

$$\lim_{k\to\infty}\|\tilde{A}_i^k - A_i\|_{W^{1,p}(U_i)} = 0\,, \tag{3.19}$$

for every $i \in I$. Then, for the corresponding L^p-curvatures we proof Corollary 1.7 saying that

$$\lim_{k\to\infty}\|F_i(\tilde{A}^k) - F_i(A)\|_{L^p(U_i)} = 0\,, \tag{3.20}$$

for every $i \in I$.

Proof. Recalling Definition 2.3, we have that

$$\begin{aligned}\|F_i(\tilde{A}^k) - F_i(A)\|_{L^p(U_i)} &= \big\|(d\tilde{A}_i^k + \tilde{A}_i^k\wedge\tilde{A}_i^k) - (dA_i + A_i\wedge A_i)\big\|_{L^p(U_i)} \\ &\le \big\|d\tilde{A}_i^k - dA_i\big\|_{L^p(U_i)} + \big\|(\tilde{A}_i^k\wedge\tilde{A}_i^k) - (A_i\wedge A_i)\big\|_{L^p(U_i)}\,,\end{aligned} \tag{3.21}$$

for every $i \in I$. As a direct consequence of the assumption (3.19), we observe that

$$\lim_{\varepsilon\to 0}\|d\tilde{A}_i^k - dA_i\|_{L^p(U_i)} = 0\,, \tag{3.22}$$

for every $i \in I$. For the second term on the right-hand side of (3.21), the continuity of the multiplication $L^{2p}\otimes L^{2p} \longrightarrow L^p$ together with the Sobolev embedding $W^{1,p} \hookrightarrow L^{2p}$ – which holds due to the assumption $p \ge n/2$ – imply that

$$\lim_{k\to\infty}\big\|(\tilde{A}_i^k\wedge\tilde{A}_i^k) - (A_i\wedge A_i)\big\|_{L^p(U_i)} = 0\,, \tag{3.23}$$

for every $i \in I$. More precisely, this can be deduced from (3.19) and the following straightforward estimate (compare with (2.15)):

$$\begin{aligned}\big\|(\tilde{A}_i^k\wedge\tilde{A}_i^k) - (A_i\wedge A_i)\big\|_{L^p(U_i)} &\le \big\|(\tilde{A}_i^k - A_i)\wedge A_i\big\|_{L^p(U_i)} + \big\|\tilde{A}_i^k\wedge(\tilde{A}_i^k - A_i)\big\|_{L^p(U_i)} \\ &\le C\,\|\tilde{A}_i^k - A_i\|_{L^{2p}(U_i)}\|A_i\|_{L^{2p}(U_i)} \\ &\quad + C\,\|\tilde{A}_i^k\|_{L^{2p}(U_i)}\|\tilde{A}_i^k - A_i\|_{L^{2p}(U_i)} \\ &\le C\,\|\tilde{A}_i^k - A_i\|_{W^{1,p}(U_i)}\|A_i\|_{W^{1,p}(U_i)} \\ &\quad + C\,\|\tilde{A}_i^k\|_{W^{1,p}(U_i)}\|\tilde{A}_i^k - A_i\|_{W^{1,p}(U_i)}\,.\end{aligned}$$

From (3.22) and (3.23) the convergence (3.20) follows easily. □

3.2 Uniqueness and Regularity of the Coulomb Gauge

In this section, we will investigate in more details the uniqueness and the regularity of the Coulomb gauge introduced in Theorem 2.8. For this purpose, since the Coulomb gauge is a local result, we can consider the particular case of some trivial principal G-bundle. For later use, we restrict moreover our investigations to the four dimensional case $n = 4$ of trivial principal G-bundles over the unit ball B^4. Note that principal G-bundles over B^4 are trivial, because B^4 is contractible (see D. Husemoller [24], Corollary 8.3 and 10.3). In the following, we will denote the space of connections on some trivial bundle over B^4 simply by $\mathcal{A}(B^4)$. Recall that they have a global representation as $\mathfrak{g}$-valued one-forms on B^4.

Before continueing with the Coulomb gauge, we want to specify the definition for Sobolev maps into G. We have already defined in Section 2.2 the Sobolev space $W^{2,p}(B^4, G)$ for a compact Lie group G. Regarding G as subset of $\mathbb{R}^m$ we recall that σ belongs to $L^p(B^4, G)$ if and only if $|\sigma| \in L^p(B^4, \mathbb{R}^m)$ with the standard norm $|\cdot|$ on $\mathbb{R}^m$. Moreover, note that in order to compute the L^p-norm of $d\sigma$ we can choose a Riemannian metric on G which is G-invariant. In particular, we hence have that

$$\|\sigma^{-1} d\sigma\|_{L^p(B^4)} = \|d\sigma\|_{L^p(B^4)} . \tag{3.24}$$

This equality will be used later several times.

In the above described setting, we reformulate Theorem 2.8 in the following way:

Theorem 3.2 (Coulomb Gauge). *Let P be a trivial principal G-bundle over B^4 and A a $W^{1,2}$-Sobolev connection on P with L^2-curvature $F = dA + A \wedge A$. Then, there exist constants $0 < \delta_{Uh} \ll 1$ and C_{Uh} such that if the L^2-norm of the curvature F is bounded by δ_{Uh}, i.e.,*

$$\|F\|_{L^2(B^4)} = \left(\int_{B^4} |F(x)|^2 \, d^4x \right)^{1/2} \leq \delta_{Uh} ,$$

we can find a gauge transformation $\sigma_{Coulomb} \in W^{2,2}(B^4, G)$ making A gauge equivalent to the so-called Coulomb gauge $A_{Coulomb} = \sigma_{Coulomb}(A)$ which satisfies

$$d_* A_{Coulomb} = 0 , \tag{3.25}$$

and

$$\|A_{Coulomb}\|_{W^{1,2}(B^4)} \leq C_{Uh} \, \|F\|_{L^2(B^4)} . \tag{3.26}$$

In the small L^2-energy regime for the curvature the Coulomb gauge turns out to be unique. This is the content of the next proposition which for simplicity we formulate for some trivial principal G-bundle over S^4.

Proposition 3.3. *Let $A \in \mathcal{A}_{W^{1,2}}(S^4)$ with $\|F(A)\|_{L^2(S^4)} \leq \varepsilon < \delta_{Uh}$ and let $\sigma_{Coulomb} \in W^{2,2}(S^4, G)$ denote the gauge transformation such that $A_{Coulomb} = \sigma_{Coulomb}(A)$ satisfies*

$$d_* A_{Coulomb} = 0 , \qquad \|A_{Coulomb}\|_{W^{1,2}(S^4)} \leq C_{Uh} \, \|F(A)\|_{L^2(S^4)} . \tag{3.27}$$

Moreover, assume that there exists another gauge transformation $\sigma'_{Coulomb} \in W^{2,2}(S^4, G)$ such that $B_{Coulomb} = \sigma'_{Coulomb}(A)$ satisfies

$$d_* B_{Coulomb} = 0 , \qquad \|B_{Coulomb}\|_{W^{1,2}(S^4)} \leq C_{Uh} \, \|F(A)\|_{L^2(S^4)} . \tag{3.28}$$

Then, for sufficiently small $\varepsilon > 0$, the Coulomb gauge is unique up to a constant gauge transformation σ_0, i.e., we have that $\sigma_0^{-1} A_{Coulomb} \sigma_0 = B_{Coulomb}$.

Proof. We observe that

$$\begin{aligned} \sigma'_{Coulomb}\sigma^{-1}_{Coulomb}(A_{Coulomb}) &= \sigma'_{Coulomb}\sigma^{-1}_{Coulomb}\big(\sigma_{Coulomb}(A)\big) \\ &= \sigma'_{Coulomb}(A) = B_{Coulomb}\,. \end{aligned}$$

This implies that

$$\sigma^{-1} A_{Coulomb}\sigma + \sigma^{-1} d\sigma = B_{Coulomb}\,, \tag{3.29}$$

for the gauge transformation $\sigma := \sigma'_{Coulomb}\sigma^{-1}_{Coulomb}$. Thus, for the uniqueness of $A_{Coulomb}$ and $B_{Coulomb}$ up to a constant gauge transformation, we have to show that $d\sigma = 0$.

Using (3.29) together with the G-invariant metric (3.24) on G, the L^4-norm of $d\sigma$ can be bounded by

$$\begin{aligned} \|d\sigma\|_{L^4(S^4)} &= \|\sigma^{-1} d\sigma\|_{L^4(S^4)} \\ &\leq \|B_{Coulomb}\|_{L^4(S^4)} + \|\sigma^{-1} A_{Coulomb}\sigma\|_{L^4(S^4)} \\ &= \|B_{Coulomb}\|_{L^4(S^4)} + \|A_{Coulomb}\|_{L^4(S^4)}\,. \end{aligned}$$

The continuous Sobolev embedding $W^{1,2} \hookrightarrow L^4$ in four dimensions implies

$$\|d\sigma\|_{L^4(S^4)} \leq C\,\|B_{Coulomb}\|_{W^{1,2}(S^4)} + C\,\|A_{Coulomb}\|_{W^{1,2}(S^4)}\,.$$

From the assumptions (3.27) and (3.28), we then obtain

$$\|d\sigma\|_{L^4(S^4)} \leq 2CC_{Uh}\,\|F(A)\|_{L^2(S^4)} \leq C\varepsilon\,. \tag{3.30}$$

In a next step, we compute

$$\begin{aligned} d_*(\sigma^{-1}d\sigma) &\overset{(3.29)}{=} d_*(B_{Coulomb} - \sigma^{-1} A_{Coulomb}\sigma) \\ &\overset{(3.28)}{=} -d_*(\sigma^{-1} A_{Coulomb}\sigma)\,. \end{aligned} \tag{3.31}$$

Using (3.27), we have

$$\begin{aligned} d_*(\sigma^{-1} A_{Coulomb}\sigma) &= d\sigma^{-1}\cdot(A_{Coulomb}\sigma) + \sigma^{-1} d_*(A_{Coulomb}\sigma) \\ &= d\sigma^{-1}\cdot(A_{Coulomb}\sigma) + \sigma^{-1}\big(d\sigma\cdot A_{Coulomb} + d_* A_{Coulomb}\sigma\big) \\ &= d\sigma^{-1}\cdot(A_{Coulomb}\sigma) + \sigma^{-1} d\sigma\cdot A_{Coulomb} \\ &= \sigma d\sigma^{-1}\cdot A_{Coulomb} + \sigma^{-1} d\sigma\cdot A_{Coulomb}\,. \end{aligned} \tag{3.32}$$

In this computation we make no difference between the one-form $A_{Coulomb}$ and its associated vector field so that, for example, we meam by $d\sigma\cdot A_{Coulomb}$ the differential of σ applied to the vector field $A_{Coulomb}$. Inserting (3.32) into (3.31), we then arrive at

$$d_*(\sigma^{-1} d\sigma) = -\sigma d\sigma^{-1}\cdot A_{Coulomb} - \sigma^{-1} d\sigma\cdot A_{Coulomb}\,. \tag{3.33}$$

Taking the L^2-norm in (3.33), it follows from (3.24) and Hölder's inequality

$$\begin{aligned}
\|d_*(\sigma^{-1}d\sigma)\|_{L^2(S^4)} &\leq \|\sigma d\sigma^{-1}\cdot A_{Coulomb}\|_{L^2(S^4)} \\
&\quad +\|\sigma^{-1}d\sigma\cdot A_{Coulomb}\|_{L^2(S^4)} \\
&\leq \|\sigma d\sigma^{-1}\|_{L^4(S^4)}\|A_{Coulomb}\|_{L^4(S^4)} \\
&\quad +\|\sigma^{-1}d\sigma\|_{L^4(S^4)}\|A_{Coulomb}\|_{L^4(S^4)} \\
&= \|d\sigma^{-1}\|_{L^4(S^4)}\|A_{Coulomb}\|_{L^4(S^4)} \\
&\quad +\|d\sigma\|_{L^4(S^4)}\|A_{Coulomb}\|_{L^4(S^4)}\,. \qquad (3.34)
\end{aligned}$$

Since the inversion satisfies $\|d\sigma^{-1}\|_{L^4(S^4)} \leq C\,\|d\sigma\|_{L^4(S^4)}$, we deduce from (3.34) that

$$\|d_*(\sigma^{-1}d\sigma)\|_{L^2(S^4)} \leq C\,\|d\sigma\|_{L^4(S^4)}\|A_{Coulomb}\|_{L^4(S^4)}\,. \qquad (3.35)$$

In a next step, we note that the first de Rham cohomology group $H^1(S^4,d)$ vanishes (see M. Nakahara, [33]). Using Hodge decomposition theory, we hence obtain the following Poincaré type inequality:

$$\|\sigma^{-1}d\sigma\|_{W^{1,2}(S^4)} \leq C\left(\|d_*(\sigma^{-1}d\sigma)\|_{L^2(S^4)} + \|d(\sigma^{-1}d\sigma)\|_{L^2(S^4)}\right)\,. \qquad (3.36)$$

For more details the reader is refered to G. Schwarz [38], especially Lemma 2.4.10. Inserting (3.35) and the estimate

$$\begin{aligned}
\|d(\sigma^{-1}d\sigma)\|_{L^2(S^4)} &= \|d\sigma^{-1}\wedge d\sigma\|_{L^2(S^4)} \\
&\leq \|d\sigma^{-1}\|_{L^4(S^4)}\|d\sigma\|_{L^4(S^4)} \leq C\,\|d\sigma\|^2_{L^4(S^4)}
\end{aligned}$$

into (3.36), we conclude

$$\|\sigma^{-1}d\sigma\|_{W^{1,2}(S^4)} \leq C\,\|d\sigma\|_{L^4(S^4)}\|A_{Coulomb}\|_{L^4(S^4)} + C\,\|d\sigma\|^2_{L^4(S^4)}\,.$$

Using again the continuous Sobolev embedding $W^{1,2}\hookrightarrow L^4$ in four dimensions, we then arrive at

$$\begin{aligned}
C^{-1}\|d\sigma\|_{L^4(S^4)} &\overset{(3.24)}{=} C^{-1}\|\sigma^{-1}d\sigma\|_{L^4(S^4)} \\
&\leq C\,\|d\sigma\|_{L^4(S^4)}\|A_{Coulomb}\|_{L^4(S^4)} + C\,\|d\sigma\|^2_{L^4(S^4)}\,. \qquad (3.37)
\end{aligned}$$

Because of $\|A_{Coulomb}\|_{L^4(S^4)} \leq C\|A_{Coulomb}\|_{W^{1,2}(S^4)} \leq C\varepsilon$ as a direct consequence of the assumption (3.27) and because of (3.30), we can rewrite (3.37) as

$$\|d\sigma\|_{L^4(S^4)} \leq C\varepsilon\,\|d\sigma\|_{L^4(S^4)}\,.$$

For sufficiently small $\varepsilon > 0$ this estimate only holds if $d\sigma = 0$. □

The next result turns out to be very useful for Section 5.3. It says that once we are dealing with a smooth connection, we can put this connection in Coulomb gauge via a smooth gauge transformation. Tough this regularity result was already known before (see K. Uhlenbeck, [46] and S. Sedlacek, [42]), we will prove it using a different strategy. More precisely, the continuity of the Coulomb gauge transformation will be obtained from Hodge decomposition and Lorentz space techniques, and we then conclude the smoothness from standard elliptic theory.

Proposition 3.4. *Let $A \in \mathcal{A}_{W^{1,2}}(B^4)$. Assume that there exists a gauge transformation $\tilde{\sigma} \in W^{2,2}(B^4, G)$ such that $\tilde{A} = \tilde{\sigma}(A)$ is smooth and another gauge transformation $\sigma_{Coulomb} \in W^{2,2}(B^4, G)$ such that $A_{Coulomb} = \sigma_{Coulomb}(A)$ given by*

$$A_{Coulomb} = \sigma_{Coulomb}^{-1} A \sigma_{Coulomb} + \sigma_{Coulomb}^{-1} d\sigma_{Coulomb}$$

is in Coulomb gauge. Then, we have that $A_{Coulomb}$ is also smooth.

Proof. Observe that

$$\begin{aligned} \sigma_{Coulomb}\tilde{\sigma}^{-1}(\tilde{A}) &= \sigma_{Coulomb}\tilde{\sigma}^{-1}\big(\tilde{\sigma}(A)\big) = \sigma_{Coulomb}\tilde{\sigma}^{-1}\tilde{\sigma}(A) \\ &= \sigma_{Coulomb}(A) = A_{Coulomb}\,. \end{aligned} \tag{3.38}$$

Since $\tilde{A}$ is by assumption smooth, the smoothness of the Coulomb gauge $A_{Coulomb}$ thus follows from the smoothness of the map $\sigma_{Coulomb}\tilde{\sigma}^{-1} : B^4 \longrightarrow G$. Hence, we have to show that $\sigma = \sigma_{Coulomb}\tilde{\sigma}^{-1} \in C^\infty(B^4, G)$. – Note that combining the assumptions on $\tilde{\sigma}$ and $\sigma_{Coulomb}$ with Lemma 2.1 it follows directly that $\sigma \in W^{2,2}(B^4, G)$.

In a first step, we decompose the $\mathfrak{g}$-valued one-form $\sigma^{-1}d\sigma$ using the Hodge decomposition theorem as

$$\sigma^{-1}d\sigma = d\alpha + d_*\beta\,, \tag{3.39}$$

where the $\mathfrak{g}$-valued function α satisfies $d_*\alpha = 0$ and the $\mathfrak{g}$-valued two-form β satisfies $d\beta = 0$. Note that for the Hodge decomposition (3.39) we also used that the first de Rham cohomology group $H^1(B^4, d)$ vanishes (see G.L. Naber [32], Corollary 5.2.5).

Denoting by $\Delta = dd_* + d_*d$ the Laplace-de Rham operator, we conclude from (3.39) and $d_*\alpha = 0$ that $\Delta\alpha$ is given by

$$\begin{aligned} \Delta\alpha &= dd_*\alpha + d_*d\alpha \\ &= d_*(\sigma^{-1}d\sigma - d_*\beta) = d_*(\sigma^{-1}d\sigma)\,. \end{aligned} \tag{3.40}$$

From the defining equation (3.25) for Coulomb gauges and (3.38), we deduce that

$$d_*(\sigma^{-1}\tilde{A}\sigma + \sigma^{-1}d\sigma) = 0\,,$$

and hence

$$d_*(\sigma^{-1}d\sigma) = -d_*(\sigma^{-1}\tilde{A}\sigma)\,. \tag{3.41}$$

Inserting this into (3.40), we then arrive at

$$\Delta\alpha = -d_*(\sigma^{-1}\tilde{A}\sigma)\,. \tag{3.42}$$

Since $\tilde{A}$ is smooth by assumption and $\sigma \in W^{2,2}(B^4, G)$, we obtain due to Lemma 2.1 that $\sigma^{-1}\tilde{A}\sigma \in W^{2,2}(B^4, T^*B^4 \otimes \mathfrak{g})$. Thus, the right-hand side of (3.42) belongs to $W^{1,2}$ and it follows that

$$d\alpha \in W^{2,2}(B^4, T^*B^4 \otimes \mathfrak{g})\,. \tag{3.43}$$

The Sobolev embedding theorem in four dimensions implies that $W^{2,2} \hookrightarrow W^{1,4}$ is a continuous embedding, since

$$2 - \frac{4}{2} \geq 1 - \frac{4}{4}\,.$$

Moreover, as a consequence of the same theorem in the critical case, we obtain that $W^{1,4} \hookrightarrow L^q$, for $4 \leq q < \infty$. Combining the last two embeddings with (3.43) leads to $d\alpha \in L^q(B^4, T^*B^4 \otimes \mathfrak{g})$, for $4 \leq q < \infty$. In particular, we have that

$$d\alpha \in L^5(B^4, T^*B^4 \otimes \mathfrak{g})\,. \tag{3.44}$$

At this stage, we use Lorentz spaces $L^{(p,q)}$. For an introduction to these spaces we refer to W.P. Ziemer [50], Section 1.8. Using the boundedness of B^4 we obtain the inclusion $L^5 = L^{(5,5)} \subset L^{(4,1)}$ (see F. Hélein [22], Chapitre 3.3). Together with (3.44) we hence end up with

$$d\alpha \in L^{(4,1)}(B^4, T^*B^4 \otimes \mathfrak{g})\,. \tag{3.45}$$

Using again (3.39) and also $d\beta = 0$, we compute

$$\begin{aligned} \Delta\beta &= dd_*\beta + d_*d\beta \\ &= d(\sigma^{-1}d\sigma - d\alpha) = d(\sigma^{-1}d\sigma) = d\sigma^{-1} \wedge d\sigma\,. \end{aligned} \tag{3.46}$$

Obviously, both forms on the right-hand side belong to $W^{1,2}$. In four dimensions, this space embeds into the Lorentz space $L^{(4,2)}$. For the results on Lorentz spaces used in this proof we refer to T. Rivière, [25] and references therein. The multiplication $L^{(4,2)} \otimes L^{(4,2)} \longrightarrow L^{(2,1)}$ of Lorentz spaces applied to the right-hand side of (3.46) then shows that $\Delta\beta \in L^{(2,1)}(B^4, \bigwedge^2(TB^4) \otimes \mathfrak{g})$. This implies that

$$d_*\beta \in L^{(4,1)}(B^4, T^*B^4 \otimes \mathfrak{g})\,. \tag{3.47}$$

Next, we insert (3.45) and (3.47) into (3.39) in order to get that

$$\sigma^{-1}d\sigma \in L^{(4,1)}(B^4, T^*B^4 \otimes \mathfrak{g})\,.$$

Due to the G-invariant metric on G and in particular (3.24), we also get that $d\sigma$ belongs to $L^{(4,1)}$. From this we then deduce the continuity of σ. Thus, we have

$$\sigma \in C^0(B^4, G) \cap W^{2,2}(B^4, G)\,. \tag{3.48}$$

We denote by $U \subset B^4$ the open subset on which σ is C^0-close to some element $\bar\sigma \in G$. Then, there exists $V \in C^0(U, \mathfrak{g}) \cap W^{2,2}(U, \mathfrak{g})$ being small in the C^0-norm such that

$$\sigma = \bar\sigma\,\exp V\,, \tag{3.49}$$

where $\exp : \mathfrak{g} \longrightarrow G$ denotes the exponential map. Hence, we have

$$\begin{aligned} \sigma^{-1}d\sigma &= (\bar\sigma\,\exp V)^{-1}d(\bar\sigma\,\exp V) \\ &= \exp(-V)\bar\sigma^{-1}\bar\sigma\,d(\exp V) = \exp(-V)d(\exp V)\,. \end{aligned} \tag{3.50}$$

Moreover, we already know that $d_*(\sigma^{-1}d\sigma)$ is an element of $W^{1,2}$ (see (3.41)). This implies that

$$d_*\big(\exp(-V)d(\exp V)\big) = d_*\big(\sigma^{-1}d\sigma\big) = \sum_{i=1}^{m}\left(\sum_{k=1}^{4}\frac{\partial f^i}{\partial x_k}\right)E_i \tag{3.51}$$

belongs to $W^{1,2}$. In the last equation the $\mathbb{C}$-valued functions f^i on U are in $W^{2,2}$, for $1 \leq i \leq l$, and $\{E_i\}_{i=1,\ldots,m}$ denotes a basis of $\mathfrak{g}$. Recall that the Lie algebra $\mathfrak{g}$ is assumed to be m-dimensional.

More precisely, for all $p \in U$, we can rewrite the right-hand side of (3.50) as $\exp\bigl(-V(p)\bigr) d\exp_{V(p)} \cdot dV_p$. Then, we define the endomorphisms $B(p,V) : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$B(p,V) = \exp\bigl(-V(p)\bigr) d\exp_{V(p)} \,. \tag{3.52}$$

Since V is continuous and since the exponential map is a diffeomorphism from a neighborhoood of $0 \in \mathfrak{g}$ into a neighboorhood of $e \in G$, we conclude that the assertion $p \longmapsto B(p,V)$ is continuous with respect to the operator norm. Thus, recalling that $d\exp_0 = \mathrm{id}_{\mathfrak{g}}$ and that V is close to 0, it follows that $B(p,V)$ is close to the identity $\mathrm{id}_{\mathfrak{g}}$.

In a next step, for all $p \in U$, we define the $m \times m$-matrices $B_{ij}(p,V)$ as the matrix representations of the endomorphisms $B(V,p)$. Moreover, we write the map $V : U \longrightarrow \mathfrak{g}$ as $V = \sum_{i=1}^m V^i E_i$ with $\mathbb{C}$-valued functions V^i on U, for $1 \leq i \leq m$. Their weak derivatives are given by $dV^i = \sum_{k=1}^4 \frac{\partial V^i}{\partial x_k}\, dx^k$. For the endomorphisms $B(p,V)$ acting on dV_p this leads to

$$\begin{aligned} B(p,V) \cdot dV_p &= \sum_{i=1}^m \left(\sum_{j=1}^m B_{ij}(p,V) \left(\sum_{k=1}^4 \frac{\partial V^j}{\partial x_k}(p)\, dx^k \right) \right) E^i \\ &= \left(B_{ij}(p,V) \frac{\partial V^j}{\partial x_k}(p)\, dx^k \right) E^i \,, \end{aligned} \tag{3.53}$$

where the Einstein summation convention is used in the second line. Applying the codifferential, we then obtain

$$\begin{aligned} d_*\Bigl(\exp\bigl(-V(p)\bigr) d\exp_{V(p)} \cdot dV_p \Bigr) &= d_*\bigl(B(p,V) \cdot dV_p \bigr) \\ &= \sum_{k=1}^4 \frac{\partial}{\partial x_k} \left(B_{ij}(p,V) \frac{\partial V^j}{\partial x_k}(p) \right) E^i \,. \end{aligned}$$

Inserting this into (3.51), the continuous function V thus satisfies the following system of partial differential equations in divergence form:

$$\sum_{k=1}^4 \frac{\partial}{\partial x_k} \left(B_{ij}(p,V) \frac{\partial V^j}{\partial x_k}(p) \right) = \sum_{k=1}^4 \frac{\partial f^i}{\partial x_k}(p) \,, \tag{3.54}$$

for $1 \leq i \leq m$, where the right-hand side belongs to $W^{1,2}$.

One can check that (3.54) defines a non-linear elliptic system. Applying the regularity theory for such systems (see for example M. Giaquinta, [15]) we then deduce that

$$V \in C^1(U,\mathfrak{g}) \cap W^{2,2}(U,\mathfrak{g}) \,. \tag{3.55}$$

Finally, the smoothness of V follows from an elliptic bootstrapping argument. This concludes the proof of the proposition. $\square$

3.3 Study of Three Different Metrics for L^2-curvatures on S^4

In this section, we study three different metrics for $W^{1,2}$-connections with associated small L^2-curvatures on some trivial principal $SU(2)$-bundle over S^4. The definitions of the metrics are motivated by Proposition 3.3 as well as by our later investigations on a strong density result for the non-Abelian case. The next proposition will be helpful for a better understanding of the difficulties occuring in the analysis of the non-Abelian case (see Section 5.3).

Proposition 3.5. *Let $A, B \in \mathcal{A}_{W^{1,2}}(S^4)$ with $\|F(A)\|_{L^2(S^4)}, \|F(B)\|_{L^2(S^4)} \leq \varepsilon < \delta_{Uh}$ and denote by $A_{Coulomb}, B_{Coulomb}$ the corresponding Coulomb gauges. Moreover, let d, δ and γ be three different metrics given by*

(i)

$$d\big(F(A), F(B)\big) = \inf_{\sigma \in L^\infty(S^4, SU(2))} \left(\int_{S^4} \big| F(A) - \sigma^{-1} F(B) \sigma \big|^2 \, d^4x \right)^{1/2}, \tag{3.56}$$

(ii)

$$\delta\big(F(A), F(B)\big) = \int_{S^4} \big| \mathrm{Tr}\big(F(A) \otimes F(A)\big) - \mathrm{Tr}\big(F(B) \otimes F(B)\big) \big| \, d^4x \,, \tag{3.57}$$

(iii)

$$\gamma\big(F(A), F(B)\big) = \inf_{\sigma_0 \in SU(2)} \| A_{Coulomb} - \sigma_0^{-1} B_{Coulomb} \sigma_0 \|_{W^{1,2}(S^4)} \,. \tag{3.58}$$

Then we have that d and δ induce the same topology, whereas δ and γ do not induce the same topology.

Proof. First, we show that d and δ are topologically equivalent metrics. – Let $(F_k)_{k \in \mathbb{N}}$ be a sequence of $\mathfrak{su}(2)$-valued two-forms on S^4 such that

$$\lim_{k \to \infty} d\big(F_k, F\big) = \lim_{k \to \infty} \inf_{\sigma \in L^\infty(S^4, SU(2))} \left(\int_{S^4} |F_k - \sigma^{-1} F \sigma|^2 \, d^4x \right)^{1/2} = 0 \,. \tag{3.59}$$

Then, we have to show that

$$\lim_{k \to \infty} \delta\big(F_k, F\big) = \lim_{k \to \infty} \int_{S^4} \big| \mathrm{Tr(F_k \otimes F_k) - Tr(F \otimes F)} \big| \, d^4x = 0 \,. \tag{3.60}$$

For doing this, we want to apply dominated convergence. – From the hypothesis (3.59) we directly obtain the existence of σ_k such that

$$\lim_{k \to \infty} \int_{S^4} |F_k - \sigma_k^{-1} F \sigma_k|^2 \, d^4x = 0 \,. \tag{3.61}$$

Using this and the inverse triangular inequality

$$\int_{S^4} \big| |F_k|^2 - |F|^2 \big| \, d^4x = \int_{S^4} \big| |F_k|^2 - |\sigma_k^{-1} F \sigma_k|^2 \big| \, d^4x \leq \int_{S^4} |F_k - \sigma_k^{-1} F \sigma_k|^2 \, d^4x \,,$$

we see that pointwise almost everywhere $|F_k|^2$ is bounded independently of k by an integrable function. From the pointwise estimate

$$\begin{aligned}\left|\mathrm{Tr}\big(F_k\otimes F_k\big)(x)-\mathrm{Tr}\big(F\otimes F\big)(x)\right| &\le \left|\mathrm{Tr}\big(F_k\otimes F_k\big)(x)\right|+\left|\mathrm{Tr}\big(F\otimes F\big)(x)\right|\\ &\le C\,|F_k(x)|^2+C\,|F(x)|^2\,,\end{aligned}$$

we then deduce that $\left|\mathrm{Tr}(F_k\otimes F_k)-\mathrm{Tr}(F\otimes F)\right|$ is pointwise almost everywhere bounded independently of k by an integrable function.

In a next step, note that as a consequence of (3.61) we have that

$$\lim_{k\to\infty}\left|F_k(x)-\sigma_k^{-1}(x)F(x)\sigma_k(x)\right|=0\,. \tag{3.62}$$

for almost every $x\in S^4$. Since

$$\begin{aligned}\mathrm{Tr}\big(F_k\otimes F_k\big)(x)-\mathrm{Tr}\big(F\otimes F\big)(x) &= \mathrm{Tr}\big(F_k\otimes F_k\big)(x)-\mathrm{Tr}\big(\sigma_k^{-1}F\sigma_k\otimes\sigma_k^{-1}F\sigma_k\big)(x)\\ &= \mathrm{Tr}\big((F_k-\sigma_k^{-1}F\sigma_k)\otimes(F_k-\sigma_k^{-1}F\sigma_k)\big)(x)\\ &\quad+\mathrm{Tr}\big(\sigma_k^{-1}F\sigma_k\otimes(F_k-\sigma_k^{-1}F\sigma_k\big)(x)\\ &\quad+\mathrm{Tr}\big((F_k-\sigma_k^{-1}F\sigma_k\big)\otimes\sigma_k^{-1}F\sigma_k\big)(x)\,,\end{aligned}$$

it follows using (3.62) that

$$\lim_{k\to\infty}\left|\mathrm{Tr}\big(F_k\otimes F_k\big)(x)-\mathrm{Tr}\big(F\otimes F\big)(x)\right|=0\,.$$

Thus, we can apply dominated convergence in order to obtain (3.60).

For the converse, we let $(F_k)_{k\in\mathbb{N}}$ be a sequence of $\mathfrak{su}(2)$-valued two-forms on S^4 such that

$$\lim_{k\to\infty}\delta\big(F_k,F\big)=\lim_{k\to\infty}\int_{S^4}\left|\mathrm{Tr}(\mathrm{F_k}\otimes\mathrm{F_k})-\mathrm{Tr}(\mathrm{F}\otimes\mathrm{F})\right|d^4x=0\,, \tag{3.63}$$

and show – using again dominated convergence – that

$$\lim_{k\to\infty}d\big(F_k,F\big)=\lim_{k\to\infty}\inf_{\sigma\in L^\infty(S^4,SU(2))}\left(\int_{S^4}|F_k-\sigma^{-1}F\sigma|^2\,d^4x\right)^{1/2}=0\,. \tag{3.64}$$

From the pointwise estimate

$$\begin{aligned}\inf_{\sigma\in SU(2)}\left|F_k(x)-\sigma^{-1}F(x)\sigma\right|^2 &\le \left|F_k(x)\right|^2+\left|F(x)\right|^2\\ &\le \left|\mathrm{Tr}\big(F_k\otimes F_k\big)(x)\right|+\left|F(x)\right|^2\end{aligned} \tag{3.65}$$

and the assumption (3.63) it follows that the left-hand side of (3.65) is pointwise almost everywhere bounded independently of k by an integrable function.

In a next step, note that as a consequence of (3.63) we have that

$$\lim_{k\to\infty}\left|\mathrm{Tr}\big(F_k\otimes F_k\big)(x)-\mathrm{Tr}\big(F\otimes F\big)(x)\right|=0\,, \tag{3.66}$$

for almost every $x\in S^4$. This implies that there exists a subsequence $(F_{k'})_{k'\in\mathbb{N}}$ – depending on x – such that $F_{k'}(x)\overset{k'\to\infty}{\longrightarrow}\bar F(x)$. Comparing this with (3.66), we obtain that $\mathrm{Tr}\big(\bar F\otimes$

$\bar{F}\big)(x) = \mathrm{Tr}\big(F \otimes F\big)(x)$. Hence, using Lemma 3.7 at the end of this section, there exists $\sigma \in SU(2)$ such that $\bar{F}(x) = \sigma^{-1}F(x)\sigma$. We have thus shown that for almost every $x \in S^4$ there exists a subsequence $(F_{k'})_{k'\in\mathbb{N}}$ such that

$$\lim_{k'\to\infty} \inf_{\sigma\in SU(2)} \big|F_{k'}(x) - \sigma^{-1}F(x)\sigma\big|^2 = 0\,. \tag{3.67}$$

We claim that this convergence holds for all subsequences and hence

$$\lim_{k\to\infty} \inf_{\sigma\in SU(2)} \big|F_{k}(x) - \sigma^{-1}F(x)\sigma\big|^2 = 0\,. \tag{3.68}$$

If this would not be the case, there would exist a subsequence $(F_{k'})_{k'\in\mathbb{N}}$ and $\varepsilon > 0$ such that

$$\inf_{\sigma\in SU(2)} \big|F_{k'}(x) - \sigma^{-1}F(x)\sigma\big|^2 \geq \varepsilon\,, \tag{3.69}$$

for all $k' \in \mathbb{N}$. Using again (3.66) and Lemma 3.7, we can extract as before a subsequence of $(F_{k'})_{k'\in\mathbb{N}}$ satisfying $F_{k''}(x) \longrightarrow \sigma^{-1}F(x)\sigma$. This contradicts (3.69) and hence (3.68) holds. Finally, we can apply dominated convergence in order to obtain as desired

$$\lim_{k\to\infty} \int_{S^4} \inf_{\sigma\in SU(2)} |F_k - \sigma^{-1}F\sigma|^2 \, d^4x = \lim_{k\to\infty} \inf_{\sigma\in L^\infty(S^4,SU(2))} \int_{S^4} |F_k - \sigma^{-1}F\sigma|^2 \, d^4x = 0\,.$$

This establishes that d and δ are topologically equivalent metrics.

As a direct consequence of the next Theorem 3.6, we obtain that the two metrics δ and γ do not induce the same topology. Hence, the proposition is shown. □

Theorem 3.6. *There exists a sequence $(A_k)_{k\in\mathbb{N}} \in \mathcal{A}_{W^{1,2}}(B^4)$ of connections with an associated sequence $(A_{Coulomb,k})_{k\in\mathbb{N}}$ of connections in Coulomb gauge such that*

$$\Big(\mathrm{Tr}\big(F(A_k) \otimes F(A_k)\big)\Big)_{k\in\mathbb{N}} \tag{3.70}$$

converges strongly in the L^1-norm for $k \to \infty$, but the sequence $(A_{Coulomb,k})_{k\in\mathbb{N}}$ of connections does not converge strongly in the $W^{1,2}$-norm.

Proof. We construct an example for which the statement of the theorem holds. – In canonical coordinates of B^4 we define for every $k \in \mathbb{N}$

$$A_k(x) = Q_k(x_1)\, dx_2\,, \tag{3.71}$$

where the $\mathfrak{su}(2)$-valued function Q_k is given by

$$Q_k(x_1) = \int_0^{x_1} \exp\big(-ks\,\sigma_2\big)\ \sigma_1\ \exp\big(ks\,\sigma_2\big)\, ds \tag{3.72}$$

with

$$\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

a basis for $\mathfrak{su}(2)$. We can rewrite (3.72) more explicitly as

$$\begin{aligned} Q_k(x_1) &= \int_0^{x_1} \begin{pmatrix} e^{-iks} & 0 \\ 0 & e^{iks} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{iks} & 0 \\ 0 & e^{-iks} \end{pmatrix} ds \\ &= \int_0^{x_1} \left(\cos(2ks) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \sin(2ks) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right) ds \\ &= \frac{\sin(2kx_1)}{2k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \left(\frac{\cos(2kx_1)}{2k} - 1 \right) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \end{aligned} \tag{3.73}$$

A direct calculation shows that A_k is in Coulomb gauge. Indeed, we have $\star A_k(x) = Q_k(x_1)\, dx_1 \wedge dx_3 \wedge dx_4$ and hence $d_* A_k(x) = - \star d \star A_k(x) = 0$.

In a next step, we let $\varepsilon > 0$ and $\phi \in C_c^\infty(B^4)$ such that $\phi \equiv 1$ on $B^4_{1/2}$. Then, we define

$$A_k^\varepsilon(x) = \varepsilon\, \phi(x)\, Q_k(x_1)\, dx_2 \,. \tag{3.74}$$

Note that A_k^ε coincides with A_k on $B^4_{1/2}$ up to a factor of ε. Moreover, we have that

$$\begin{aligned} F\big(A_k^\varepsilon\big)(x) &= dA_k^\varepsilon(x) + A_k^\varepsilon(x) \wedge A_k^\varepsilon(x) = dA_k^\varepsilon(x) \\ &= \varepsilon\, d\phi(x) \wedge Q_k(x_1)\, dx_2 + \varepsilon\, \phi(x)\, dQ_k(x_1) \wedge dx_2 \\ &\overset{(3.73)}{=} \varepsilon\, d\phi(x) \wedge Q_k(x_1)\, dx_2 \\ &\quad + \varepsilon\, \phi(x) \begin{pmatrix} e^{-ikx_1} & 0 \\ 0 & e^{ikx_1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{ikx_1} & 0 \\ 0 & e^{-ikx_1} \end{pmatrix} dx_1 \wedge dx_2 \,. \end{aligned} \tag{3.75}$$

It is not difficult to check that Q_k converges to zero in L^2 for $k \to \infty$. Defining $P_k : B^4 \longrightarrow SU(2)$ by $P_k(x) = \exp(-kx_1\, \sigma_2)$, we thus obtain from (3.75) that

$$P_k^{-1}(x) F\big(A_k^\varepsilon\big)(x) P_k(x) = \begin{pmatrix} e^{ikx_1} & 0 \\ 0 & e^{-ikx_1} \end{pmatrix} F\big(A_k^\varepsilon\big)(x) \begin{pmatrix} e^{-ikx_1} & 0 \\ 0 & e^{ikx_1} \end{pmatrix}$$

converges to $\varepsilon\, \phi(x) \sigma_1\, dx_1 \wedge dx_2$ in L^2 for $k \to \infty$. This then implies that

$$\mathrm{Tr}\big(F(A_k^\varepsilon) \otimes F(A_k^\varepsilon)\big) \longrightarrow -\varepsilon^2 \phi^2(x)\, dx_1 \wedge dx_2 \otimes dx_1 \wedge dx_2 \,,$$

and hence the example constructed in (3.74) has the desired convergence property (3.70).

Now we apply Uhlenbeck's Theorem 3.2 to A_k^ε for sufficiently small ε in order to get a sequence $(A^\varepsilon_{Coulomb,k})_{k\in\mathbb{N}}$ of Coulomb gauges. For an argument by contradiction we now assume that

$$A^\varepsilon_{Coulomb,k} \longrightarrow A_\infty \qquad \text{strongly in } W^{1,2}(B^4_{1/2}) \,. \tag{3.76}$$

Since A_k^ε and $A^\varepsilon_{Coulomb,k}$ are gauge equivalent, there exists a gauge transformation $g_k \in W^{2,2}(B^4, SU(2))$ such that

$$A_k^\varepsilon = g_k^{-1}\, A^\varepsilon_{Coulomb,k}\, g_k + g_k^{-1} dg_k \,, \tag{3.77}$$

for every $k \in \mathbb{N}$. This together with the invariant metric (3.24) leads to

$$\begin{aligned} \|dg_k\|_{L^4(B^4)} &= \|g_k^{-1} dg_k\|_{L^4(B^4)} \\ &\leq \|A_k^\varepsilon\|_{L^4(B^4)} + \|g_k^{-1}\, A^\varepsilon_{Coulomb,k}\, g_k\|_{L^4(B^4)} \\ &= \|A_k^\varepsilon\|_{L^4(B^4)} + \|A^\varepsilon_{Coulomb,k}\|_{L^4(B^4)} \,. \end{aligned}$$

The continuous Sobolev embedding $W^{1,2} \hookrightarrow L^4$ in four dimensions implies

$$\|dg_k\|_{L^4(B^4)} \leq C\,\|A_k^\varepsilon\|_{W^{1,2}(B^4)} + C\,\|A^\varepsilon_{Coulomb,k}\|_{W^{1,2}(B^4)}\,.$$

From (3.74) and the $W^{1,2}$-norm estimate (3.26) for Coulomb gauges we then conclude

$$\|dg_k\|_{L^4(B^4)} \leq C\,\varepsilon\,. \tag{3.78}$$

Next, we decompose the $\mathfrak{su}(2)$-valued one-form $g_k^{-1}dg_k$ using the Hodge decomposition theorem as

$$g_k^{-1}dg_k = d\alpha_k + d_*\beta_k\,, \tag{3.79}$$

where the $\mathfrak{su}(2)$-valued function α satisfies $d_*\alpha = 0$ and the $\mathfrak{su}(2)$-valued two-form β satisfies $d\beta = 0$. A straightforward computation for $k, l \in \mathbb{N}$ then yields

$$\begin{aligned}
\|d(d_*\beta_k) - d(d_*\beta_l)\|_{L^2} &= \|d(g_k^{-1}dg_k) - d(g_l^{-1}dg_l)\|_{L^2} \\
&= \|dg_k^{-1}\wedge dg_k - dg_l^{-1}\wedge dg_l\|_{L^2} \\
&= \|(dg_k^{-1} - dg_l^{-1})\wedge dg_k - dg_l^{-1}\wedge(dg_l - dg_k)\|_{L^2} \\
&\leq \|dg_k^{-1} - dg_l^{-1}\|_{L^4}\|dg_k\|_{L^4} + \|dg_l^{-1}\|_{L^4}\|dg_l - dg_k\|_{L^4} \\
&\leq C\varepsilon\,\|dg_k - dg_l\|_{L^4}\,,
\end{aligned}$$

where we used (3.78) for last inequality. Again owing to the continuous Sobolev embedding $W^{1,2} \hookrightarrow L^4$ in four dimensions, it then follows

$$\|d(d_*\beta_k) - d(d_*\beta_l)\|_{L^2} \leq C\varepsilon\,\|dg_k - dg_l\|_{W^{1,2}}\,. \tag{3.80}$$

This then leads to

$$\begin{aligned}
\|dg_k - dg_l\|_{W^{1,2}} &\leq C\big(\|d_*(d\alpha_k) - d_*(d\alpha_l)\|_{L^2} + \|d(d_*\beta_k) - d(d_*\beta_l)\|_{L^2}\big) \\
&\leq C\,\|d_*(g_k^{-1}dg_k) - d_*(g_l^{-1}dg_l)\|_{L^2} + C\varepsilon\,\|dg_k - dg_l\|_{W^{1,2}}\,.
\end{aligned} \tag{3.81}$$

On the other hand, we deduce from (3.77) and $d_*A_k^\varepsilon = 0$ on $B^4_{1/2}$ that

$$d_*(g_k^{-1}dg_k) = -d_*(g_k^{-1}\,A^\varepsilon_{Coulomb,k}\,g_k) \qquad \text{on } B^4_{1/2}\,. \tag{3.82}$$

Hence, the left-hand side of (3.82) converges in L^2-norm, since the right-hand side converges by the convergence assumption (3.76). Using this fact for the first term on the right-hand side of (3.81), we conclude that for sufficiently small $\varepsilon > 0$ the sequence $(dg_k)_{k\in\mathbb{N}}$ is a Cauchy sequence for the $W^{1,2}$-norm and hence converges. Inserting this converges in (3.77) then implies that the sequence $(A_k^\varepsilon)_{k\in\mathbb{N}}$ converges strongly in the $W^{1,2}$-norm on $B^4_{1/2}$. But such a strong convergence is not possible as can be seen from the definition (3.74) of A_k^ε (see also (3.73)). Thus, the strong convergence assumption (3.76) for the sequence $(A^\varepsilon_{Coulomb,k})_{k\in\mathbb{N}}$ of Coulomb gauges gives a contradiction and the theorem is shown. □

Lemma 3.7. *Let F and G be two elements in $\Omega^2_{L^2}(S^4, \mathfrak{su}(2))$. Then there exists a map $\sigma \in L^\infty(S^4, SU(2))$ such that $F = \sigma^{-1}G\sigma$ if and only if*

$$\mathrm{Tr}(F\otimes F) = \mathrm{Tr}(G\otimes G)\,. \tag{3.83}$$

Proof. We first consider the case of F and G being alternated $\mathfrak{su}(2)$-valued two-forms on $\mathbb{R}^4$ satisfying (3.83). Let $\{e_i\}_{i=1,\dots,4}$ denote the canonical basis of $\mathbb{R}^4$. Then we define

$$F_{ij} = F(e_i, e_j) \qquad \text{and} \qquad G_{ij} = G(e_i, e_j)\,,$$

for $i = 1, \dots, 4$, being matrices in $\mathfrak{su}(2)$. Since $F, G \in \bigwedge^2(\mathbb{R}^4) \otimes \mathfrak{su}(2)$, we observe that $F_{ii} = G_{ii} = 0$ and also that

$$F_{ij} = -F_{ji} \qquad \text{and} \qquad G_{ij} = -G_{ji}\,. \tag{3.84}$$

Moreover, the assumption now reads as

$$\mathrm{Tr}(F_{ij}F_{kl}) = \mathrm{Tr}(G_{ij}G_{kl})\,, \tag{3.85}$$

for every $i, j, k, l = 1, \dots, 6$.

Next, we define the four-form $\Omega = \mathrm{Tr}(F \otimes F)$ on $\mathbb{R}^4$ with components

$$\begin{aligned} \Omega_{ijkl} &= \mathrm{Tr}\big(F \otimes F\big)(e_i, e_j, e_k, e_l) \\ &= \mathrm{Tr}\big(F(e_i, e_j)F(e_k, e_l)\big) = \mathrm{Tr}(F_{ij}F_{kl})\,, \end{aligned} \tag{3.86}$$

for $i, j, k, l = 1, \dots, 6$. By definition, we have that

$$\Omega_{ijkl} = \Omega_{klij}\,,$$

and together with (3.84) it follows that

$$\Omega_{ijkl} = -\Omega_{jikl} \qquad \text{and} \qquad \Omega_{ijkl} = -\Omega_{ijlk}\,. \tag{3.87}$$

In particular, we note that

$$\Omega_{ijij} = \mathrm{Tr}(F_{ij}^2) = \mathrm{Tr}(F_{ji}^2) = \Omega_{jiji}\,. \tag{3.88}$$

Consider now the Pauli matrices $\{\sigma_1, \sigma_2, \sigma_3\}$ as defined in Appendix A. It is not difficult to check that

$$\sigma_i^2 = \mathbb{1} \qquad \text{and} \qquad \sigma_\alpha\sigma_\beta = -\sigma_\beta\sigma_\alpha = i\,\sigma_\gamma\,, \tag{3.89}$$

where $\{\alpha, \beta, \gamma\}$ is an even permutation of $\{1, 2, 3\}$. Since the Pauli matrices are a basis for $\mathfrak{su}(2)$, every matrix $A \in \mathfrak{su}(2)$ can be written as

$$A = a^1\sigma_1 + a^2\sigma_2 + a^3\sigma_3 =: \vec{a} \cdot \vec{\sigma}\,.$$

This implies that A is uniquely determined by the vector $\vec{a} = (a^1, a^2, a^3) \in \mathbb{R}^3$. With another matrix $B = \vec{b} \cdot \vec{\sigma} \in \mathfrak{su}(2)$ it follows by a straightforward calculation from (3.89) that

$$AB = (\vec{a} \cdot \vec{b})\mathbb{1} + i(\vec{a} \wedge \vec{b}) \cdot \vec{\sigma}\,.$$

Taking the trace and noting that the Pauli matrices have vanishing trace, we obtain

$$\mathrm{Tr}(AB) = 2\,\vec{a} \cdot \vec{b}\,. \tag{3.90}$$

As described before, we can now uniquely associated vectors $\vec{a}_{ij} \in \mathbb{R}^3$ and $\vec{b}_{ij} \in \mathbb{R}^3$ to the matrices $F_{ij} \in \mathfrak{su}(2)$ and $G_{ij} \in \mathfrak{su}(2)$, respectively. From (3.84), we observe that $\vec{a}_{ij} = -\vec{a}_{ji}$ and $\vec{b}_{ij} = -\vec{b}_{ji}$. Combining (3.88) with (3.90), we deduce that

$$|\vec{a}_{ij}|^2 = |\vec{a}_{ji}|^2 \qquad \text{and} \qquad |\vec{b}_{ij}|^2 = |\vec{b}_{ji}|^2 \,, \tag{3.91}$$

for every $i, j = 1, \ldots, 6$. Moreover, exploiting the assumption (3.85), it follows again from (3.90) that

$$|\vec{a}_{ij}|^2 = |\vec{b}_{ij}|^2 \qquad \text{and} \qquad \vec{a}_{ij} \cdot \vec{a}_{kl} = \vec{b}_{ij} \cdot \vec{b}_{kl} \,. \tag{3.92}$$

In summary, we end up with two families $\{\vec{a}_{ij}\}$ and $\{\vec{b}_{ij}\}$ of 12 vectors in $\mathbb{R}^3$ such that the following holds:

a) The vectors in both families have the same scalar products (see (3.92)).

b) The two vectors $\vec{a}_{ij}$, $\vec{a}_{ji}$ in the first family and the two vectors $\vec{b}_{ij}$, $\vec{b}_{ji}$ in the second family have all the same length (see (3.91) and (3.92)).

In a next step, we can relabel without loss of generality the two families of vectors $\{\vec{a}_i\}_{i=1,\ldots,12}$ and $\{\vec{b}_i\}_{i=1,\ldots,12}$ such that the following is true:

a) We have that $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ and $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ are two basis with same orientation for $\mathbb{R}^3$.

b) Applying the Gram-Schmidt process we can orthonormalize $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ and $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ without changing (3.92).

Then, it is well-known that there exists a rotation $R \in SO(3)$ such that

$$R\vec{a}_i = \vec{b}_i \qquad \text{for } \ i = 1, 2, 3 \,. \tag{3.93}$$

Next, we claim that the same rotation R satisfies

$$R\vec{a}_i = \vec{b}_i \qquad \text{for } \ i = 4, \ldots, 12 \,. \tag{3.94}$$

In order to see this, we use the representations

$$\vec{a}_j = \sum_{i=1}^{3} (\vec{a}_j \cdot \vec{a}_i)\, \vec{a}_i \qquad \text{and} \qquad \vec{b}_j = \sum_{i=1}^{3} (\vec{b}_j \cdot \vec{b}_i)\, \vec{b}_i \,,$$

for $j = 4, \ldots, 12$. A straightforward computation then yields

$$R\vec{a}_j = \sum_{i=1}^{3} (\vec{a}_j \cdot \vec{a}_i)\, R\vec{a}_i \overset{(3.93)}{=} \sum_{i=1}^{3} (\vec{a}_j \cdot \vec{a}_i)\, \vec{b}_i \overset{(3.92)}{=} \sum_{i=1}^{3} (\vec{b}_j \cdot \vec{b}_i)\, \vec{b}_i = \vec{b}_j \,,$$

showing the claim (3.94).

So far we have thus shown that there exists a rotation $R \in SO(3)$ such that $R\vec{a}_{ij} = \vec{b}_{ij}$. In terms of matrices this translates to the existence of a transformation $\sigma \in SU(2)$ such that

$$F_{ij} = \sigma^{-1} G_{ij} \sigma \,, \tag{3.95}$$

for every $i, j = 1, \ldots, 4$. For more details the reader should consult G.L. Naber [31], Appendix. – From (3.95) it is not difficult to conclude the statement of the lemma for $F, G \in \Omega^2_{L^2}(S^4, \mathfrak{su}(2))$.

For the converse, we only mention that the invariance of the trace under cyclic permutations directly gives the result. $\square$

Chapter 4

The Abelian Case

As explained in the introduction, we are interested in finding a closure of the space $\mathcal{F}_R(B^5)$ of weak curvatures for principal $SU(2)$-bundles over B^5 with a finite number of topological singularities. For this purpose, we first consider the simplified Abelian case of principal $U(1)$-bundles over B^3. The aim of this chapter is then to give a rigorous proof of Theorem 1.1 characterizing the closure of weak curvatures with topological singularities in the Abelian case.

Let us first explain how the Abelian case can be naturally constructed from Section 2.3. – One of the fundamental objects in the investigations on $W^{1,2}$-Sobolev maps $u : B^3 \longrightarrow S^2$ is the D-field $D(u) = u^*\omega_{S^2} \in \Omega^2_{L^1}(B^3)$. Recall that its integration over small spheres indicates the topological character of the singularities of the maps. As a generalization, consider now arbitrary integrable two-forms on B^3 denoted by F which have – in contrast to the D-field – not necessarily the structure of a pull-back. The integration of such two-forms F over spheres gets a topological sense if we interpret them as curvature forms of some principal $U(1)$-bundle over B^3. In order to see this recall from Section 2.1 that there is a one-to-one correspondence between the topological classification of principal $U(1)$-bundles over S^2 and their first Chern number defined by integration of the curvature. The interpretation of F as curvature yields also $dF = 0$ in the smooth case. This is in analogy with $dD(u) = 0$ for smooth u. Note also that in the Abelian $U(1)$-case curvatures are simply ordinary real-valued two-forms.

4.1 Weak Curvatures in the Abelian Case

In this section, we introduce classes of weak curvatures for the Abelian case with very similar properties to the three classes $C^\infty(\bar{B}^3, S^2)$, $R^{1,2}(B^3, S^2)$ and $W^{1,2}(B^3, S^2)$ of maps from B^3 into S^2 (see Section 2.3).

We denote by $\mathcal{F}_\infty(B^3)$ the class of smooth closed differential two-forms on B^3, i.e.,

$$\mathcal{F}_\infty(\bar{B}^3) = \left\{ F \in \Omega^2_\infty(\bar{B}^3) \ : \ dF = 0 \right\}. \tag{4.1}$$

From Poincaré's lemma, we obtain that $F \in \mathcal{F}_\infty(\bar{B}^3)$ is also exact. More precisely, there exists $A \in \Omega^1_\infty(\bar{B}^3)$ such that $F = dA$. An element in $\mathcal{F}_\infty(\bar{B}^3)$ can hence be seen as the curvature of some smooth principal $U(1)$-bundle over B^3 (see Example B.1). Note that since B^3 is contractible and paracompact or, equivalently, since there exists a global cross section, this bundle must be trivial. Proofs are given, for example, in D. Husemoller [24], Corollary 8.3 and 10.3. Using $dF = 0$ and Stokes' theorem, we obtain

$$c_1(\iota_{S^2}^* P) = \frac{1}{2\pi} \int_{S^2} \iota_{S^2}^* F = 0 \,, \tag{4.2}$$

showing that the first Chern number of the restricted principal $U(1)$-bundle $\iota_{S^2}^* P$ over S^2 vanishes which is equivalent to the triviality of $\iota_{S^2}^* P$.

Next, let $a_1, \ldots, a_N$ denote a finite number of points in B^3 and let $d_1, \ldots, d_N \in \mathbb{Z}$. Then, we introduce the following class of differential two-forms on B^3:

$$\begin{aligned} \mathcal{F}_R(B^3) = \Big\{ F \in \Omega^2_{L^1}(B^3) \;\; : \;\; & \exists \; a_1, \ldots, a_N \in B^3 \text{ st. } F \in \Omega^2_\infty\left(B^3 \setminus \bigcup_{i=1}^N a_i\right) , \\ & F \overset{\text{loc.}}{=} dA \text{ locally on } B^3 \setminus \{a_1, \ldots, a_N\} , \\ & dF = 2\pi \left(\sum_{i=1}^N d_i \delta_{a_i} \right) dx^1 \wedge dx^2 \wedge dx^3 \text{ in } \mathcal{D}' \Big\} . \end{aligned} \tag{4.3}$$

For $F \in \mathcal{F}_R(B^3)$, we claim that

$$\frac{1}{2\pi} \int_{\partial B_r^3(x)} \iota_{\partial B_r^3(x)}^* F \in \mathbb{Z} \,, \tag{4.4}$$

where x is an arbitrary point in B^3 and $0 < r \leq \operatorname{dist}(x, \partial B^3)$. In order to show the claim, we can assume without loss of generality that $F \in \mathcal{F}_R(B^3)$ is such that $F \in \Omega^2_\infty(B^3 \setminus \{0\})$ and $dF = 2\pi \delta_0 \, dx^1 \wedge dx^2 \wedge dx^3$ in the sense of distributions. Then, we take a test function $\phi_\varepsilon \in C_c^\infty(B^3)$ with

$$\phi_\varepsilon = \begin{cases} 0 & \text{on } B^3 \setminus B_r^3 \\ 1 & \text{on } B_{r-\varepsilon}^3 \end{cases} ,$$

where $0 < r < 1$. By definition of the derivative in the sense of distributions, we have

$$\langle dF, \phi_\varepsilon \rangle = -\langle F, d\phi_\varepsilon \rangle = -\int_{B^3} F \wedge d\phi_\varepsilon \,,$$

and hence, using the assumption on dF,

$$2\pi \phi_\varepsilon(0) = -\int_{B^3} F \wedge d\phi_\varepsilon \,. \tag{4.5}$$

From the choice of the test function and Stokes' theorem which can be applied since F is smooth on $B_r^3 \setminus B_{r-\varepsilon}^3$, it follows that

$$\begin{aligned} \int_{B^3} F \wedge d\phi_\varepsilon &= \int_{\partial(B_r^3 \setminus B_{r-\varepsilon}^3)} \iota^*(F\phi_\varepsilon) - \int_{B_r^3 \setminus B_{r-\varepsilon}^3} dF \, \phi_\varepsilon \\ &= \int_{\partial(B_r^3 \setminus B_{r-\varepsilon}^3)} \iota^*(F\phi_\varepsilon) = -\int_{\partial B_{r-\varepsilon}^3} \iota^*(F\phi_\varepsilon) \,. \end{aligned} \tag{4.6}$$

Here, we used also that $d(F\phi_\varepsilon) = dF\,\phi_\varepsilon + F \wedge d\phi$. Inserting (4.6) into (4.5), we find

$$2\pi\phi_\varepsilon(0) = \int_{\partial B^3_{r-\varepsilon}} \iota^*(F\phi_\varepsilon)\,.$$

Thus, in the limit $\varepsilon \to 0$, we obtain

$$1 = \frac{1}{2\pi}\int_{\partial B^3_r} \iota^*_{\partial B^3_r} F\,,$$

showing the claim.

An element in $F \in \mathcal{F}_R(B^3)$ has an geometrical interpretation, if we consider a smooth principal $U(1)$-bundle $P_{U,g}$ over $B^3 \setminus \bigcup_{i=1}^N a_i$ with globally defined smooth curvature form F given by $F = dA_i$ on U_i, where $A = \{A_i\}_{i\in I} \in \mathcal{A}_\infty(P_{U,g})$ denotes a smooth connection on $P_{U,g}$. The assumption $dF = 2\pi\left(\sum_{i=1}^N d_i\delta_{a_i}\right) dx^1 \wedge dx^2 \wedge dx^3$ in (1.9) implies that

$$c_1(\iota^*_{\partial B^3_r(a_i)} P_{U,g}) = \frac{1}{2\pi}\int_{\partial B^3_r(a_i)} \iota^*_{\partial B^3_r(a_i)} F = d_i\,, \tag{4.7}$$

for $i = 1,\ldots,N$. In words, we have that the first Chern numbers of the principal $U(1)$-bundles over small spheres around the singularities $a_1,\ldots,a_N$ of the principal $U(1)$-bundle $P_{U,g}$ over $B^3 \setminus \bigcup_{i=1}^N a_i$ are prescribed by the integers $d_1,\ldots,d_N$. We will later denote them by $c_1(P_{U,g}, a_i)$, for $i = 1,\ldots,N$.

The next proposition shows that the property (4.4) passes to the limit in the case of L^1-convergence.

Proposition 4.1. *Let $(F_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}_R(B^3)$. Assume that $F_k \longrightarrow F$ in L^1 $(k \to \infty)$ for some $F \in \Omega^2_{L^1}(B^3)$, i.e.,*

$$\lim_{k\to\infty}\int_{B^3} |F_k(x) - F(x)|\, d^3x = 0\,.$$

Then, for every $x \in B^3$ and a.e. $0 < r \le dist(x, \partial B^3)$, we have that

$$\int_{\partial B^3_r(x)} \iota^*_{\partial B^3_r(x)} F \in \mathbb{Z}\,.$$

Proof. Let $x \in B^3$ and $\tilde{r} = \mathrm{dist}(x, \partial B^3)$. By assumption, we have that

$$\lim_{k\to 0}\int_{B^3_{\tilde{r}}(x)} |F_k(x) - F(x)|\, d^3x = 0\,. \tag{4.8}$$

On the other hand, we obtain from Fubini's theorem that

$$\begin{aligned}\int_{B^3_{\tilde{r}}(x)} |F_k(x) - F(x)|\, d^3x &= \int_{(0,\tilde{r}]}\left(\int_{\partial B^3_r(x)} |F_k(x) - F(x)|\, d^2x\right) dr \\ &\ge \int_{(0,\tilde{r}]}\left|\int_{\partial B^3_r(x)} \iota^*_{\partial B^3_r(x)} F_k - \int_{\partial B^3_r(x)} \iota^*_{\partial B^3_r(x)} F\right| dr\,.\end{aligned}$$

Next, we set

$$f_k(r) = \int_{\partial B^3_r(x)} \iota^*_{\partial B^3_r(x)} F_k \qquad \text{and} \qquad f(r) = \int_{\partial B^3_r(x)} \iota^*_{\partial B^3_r(x)} F\,.$$

Note that $f_k(r) \in \mathbb{Z}$ by assumption. Thus (4.8) implies that

$$\int_{(0,\tilde{r}]} |f_k(r) - f(r)|\, dr \longrightarrow 0 \qquad (k \to \infty)\,.$$

From this L^1-convergence we deduce that, for a.e. $r \in (0, \tilde{r}]$,

$$f_k(r) \longrightarrow f(r) \qquad (k \to \infty)\,.$$

Since the sequence $\big(f_k(r)\big)_{k\in\mathbb{N}}$ of integers converges, the limit must also be an integer. Hence, we have shown that

$$f(r) = \int_{\partial B^3_r(x)} \iota^*_{\partial B^3_r(x)} F \in \mathbb{Z}\,.$$

□

Motivated by the previous proposition, we introduce the following third class of differential two-forms on B^3:

$$\begin{aligned} \mathcal{F}_{\mathbb{Z}}(B^3) \;=\; \Big\{ F \in \Omega^2_{L^1}(B^3) \;:\; \frac{1}{2\pi}\int_{\partial B^3_r(x)} \iota^*_{\partial B^3_r(x)} F \in \mathbb{Z}\,, \\ \text{for } \forall x \in B^3 \text{ and a.e } 0 < r \le \operatorname{dist}(x, \partial B^3) \Big\}\,. \end{aligned} \tag{4.9}$$

We give some explanations and a geometrical interpretation for $\mathcal{F}_{\mathbb{Z}}(B^3)$. From the assumptions in (4.9), it can be deduced that $\iota^*_{\partial B^3_r(x)} F$ is a curvature for a principal $U(1)$-bundle over $\partial B^3_r(x)$ (see R. Bott and L.W. Tu, [1]). This means the following: For all $x \in B^3$ and a.e. $0 < r \le \operatorname{dist}(x, \partial B^3)$, there exists some principal $U(1)$-bundle $P_{U,g} \in \mathcal{P}^{U(1)}_{W^{2,1}}(\partial B^3_r(x))$ and a connection form $a = \{a_i\}_{i\in I} \in \mathcal{A}_{W^{1,1}}(P_{U,g})$ whose globally defined curvature is given by $\iota^*_{\partial B^3_r(x)} F$. More precisely, we have that

$$\iota^*_{\partial B^3_r(x)} F = da_i \qquad \text{on } U_i\,. \tag{4.10}$$

As in the last equation, we will denote from now on in the thesis the geometric quantities of the gauge field theories with capital letters whereas quantities on the "boundary" will be denoted by small letters.

At this place, we mention that considering $W^{2,1}$-Sobolev bundles – or also $W^{2,1}$-Sobolev gauge transformations – in the three-dimensional Abelian case shows that we are dealing with a gauge theory in a non-critical higher dimensional setting (compare with Section 3.1).

4.2 Strong Density Result in the Abelian Case

This section is devoted to the proof of Theorem 1.1 in the introduction. The first part of this theorem was already shown in Proposition 4.1. So it remains to show the second part, namely that every element in $\mathcal{F}_{\mathbb{Z}}(B^3)$ can be strongly approximated in L^1 by weak curvatures in $\mathcal{F}_R(B^3)$. For this purpose, following F. Bethuel, [9], we first introduce the concept of a cubic decomposition. Then we apply the results of Section 3.1 on the boundary of the cubic decomposition.

4.2.1 Cubic Decomposition and Smoothing

For $\varepsilon > 0$ and $a \in [0,\varepsilon]^3$, we divide B^3 into open cubes $\{C^i_{\varepsilon,a}\}_{i=1}^{N_\varepsilon}$ of side length ε centered at the points $a^i \in a + \varepsilon\,\mathbb{Z}^3$. In the following, we will write $C_{\varepsilon,a}$ and $\partial C_{\varepsilon,a}$ for the sets $\bigcup_{i=1}^{N_\varepsilon} C^i_{\varepsilon,a}$ and $\bigcup_{i=1}^{N_\varepsilon} \partial C^i_{\varepsilon,a}$, respectively, where N_ε denotes the number of cubes with side length ε obtained by the *cubic decomposition* of B^3. – The next lemma states how this cubic decomposition can be chosen.

Lemma 4.2. *Let $F \in \Omega^2_{L^1}(B^3)$ and define the piecewise constant two-form $\bar{F}_{\varepsilon,a}$ on B^3 by $\bar{F}_{\varepsilon,a} = \bar{F}^i_{\varepsilon,a}$ on $C^i_{\varepsilon,a}$, where $\bar{F}^i_{\varepsilon,a}$ denotes the average of F in the cube $C^i_{\varepsilon,a}$. Then, for every $\varepsilon > 0$, there exists $\bar{a}_\varepsilon \in [0,\varepsilon]^3$ such that*

$$\lim_{\varepsilon\to 0} \varepsilon\, \|F - {}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon}\|_{L^1(\partial C_{\varepsilon,\bar{a}_\varepsilon})} = 0\,, \tag{4.11}$$

where ${}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon}$ is defined as the "left" trace of $\bar{F}_{\varepsilon,a}$ on $\partial C_{\varepsilon,\bar{a}_\varepsilon}$, and also

$$\|F\|_{L^1(\partial C_{\varepsilon,\bar{a}_\varepsilon})} \le \frac{C}{\varepsilon}\,\|F\|_{L^1(B^3)} + \|F - {}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon}\|_{L^1(\partial C_{\varepsilon,\bar{a}_\varepsilon})}\,. \tag{4.12}$$

Proof. Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$ and write $a = (a_1, a_2, a_3)$. For $j = 1, \ldots, 3$, we then denote by $\partial C^{e_j}_{\varepsilon,a_j}$ the subset of $\partial C_{\varepsilon,a}$ consisting of those hyperplanes orthogonal to e_j and passing through the points $\big((a_j + \varepsilon/2) + \varepsilon\,\mathbb{Z}\big)\, e_j \in B^3$. Hence, we get

$$\partial C_{\varepsilon,a} = \bigcup_{j=1}^{3} \partial C^{e_j}_{\varepsilon,a_j}\,. \tag{4.13}$$

By definition of the piecewise constant two-form $\bar{F}_{\varepsilon,a} \in \Omega^2_\infty(B^3)$, it is not difficult to check that

$$\int_{C^i_{\varepsilon,a}} \big|\bar{F}^i_{\varepsilon,a}(x)\big|\, d^3x \le \int_{C^i_{\varepsilon,a}} \big|F(x)\big|\, d^3x\,,$$

for every $i = 1, \ldots, N_\varepsilon$ and $a \in [0,\varepsilon]^3$. Summation and Fubini's theorem then lead to

$$\varepsilon \int_{\partial C^{e_j}_{\varepsilon,a_j}} \big|\,{}^l\bar{F}_{\varepsilon,a}(x)\big|\, d^2x \le \|F\|_{L^1(B^3)}\,,$$

for $j = 1, \ldots, 3$. Because of (4.13) this gives

$$\int_{\partial C_{\varepsilon,a}} \big|\,{}^l\bar{F}_{\varepsilon,a}(x)\big|\, d^2x \le \frac{3}{\varepsilon}\,\|F\|_{L^1(B^3)}\,. \tag{4.14}$$

Next, we claim that, for every $k > 0$, there exists $a_\varepsilon^k \in [0,\varepsilon]^3$, for each $\varepsilon > 0$, such that

$$\limsup_{\varepsilon\to 0} \varepsilon \, \|F - {}^l\bar{F}_{\varepsilon,a_\varepsilon^k}\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} \leq \frac{1}{k}\,. \tag{4.15}$$

In order to prove the claim, we first take by density a smooth two-form $G \in \Omega_\infty^2(B^3)$ satisfying

$$\|F - G\|_{L^1(B^3)} \leq \frac{1}{3\cdot 2k}\,. \tag{4.16}$$

By Fubini's theorem, we then deduce, for every $\varepsilon > 0$, the existence of $a_j^k \in [0,\varepsilon]$ such that

$$\begin{aligned} \int_{B^3} \big|F(x) - G(x)\big|\, d^3x &= \int_0^\varepsilon \left(\int_{\partial C^{e_j}_{\varepsilon,a_j}} \big|F(x) - G(x)\big|\, d^2x \right) da_j \\ &\geq \varepsilon \int_{\partial C^{e_j}_{\varepsilon,a_j^k}} \big|F(x) - G(x)\big|\, d^2x\,, \end{aligned} \tag{4.17}$$

for all $j = 1, \ldots, 3$. Hence, denoting $a_\varepsilon^k = (a_1^k, a_2^k, a_3^k)$, it follows that

$$\|F - G\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} \overset{(4.13)}{=} \sum_{j=1}^{3} \int_{\partial C^{e_j}_{\varepsilon,a_j^k}} \big|F(x) - G(x)\big|\, d^2x \leq \frac{3}{\varepsilon}\, \|F - G\|_{L^1(B^3)}\,.$$

Using (4.16), we obtain

$$\|F - G\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} \leq \frac{1}{2k\,\varepsilon}\,. \tag{4.18}$$

Since the operation of taking the average on each cube in the decomposition is linear, we conclude from (4.14) that

$$\begin{aligned} \| {}^l\bar{G}_{\varepsilon,a_\varepsilon^k} - {}^l\bar{F}_{\varepsilon,a_\varepsilon^k}\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} &= \| {}^l\overline{(G - F)}_{\varepsilon,a_\varepsilon^k}\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} \\ &\leq \frac{3}{\varepsilon}\, \|F - G\|_{L^1(B^3)} \overset{(4.16)}{\leq} \frac{1}{2k\,\varepsilon}\,. \end{aligned} \tag{4.19}$$

Next, we observe from the triangular inequality that

$$\begin{aligned} \varepsilon\, \|F - {}^l\bar{F}_{\varepsilon,a_\varepsilon^k}\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} &\leq \varepsilon\, \|F - G\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} + \varepsilon\, \|G - {}^l\bar{G}_{\varepsilon,a_\varepsilon^k}\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} \\ &\quad + \varepsilon\, \| {}^l\bar{G}_{\varepsilon,a_\varepsilon^k} - {}^l\bar{F}_{\varepsilon,a_\varepsilon^k}\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} \\ &\leq \frac{1}{2k} + \varepsilon\, \|G - {}^l\bar{G}_{\varepsilon,a_\varepsilon^k}\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} + \frac{1}{2k}\,, \end{aligned}$$

where the estimates (4.18) and (4.19) are also used. By the smoothness assumption on G, we then arrive at

$$\limsup_{\varepsilon\to 0} \varepsilon \, \|F - {}^l\bar{F}_{\varepsilon,a_\varepsilon^k}\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} \leq \frac{1}{k}\,,$$

showing the claim (4.15).

In a next step, we can pass by a standard diagonal argument from (4.15) to (4.11). More precisely, due to (4.15), for every $k > 0$, it is possible to choose $\delta_k > 0$ sufficiently small such that

$$\varepsilon \, \|F - {}^l\bar{F}_{\varepsilon,a_\varepsilon^k}\|_{L^1(\partial C_{\varepsilon,a_\varepsilon^k})} \leq \frac{2}{k}\,,$$

for all $0 < \varepsilon < \delta_k$. Moreover, we can assume that $\delta_{k+1} < \delta_k$. Thus, defining $\bar{a}_\varepsilon = a_\varepsilon^k$ in the case of $\varepsilon \in [\delta_{k+1}, \delta_k)$, we end up with (4.11).

It remains to show that (4.12) holds. For this purpose, the triangular inequality and (4.14) give the bound

$$\begin{aligned}
\|F\|_{L^1(\partial C_{\varepsilon,\bar{a}_\varepsilon})} &\leq \|F - {}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon}\|_{L^1(\partial C_{\varepsilon,\bar{a}_\varepsilon})} + \|{}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon}\|_{L^1(\partial C_{\varepsilon,\bar{a}_\varepsilon})} \\
&\leq \|F - {}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon}\|_{L^1(\partial C_{\varepsilon,\bar{a}_\varepsilon})} + \frac{3}{\varepsilon}\,\|F\|_{L^1(B^3)}\,,
\end{aligned}$$

concluding the proof of the lemma. □

Good and Bad Cubes in the Cubic Decomposition

In a next step, we consider $F \in \mathcal{F}_{\mathbb{Z}}(B^3)$ and divide the cubes $\{C^i_{\varepsilon,a}\}_{i=1}^{N_\varepsilon}$ of the cubic decomposition into "good" and "bad" ones. More precisely, we say that a cube $C^{g_i}_{\varepsilon,a}$ in $\{C^i_{\varepsilon,a}\}_{i=1}^{N_\varepsilon}$ is a *good* cube if

$$\int_{\partial C^{g_i}_{\varepsilon,a}} \iota^*_{\partial C^{g_i}_{\varepsilon,a}} F = 0\,. \tag{4.20}$$

The *bad* cubes $C^{b_i}_{\varepsilon,a}$ are the remaining ones, i.e.,

$$0 \neq \int_{\partial C^{b_i}_{\varepsilon,a}} \iota^*_{\partial C^{b_i}_{\varepsilon,a}} F \in \mathbb{Z}\,. \tag{4.21}$$

Hence, we can write

$$\bigcup_{i=1}^{N_\varepsilon} C^i_{\varepsilon,a} = \bigcup_{i=1}^{N^g_\varepsilon} C^{g_i}_{\varepsilon,a} \cup \bigcup_{i=1}^{N^b_\varepsilon} C^{b_i}_{\varepsilon,a}\,,$$

where $N_\varepsilon = N^g_\varepsilon + N^b_\varepsilon$ with N^g_ε and N^b_ε the number of good, respectively, bad cubes in B^3. In the following, the sets $\bigcup_{i=1}^{N^g_\varepsilon} C^{g_i}_{\varepsilon,a}$ and $\bigcup_{i=1}^{N^b_\varepsilon} C^{b_i}_{\varepsilon,a}$ will be denoted by $C^g_{\varepsilon,a}$ and $C^b_{\varepsilon,a}$, respectively. – Though we take spheres in the defining equation (4.9) of $\mathcal{F}_{\mathbb{Z}}(B^3)$, we choose a cubic decomposition of B^3 in order to make the presentation simpler. The construction of a decomposition into spheres for maps $W^{2,2}(B^5, S^3)$ is described in B. Hardt and T. Rivière, [21].

Lemma 4.3. *Let $F \in \mathcal{F}_{\mathbb{Z}}(B^3)$. Then, for every $\varepsilon > 0$, there exists $\bar{a}_\varepsilon \in [0,\varepsilon]^3$ such that the volume of the bad cubes vanishes in the limit $\varepsilon \to 0$, i.e.,*

$$\lim_{\varepsilon \to 0} \mu(C^b_{\varepsilon,\bar{a}_\varepsilon}) = 0\,,$$

where μ denotes the Lebesgue measure.

Proof. For $F \in \mathcal{F}_{\mathbb{Z}}(B^3)$, we observe that

$$1 \leq \left| \int_{\partial C^{b_i}_{\varepsilon,a}} \iota^*_{\partial C^{b_i}_{\varepsilon,a}} F \right| \leq \int_{\partial C^{b_i}_{\varepsilon,a}} |F(x)| \, d^2x \,, \tag{4.22}$$

for all bad cubes. Choosing $\bar{a}_\varepsilon \in [0,\varepsilon]^3$ as in Lemma 4.2, we then obtain the following estimate:

$$\begin{aligned} \frac{C}{\varepsilon} \|F\|_{L^1(B^3)} + \|F - {}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon}\|_{L^1(\partial C_{\varepsilon,\bar{a}_\varepsilon})} &\overset{(4.12)}{\geq} \|F\|_{L^1(\partial C_{\varepsilon,\bar{a}_\varepsilon})} \\ &= \frac{1}{2} \sum_{i=1}^{N^b_\varepsilon} \int_{\partial C^i_{\varepsilon,\bar{a}_\varepsilon}} |F(x)| \, d^2x \overset{(4.22)}{\geq} \frac{N^b_\varepsilon}{2} \,. \end{aligned} \tag{4.23}$$

For the volume of the bad cubes $C^b_{\varepsilon,\bar{a}_\varepsilon}$, we hence deduce

$$\begin{aligned} \mu(C^b_{\varepsilon,\bar{a}_\varepsilon}) = N^b_\varepsilon \, \varepsilon^3 &\overset{(4.23)}{\leq} \frac{C}{\varepsilon} \|F\|_{L^1(B^3)} \, \varepsilon^3 + \|F - {}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon}\|_{L^1(\partial C_{\varepsilon,\bar{a}_\varepsilon})} \, \varepsilon^3 \\ &= C \|F\|_{L^1(B^3)} \, \varepsilon^2 + \|F - {}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon}\|_{L^1(\partial C_{\varepsilon,\bar{a}_\varepsilon})} \, \varepsilon^3 \,. \end{aligned}$$

Because of (4.11) the right-hand side vanishes in the limit $\varepsilon \to 0$. □

Smoothing on the Boundary of the Cubic Decomposition

The following considerations will be used in Section 4.2.2 for the smoothing on the boundary of the cubic decomposition. – In the Abelian case we are dealing with $W^{1,2}$-Sobolev $U(1)$-bundles. They are critical in two-dimensions and hence the density results of Section 3.1 apply. For completeness, we now give a precise formulation of these results in the Abelian case.

Let M be a two-dimensional compact Riemannian manifold and $\mathcal{P}^{U(1)}_{W^{2,1}}(M)$ the space of $W^{2,1}$-Sobolev principal $U(1)$-bundles over M. Connections $A \in \mathcal{A}_{W^{1,1}}(P_{U,g})$ on $P_{U,g} \in \mathcal{P}^{U(1)}_{W^{2,1}}(M)$ are characterized by the Abelian compatibility condition (see Example B.1)

$$A_j = A_i + g_{ij}^{-1} dg_{ij} \qquad \text{on } U_i \cap U_j \,. \tag{4.24}$$

The next smooth approximation result for connections in the Abelian case is a particular case of Proposition 1.6.

Proposition 4.4. *Let $P_{U,g} \in \mathcal{P}^{U(1)}_\infty(M)$ and $A = \{A_i\}_{i \in I}$ a $W^{1,1}$-Sobolev connection on the smooth principal $U(1)$-bundle $P_{U,g}$. Then, there exists a sequence $(\tilde{A}^\varepsilon)_{\varepsilon>0}$ of smooth connections on $P_{U,g}$ such that*

$$\lim_{\varepsilon \to 0} \|\tilde{A}^\varepsilon_i - A_i\|_{W^{1,1}(U_i)} = 0 \,, \tag{4.25}$$

for every $i \in I$.

The L^1-Sobolev curvature associated to $A\mathcal{A}_{W^{1,1}}(P_{U,g})$ is defined as the family $F(A) = \{F_i(A)\}_{i\in I}$ of two-forms on U_i given by

$$F_i(A) = dA_i \qquad \text{on } U_i\,. \tag{4.26}$$

Recall also that the curvature form is *globally* defined on M as explained in Example B.1. As a direct consequence of Proposition 4.4, there is also a smooth approximation for L^1-curvatures in the two dimensions.

Corollary 4.5. *Let $A = \{A_i\}_{i\in I} \in \mathcal{A}_{W^{1,1}}(P_{U,g})$ and let $F(A)$ be its curvature. Assume also that there exists a sequence $(\tilde{A}^\varepsilon)_{\varepsilon>0}$ of smooth connections such that $\lim_{\varepsilon\to 0}\|\tilde{A}^\varepsilon_i - A_i\|_{W^{1,1}(U_i)} = 0$, for every $i \in I$. Then, for the sequence of smooth curvatures $\big(F(\tilde{A}^\varepsilon)\big)_{\varepsilon>0}$ associated to $\tilde{A}^\varepsilon$, we have that*

$$\lim_{\varepsilon\to 0}\|F_i(\tilde{A}^\varepsilon) - F_i(A)\|_{L^1(U_i)} = 0\,, \tag{4.27}$$

for every $i \in I$.

4.2.2 Proof of the Strong Density Result

Now, we are ready to give a proof of the following density result announced before:

Theorem 4.6. *Let $F \in \mathcal{F}_{\mathbb{Z}}(B^3)$. Then there exists a sequence $(F_k)_{k\in\mathbb{N}}$ in $\mathcal{F}_R(B^3)$ such that*

$$F_k \longrightarrow F \quad \text{in } L^1 \qquad (k\to\infty)\,.$$

Proof. We divide B^3 into open cubes $\{C^i_{\varepsilon,\bar{a}_\varepsilon}\}_{i=1}^{N_\varepsilon}$ in such a way that Lemma 4.2 and 4.3 hold for $F \in \mathcal{F}_{\mathbb{Z}}(B^3)$. By abuse of notation we will simply denote by C_ε, for every $\varepsilon > 0$, the chosen cubic decomposition.

Mollification on the Boundary of the Cubic Decomposition.

Now, we fix some $\varepsilon > 0$. We already know that $F \in \mathcal{F}_{\mathbb{Z}}(B^3)$ is a curvature on the boundary $\partial C_\varepsilon = \bigcup_{i=1}^{N_\varepsilon}\partial C^i_\varepsilon$ of the cubic decomposition. Denote by $a_\varepsilon = \{a_{\varepsilon,i}\}_{i\in I(\varepsilon)}$ the connection on some $P_{U_\varepsilon,g_\varepsilon} \in \mathcal{P}^{U(1)}_{W^{2,1}}(\partial C_\varepsilon)$ whose curvature form is given by the restriction of F to the boundary of the cubic decomposition C_ε. Applying Proposition 4.4, we deduce the existence of a smooth connection $\tilde{a}^k_\varepsilon = \{\tilde{a}^k_{\varepsilon,i}\}_{i\in I(\varepsilon)}$ on $P_{U_\varepsilon,g_\varepsilon}$ such that

$$\|\tilde{a}^k_{\varepsilon,i} - a_{\varepsilon,i}\|_{W^{1,1}(U_{\varepsilon,i})} \le \frac{1}{k}\,, \tag{4.28}$$

for every $i \in I(\varepsilon)$. For the associated *smooth* curvature $f(\tilde{a}^k_\varepsilon) = \{f_i(\tilde{a}^k_\varepsilon)\}_{i\in I(\varepsilon)}$, it follows from (4.28) that (see also Corollary 4.5)

$$\|f_i(\tilde{a}^k_\varepsilon) - \iota^*_{U_{\varepsilon,i}}F\|_{L^1(U_{\varepsilon,i})} \le \frac{C}{k}\,,$$

for every $i \in I(\varepsilon)$. After summation over $i \in I(\varepsilon)$ and choosing $1/k$ – in function of $\varepsilon > 0$ – small enough, we can achieve that the *globally* well-defined curvature $f(\tilde{a}^k_\varepsilon)$ satisfy

$$\|f(\tilde{a}^k_\varepsilon) - F\|_{L^1(\partial C_\varepsilon)} \le \varepsilon\,. \tag{4.29}$$

Repeating this construction for every $\varepsilon > 0$, we can show as a consequence of (4.29) that there exists a sequence $(\tilde{f}_\varepsilon)_{\varepsilon>0}$ of smooth curvatures on ∂C_ε such that

$$\lim_{\varepsilon\to 0} \|\tilde{f}_\varepsilon - F\|_{L^1(\partial C_\varepsilon)} = 0 \,. \tag{4.30}$$

Density on the Good Cubes.

We first show the L^1-convergence on the good cubes C^g_ε, i.e.,

$$\int_{C^g_\varepsilon} \big|F^g_\varepsilon(x) - F(x)\big|\, d^3x \longrightarrow 0 \qquad (\varepsilon \to 0)\,, \tag{4.31}$$

for a sequence $(F^g_\varepsilon)_{\varepsilon>0}$ in $\mathcal{F}_R(C^g_\varepsilon)$. It will turn out that – on the good cubes – $(F^g_\varepsilon)_{\varepsilon>0}$ is actually a sequence in $\mathcal{F}_\infty(C^g_\varepsilon)$.

First step: For a *fixed* good cube $C^{g_{i_0}}_\varepsilon$, we consider the constant two-form $\bar{F}^{g_{i_0}}_\varepsilon$ defined by the average of F in $C^{g_{i_0}}_\varepsilon$. Since $\bar{F}^{g_{i_0}}_\varepsilon$ is constant, we can easily find an one-form $\bar{A}^{g_{i_0}}_\varepsilon$ such that

$$\begin{cases} d\bar{A}^{g_{i_0}}_\varepsilon &= \bar{F}^{g_{i_0}}_\varepsilon \\ d_*\bar{A}^{g_{i_0}}_\varepsilon &= 0 \end{cases} \qquad \text{on } \; C^{g_{i_0}}_\varepsilon \,. \tag{4.32}$$

The restriction of $\bar{A}^{g_{i_0}}_\varepsilon$ to $\partial C^{g_{i_0}}_\varepsilon$ will be denoted by $\bar{a}^{g_{i_0}}_\varepsilon$. It satisfies $d\bar{a}^{g_{i_0}}_\varepsilon = \iota^*_{\partial C}\bar{F}^{g_{i_0}}_\varepsilon$ and $d_*\bar{a}^{g_{i_0}}_\varepsilon = 0$.

As a consequence of the defining equation (4.20) of the good cubes, the principal $U(1)$-bundle over its boundary for which $\iota^*_{\partial C}F$ represents a curvature is *trivial.* Thus, the smooth approximation $(\tilde{f}_\varepsilon)_{\varepsilon>0}$ in the L^1-norm of $\iota^*_{\partial C}F$ has the form $\tilde{f}_\varepsilon = d\tilde{a}^{g_{i_0}}_\varepsilon$ on $\partial C^{g_{i_0}}_\varepsilon$. Next, we apply a smooth gauge transformation $\sigma = e^{-i\chi}$ to $\tilde{a}^{g_{i_0}}_\varepsilon$, where χ denotes some smooth real-valued function on $\partial C^{g_{i_0}}_\varepsilon$, and impose that $\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon} = \sigma(\tilde{a}^{g_{i_0}}_\varepsilon)$ satisfies

$$d_*\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon} = 0\,.$$

More precisely, for the function χ, we want to solve

$$d_*\sigma(\tilde{a}^{g_{i_0}}_\varepsilon) = d_*\tilde{a}^{g_{i_0}}_\varepsilon - i\, d_*d\chi = 0\,,$$

or, equivalently,

$$d_*d\chi = -i\, d_*\tilde{a}^{g_{i_0}}_\varepsilon\,.$$

For the existence result we refer to S.K. Donaldson and P.B. Kronheimer [11], Section 2.3, and also to G. Schwarz [38], Section 3.2, for more details. It is very important to mention that $d\tilde{a}^{g_{i_0}}_\varepsilon = d\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon}$, since the curvatures remain unchanged under a gauge transformation in the $U(1)$-case (see Example B.1). We have thus shown that there exists a *Abelian Coulomb gauge* $\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon} \in \Omega^1_\infty(\partial C^{g_{i_0}}_\varepsilon)$ such that

$$\begin{cases} d\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon} &= \tilde{f}_\varepsilon \\ d_*\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon} &= 0 \end{cases} \qquad \text{on } \; \partial C^{g_{i_0}}_\varepsilon \,. \tag{4.33}$$

Next, we denote by $\tilde{A}^{g_{i_0}}_\varepsilon$ the smooth harmonic extension of $\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon}$ on the fixed good cube $C^{g_{i_0}}_\varepsilon$ satisfying

$$\begin{cases} \Delta\tilde{A}^{g_{i_0}}_\varepsilon = 0 & \text{on } \; C^{g_{i_0}}_\varepsilon \\ \iota^*_{\partial C}\tilde{A}^{g_{i_0}}_\varepsilon = \tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon}\,, \quad \iota^*_{\partial C}(d_*\tilde{A}^{g_{i_0}}_\varepsilon) = 0 & \text{on } \; \partial C^{g_{i_0}}_\varepsilon\,. \end{cases} \tag{4.34}$$

Then, we define the one-form

$$A_\varepsilon^{g_{i_0}} = \tilde{A}_\varepsilon^{g_{i_0}} - \bar{A}_\varepsilon^{g_{i_0}} \,,$$

whose restriction $a_\varepsilon^{g_{i_0}}$ to the boundary equals $\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon} - \bar{a}_\varepsilon^{g_{i_0}}$. We deduce easily from (4.32) – (4.34) that

$$\begin{cases} \Delta A_\varepsilon^{g_{i_0}} = 0 & \text{on } C_\varepsilon^{g_{i_0}} \\ \iota^*_{\partial C} A_\varepsilon^{g_{i_0}} = \tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon} - \bar{a}_\varepsilon^{g_{i_0}} \,, \quad \iota^*_{\partial C}(d_* A_\varepsilon^{g_{i_0}}) = 0 & \text{on } \partial C_\varepsilon^{g_{i_0}} \,, \end{cases}$$

and that

$$\begin{cases} da_\varepsilon^{g_{i_0}} & = \tilde{f}_\varepsilon - \iota^*_{\partial C} \bar{F}_\varepsilon^{g_{i_0}} \\ d_* a_\varepsilon^{g_{i_0}} & = 0 \end{cases} \qquad \text{on } \partial C_\varepsilon^{g_{i_0}} \,.$$

Thus, we can apply Lemma 4.7 to $A^{g_i}_{\varepsilon,a}$ in order to get

$$\begin{aligned} \|dA_\varepsilon^{g_{i_0}}\|_{L^1(C_\varepsilon^{g_{i_0}})} &= \|d\tilde{A}_\varepsilon^{g_{i_0}} - d\bar{A}_\varepsilon^{g_{i_0}}\|_{L^1(C_\varepsilon^{g_{i_0}})} \\ &\leq C\varepsilon \, \|\tilde{f}_\varepsilon - \iota^*_{\partial C} \bar{F}_\varepsilon^{g_{i_0}}\|_{L^1(\partial C_\varepsilon^{g_{i_0}})} \,, \end{aligned} \tag{4.35}$$

where the constant C is independent of $\varepsilon > 0$.

Second step: Now, we pass from one fixed good cube to the union $C_\varepsilon^g = \bigcup_{i=1}^{N_\varepsilon^g} C_\varepsilon^{g_i}$ of good cubes and define the *piecewise* smooth one-form $\tilde{A}_\varepsilon^g$ on C_ε^g by

$$\tilde{A}_\varepsilon^g = \tilde{A}_\varepsilon^{g_i} \qquad \text{on } C_\varepsilon^{g_i} \,. \tag{4.36}$$

Similarly, the piecewise constant two-form $\bar{F}_\varepsilon^g$ on C_ε^g is obtained. Then, we compute

$$\|d\tilde{A}_\varepsilon^g - F\|_{L^1(C_\varepsilon^g)} \leq \|d\tilde{A}_\varepsilon^g - \bar{F}_\varepsilon^g\|_{L^1(C_\varepsilon^g)} + \|\bar{F}_\varepsilon^g - F\|_{L^1(C_\varepsilon^g)} \,. \tag{4.37}$$

For the first term on the right-hand side of (4.37), we have

$$\begin{aligned} \|d\tilde{A}_\varepsilon^g - \bar{F}_\varepsilon^g\|_{L^1(C_\varepsilon^g)} &= \|d\tilde{A}_\varepsilon^g - d\bar{A}_\varepsilon^g\|_{L^1(C_\varepsilon^g)} \\ &= \sum_{i=1}^{N_\varepsilon^g} \|d\tilde{A}_\varepsilon^{g_i} - d\bar{A}_\varepsilon^{g_i}\|_{L^1(C_\varepsilon^{g_i})} \,. \end{aligned}$$

Applying the estimate (4.35) to each good cube $C_\varepsilon^{g_i}$, we then conclude that

$$\begin{aligned} \|d\tilde{A}_\varepsilon^g - \bar{F}_\varepsilon^g\|_{L^1(C_\varepsilon^g)} &\leq C\varepsilon \sum_{i=1}^{N_\varepsilon^g} \|\tilde{f}_\varepsilon - \iota^*_{\partial C} \bar{F}_\varepsilon^{g_i}\|_{L^1(\partial C_\varepsilon^{g_i})} \\ &= 2\, C\varepsilon \big(\|\tilde{f}_\varepsilon - {}^l\bar{F}_\varepsilon^g\|_{L^1(\partial C_\varepsilon^g)} + \|\tilde{f}_\varepsilon - {}^r\bar{F}_\varepsilon^g\|_{L^1(\partial C_\varepsilon^g)} \big) \,, \end{aligned}$$

where ${}^l\bar{F}_\varepsilon^g$, respectively ${}^r\bar{F}_\varepsilon^g$ denote the "left", respectively "right" trace of $\bar{F}_\varepsilon^g$ on ∂C_ε^g. Inserting this estimate into (4.37), we arrive at

$$\begin{aligned} \|d\tilde{A}_\varepsilon^g - F\|_{L^1(C_\varepsilon^g)} &\leq 2\, C\varepsilon \big(\|\tilde{f}_\varepsilon - {}^l\bar{F}_\varepsilon^g\|_{L^1(\partial C_\varepsilon^g)} + \|\tilde{f}_\varepsilon - {}^r\bar{F}_\varepsilon^g\|_{L^1(\partial C_\varepsilon^g)} \big) \\ &\quad + \|\bar{F}_\varepsilon^g - F\|_{L^1(C_\varepsilon^g)} \,. \end{aligned} \tag{4.38}$$

Third Step: We take the limit $\varepsilon \to 0$ in (4.38). For the first term on the right-hand side, it follows from the choice (4.11) of the cubic decomposition – which holds for both the "left" and the "right" trace – and from (4.30) that

$$\begin{aligned} &\lim_{\varepsilon\to 0} 2\, C\varepsilon\big(\|\tilde f_\varepsilon - {}^l\bar F^g_\varepsilon\|_{L^1(\partial C^g_\varepsilon)} + \|\tilde f_\varepsilon - {}^r\bar F^g_\varepsilon\|_{L^1(\partial C^g_\varepsilon)}\big) \\ \le\ &\lim_{\varepsilon\to 0} 2\, C\varepsilon\Big(\|\tilde f_\varepsilon - F\|_{L^1(\partial C^g_\varepsilon)} + \|F - {}^l\bar F^g_\varepsilon\|_{L^1(\partial C^g_\varepsilon)} \\ &\qquad + \|\tilde f_\varepsilon - F\|_{L^1(\partial C^g_\varepsilon)} + \|F - {}^r\bar F^g_\varepsilon\|_{L^1(\partial C^g_\varepsilon)}\Big) = 0\,. \end{aligned}$$

Because of Lebesgue's theorem, the second term on the right-hand side of (4.38) goes also to zero in the limit $\varepsilon \to 0$. Thus, we end up with

$$\lim_{\varepsilon\to 0} \|d\tilde A^g_\varepsilon - F\|_{L^1(C^g_\varepsilon)} = 0\,. \tag{4.39}$$

For every $\varepsilon > 0$, we then define

$$\tilde F^g_\varepsilon = d\tilde A^g_\varepsilon\,. \tag{4.40}$$

The resulting sequence $(\tilde F^g_\varepsilon)_{\varepsilon>0}$ hence satisfies the desired L^1-convergence (4.31) on the good cubes. We emphasize that – in contrast to $\tilde A^g_\varepsilon$ – the two-form $\tilde F^g_\varepsilon$ even has smooth tangential components on the faces of the cubic decomposition which is a consequence of the gauge invariance for the curvature in the Abelian $U(1)$-case. However, note that the normal components of $\tilde F^g_\varepsilon$ are not necessarily smooth on these boundaries.

Forth step: Since $\tilde F^g_\varepsilon$ is exact on every single good cube, we have that $d\tilde F^g_\varepsilon = 0$ on $C^{g_i}_\varepsilon$, for every $\varepsilon > 0$ and $i = 1, \ldots, N^g_\varepsilon$. We next show that the closeness $d\tilde F^g_\varepsilon = 0$ also holds on the union C^g_ε of the good cubes.

For this purpose, we consider now two neighboring good cubes $C^{g_i}_\varepsilon$ and $C^{g_j}_\varepsilon$ with common boundary $\partial C^{g_{i,j}}_\varepsilon$ and we claim that $d\tilde F^g_\varepsilon = 0$ is still true on the union $\overline{C^{g_i}_\varepsilon \cup C^{g_j}_\varepsilon}$. In order to establish the claim, we deduce from Stokes' theorem that

$$0 = \int_{C^{g_i}_\varepsilon \cup C^{g_j}_\varepsilon} d(\tilde F^g_\varepsilon\, \phi) = \int_{C^{g_i}_\varepsilon \cup C^{g_j}_\varepsilon} d\tilde F^g_\varepsilon\, \phi + \int_{C^{g_i}_\varepsilon \cup C^{g_j}_\varepsilon} \tilde F^g_\varepsilon \wedge d\phi\,,$$

for every $\phi \in C^\infty_0(C^{g_i}_\varepsilon \cup C^{g_j}_\varepsilon)$. This can be rewritten as

$$\begin{aligned} \int_{C^{g_i}_\varepsilon \cup C^{g_j}_\varepsilon} d\tilde F^g_\varepsilon\, \phi &= -\int_{C^{g_i}_\varepsilon \cup C^{g_j}_\varepsilon} \tilde F^g_\varepsilon \wedge d\phi \\ &= -\int_{C^{g_i}_\varepsilon} d\tilde A^{g_i}_\varepsilon \wedge d\phi - \int_{C^{g_j}_\varepsilon} d\tilde A^{g_j}_\varepsilon \wedge d\phi\,. \end{aligned}$$

Using again Stokes' theorem for both terms on the right-hand side and taking care about the orientation on the common boundary $\partial C^{g_{i,j}}_\varepsilon$, we arrive at

$$\begin{aligned} \int_{C^{g_i}_\varepsilon \cup C^{g_j}_\varepsilon} d\tilde F^g_\varepsilon \phi &= -\int_{\partial C^{g_i}_\varepsilon} \iota^*_{\partial C^{g_i}_\varepsilon}\big(d\tilde A^{g_i}_\varepsilon \phi\big) - \int_{\partial C^{g_j}_\varepsilon} \iota^*_{\partial C^{g_j}_\varepsilon}\big(d\tilde A^{g_j}_\varepsilon \phi\big) \\ &= -\int_{\partial C^{g_{i,j}}_\varepsilon} \iota^*_{\partial C^{g_i}_\varepsilon}\big(\tilde F^{g_i}_\varepsilon \phi\big) + \int_{\partial C^{g_{i,j}}_\varepsilon} \iota^*_{\partial C^{g_j}_\varepsilon}\big(\tilde F^{g_j}_\varepsilon \phi\big) = 0\,, \end{aligned}$$

showing the claim. – This completes the proof of the density on the good cubes except for the smoothness of the normal components (see Lemma 4.8).

Density on the Bad Cubes.

We now show the L^1-convergence on the bad cubes C_ε^b, i.e.,

$$\int_{C_\varepsilon^b} \big|F_\varepsilon^b(x) - F(x)\big|\, d^3x \longrightarrow 0 \qquad (\varepsilon \to 0)\,, \tag{4.41}$$

for a sequence $(F_\varepsilon^b)_{\varepsilon>0}$ in $\mathcal{F}_R(C_\varepsilon^b)$. – The triangular inequality gives

$$\int_{C_\varepsilon^b} \big|F_\varepsilon^b(x) - F(x)\big|\, d^3x \le \int_{C_\varepsilon^b} \big|F_\varepsilon^b(x)\big|\, d^3x + \int_{C_\varepsilon^b} \big|F(x)\big|\, d^3x\,. \tag{4.42}$$

As a consequence of Lemma 4.3 and dominated convergence, the second term on the right-hand side vanishes in the limit $\varepsilon \to 0$. Hence, for the density result on the bad cubes it remains to show that the first term on the right-hand side of (4.42) satisfies

$$\int_{C_\varepsilon^b} \big|F_\varepsilon^b(x)\big|\, d^3x \longrightarrow 0 \qquad (\varepsilon \to 0)\,. \tag{4.43}$$

We thus have to construct a sequence $(F_\varepsilon^b)_{\varepsilon>0}$ in $\mathcal{F}_R(C_\varepsilon^b)$ such that (4.43) holds.

For every bad cube $C_\varepsilon^{b_i}$, we assume that (see (4.21))

$$\frac{1}{2\pi} \int_{\partial C_\varepsilon^{b_i}} \iota^*_{\partial C_\varepsilon^{b_i}} F = d_\varepsilon^{b_i}\,, \tag{4.44}$$

where $0 \neq d_\varepsilon^{b_i} \in \mathbb{Z}$. This implies that the principal $U(1)$-bundles over the boundaries of the bad cubes are *not* trivial. However, the smooth approximation (4.30) for the restriction of $F \in \mathcal{F}_\mathbb{Z}(B^3)$ to the boundaries of the bad cubes piece together to give a globally well-defined smooth two-form. On the bad cubes, we can thus take the radial extension of the smooth approximation.

To be more precise, let $\pi_\varepsilon^{b_i}$ denote the radial projection on the boundary of $C_\varepsilon^{b_i}$. For some suitably choice of coordinates, this projection is explicitly given by the following expression:

$$\begin{aligned} \pi_\varepsilon^{b_i} : C_\varepsilon^{b_i} &\longrightarrow \partial C_\varepsilon^{b_i} \\ x &\longmapsto \frac{\varepsilon}{2} \frac{x - a_\varepsilon^i}{\|x - a_\varepsilon^i\|_*} + a_\varepsilon^i\,, \end{aligned}$$

where a_ε^i denotes the center of $C_\varepsilon^{b_i}$ and $\|x\|_* = \sup_{j=1,\dots,3} |x_j|$. Then, we define the radial extension

$$F_\varepsilon^{b_i} = (\pi_\varepsilon^{b_i})^* \tilde{f}_\varepsilon^{b_i} \qquad \text{on } C_\varepsilon^{b_i}\,, \tag{4.45}$$

where $\tilde{f}_\varepsilon^{b_i}$ denotes the smooth approximation of F on the boundary $\partial C_\varepsilon^{b_i}$. Since the radial projection $\pi_\varepsilon^{b_i}$ has only one point singularity at a_ε^i, we deduce that $F_\varepsilon^{b_i}$ is smooth on all of $C_\varepsilon^{b_i}$ except at the point a_ε^i. Moreover, using (4.44), a straightforward computation yields

$$dF_\varepsilon^{b_i} = 2\pi\, d_\varepsilon^{b_i} \delta_{a_\varepsilon^i}\, dx^1 \wedge dx^2 \wedge dx^3 \quad \text{in } \mathcal{D}'\,. \tag{4.46}$$

On the union $C_\varepsilon^b = \bigcup_{i=1}^{N_\varepsilon^b} C_\varepsilon^{b_i}$ of bad cubes, we define the two-form F_ε^b by

$$F_\varepsilon^b = F_\varepsilon^{b_i} \qquad \text{on } C_\varepsilon^{b_i}\,. \tag{4.47}$$

Because of the gauge invariance for the curvature in the $U(1)$-case and due to (4.46), we obtain that F^b_ε belongs to $\mathcal{F}_R(C^b_\varepsilon)$ except for the smoothness of the normal components on the boundaries of the cubic decomposition. – For simplicity, we will work from now on with F itself instead of its smooth approximation on the boundary.

In a next step, we have to check if the sequence $(F^b_\varepsilon)_{\varepsilon>0}$ in $\mathcal{F}_R(C^b_\varepsilon)$ just constructed satisfies (4.43). For this purpose, we compute

$$\int_{C^{b_i}_\varepsilon} \big|F^{b_i}_\varepsilon(x)\big|\, d^3x \leq C\varepsilon \int_{\partial C^{b_i}_\varepsilon} \big|F^{b_i}_\varepsilon(x)\big|\, d^2x = C\varepsilon \int_{\partial C^{b_i}_\varepsilon} \big|F(x)\big|\, d^2x\,. \tag{4.48}$$

Here, we used Fubini's theorem together with the fact that $F^{b_i}_\varepsilon$ is radially constant by the defining equation (4.45). This gives

$$\begin{aligned} \int_{C^b_\varepsilon} \big|F^b_\varepsilon(x)\big|\, d^3x \;&=\; \sum_{i=1}^{N^b_\varepsilon} \int_{C^{b_i}_\varepsilon} \big|F^{b_i}_\varepsilon(x)\big|\, d^3x \\ &\leq\; C\varepsilon \sum_{i=1}^{N^b_\varepsilon} \int_{\partial C^{b_i}_\varepsilon} \big|F(x)\big|\, d^2x = 2\, C\varepsilon\, \|F\|_{L^1(\partial C^b_\varepsilon)}\,. \end{aligned} \tag{4.49}$$

From (4.12), we then obtain

$$\int_{C^b_\varepsilon} \big|F^b_\varepsilon(x)\big|\, d^3x \leq C\, \|F\|_{L^1(B^3)} + C\varepsilon\, \|F - {}^l\bar{F}_\varepsilon\|_{L^1(\partial C_\varepsilon)}\,.$$

Though the second term on the right-hand side converges to zero in the limit $\varepsilon \to 0$ (see (4.11)), we conclude that (4.43) does not hold for (4.47), because of the first term on the right-hand side. Therefore, we are led to divide the bad cubes C^b_ε into smaller cubes.

For $0 < \varepsilon' \ll \varepsilon$ and $a' \in [0,\varepsilon']^3$, we divide one fixed bad cube $C^{b_{i_0}}_\varepsilon$ into open cubes $\{c^j_{\varepsilon',a'}\}_{j=1}^{N_{\varepsilon'}}$ of side length ε' centered at the points $a^j \in a' + \varepsilon'\,\mathbb{Z}^3$. The cubic decomposition of the fixed bad cube can be chosen in such a way that on its boundary $\partial c_{\varepsilon'}$ the following estimate holds:

$$\|F\|_{L^1(\partial c_{\varepsilon'})} \leq \frac{C}{\varepsilon'}\, \|F\|_{L^1(C^{b_{i_0}}_\varepsilon)}\,. \tag{4.50}$$

This follows from a Fubini type argument and a particular choice of $\bar{a}'_{\varepsilon'} \in [0,\varepsilon']^3$ very similar to the proof of Lemma 4.2.

For every little cube $c^j_{\varepsilon'}$, we now define the radial extension

$$\tilde{F}^j_{\varepsilon'} = (\pi^j_{\varepsilon'})^*\big(\iota^*_{\partial c^j_{\varepsilon'}} F\big)\,, \tag{4.51}$$

where $\pi^j_{\varepsilon'} : c^j_{\varepsilon'} \longrightarrow \partial c^j_{\varepsilon'}$ denotes the radial projection on the boundary. On the union $C^{b_{i_0}}_\varepsilon = \bigcup_{j=1}^{N_{\varepsilon'}} c^j_{\varepsilon'}$, we then define the two-form $\tilde{F}^{b_{i_0}}_\varepsilon$ by

$$\tilde{F}^{b_{i_0}}_\varepsilon = \tilde{F}^j_{\varepsilon'} \qquad \text{on } c^j_{\varepsilon'}\,. \tag{4.52}$$

By the same arguments as before, we see that $\tilde{F}^{b_{i_0}}_\varepsilon \in \mathcal{F}_R(C^{b_{i_0}}_\varepsilon)$.

Calculating as for (4.48) and (4.49), we obtain

$$\int_{C_\varepsilon^{b_{i_0}}} \left|\tilde{F}_\varepsilon^{b_{i_0}}(x)\right| d^3x \leq C\varepsilon' \, \|F\|_{L^1(\partial c_{\varepsilon'})} \,.$$

From (4.50), we conclude that

$$\int_{C_\varepsilon^{b_{i_0}}} \left|\tilde{F}_\varepsilon^{b_{i_0}}(x)\right| d^3x \leq C \int_{C_\varepsilon^{b_{i_0}}} \left|F(x)\right| d^3x \,. \tag{4.53}$$

In a next step, we repeat the same construction for every bad cube and then define on C_ε^b the two-form $\tilde{F}_\varepsilon^b$ by

$$\tilde{F}_\varepsilon^b = \tilde{F}_\varepsilon^{b_i} \qquad \text{on } C_\varepsilon^{b_i} \,. \tag{4.54}$$

Using (4.53), we then end up with

$$\int_{C_\varepsilon^b} \left|\tilde{F}_\varepsilon^b(x)\right| d^3x \leq C \int_{C_\varepsilon^b} \left|F(x)\right| d^3x \,. \tag{4.55}$$

As a consequence of Lemma 4.3 and dominated convergence, the sequence $(\tilde{F}_\varepsilon^b)_{\varepsilon>0}$ in $\mathcal{F}_R(C_\varepsilon^b)$ defined by (4.54) satisfies (4.43). Hence, the L^1-convergence (4.41) on the bad cubes is established.

Conclusion.

In summary, using (4.40) and (4.54) for the good and bad cubes, respectively, we define

$$F_k = \begin{cases} \tilde{F}_{1/k}^g & \text{on } C_{1/k}^g \\ \tilde{F}_{1/k}^b & \text{on } C_{1/k}^b \,, \end{cases} \tag{4.56}$$

for every $k \in \mathbb{N}$. We have shown that the resulting sequence $(F_k)_{k\in\mathbb{N}}$ approximates the given two-form $F \in \mathcal{F}_\mathbb{Z}(B^3)$ in the L^1-norm. Using Lemma 4.8 in order to mollify the normal components of F_k on the boundaries of the cubic decomposition we can replace each F_k by an element in $\mathcal{F}_R(B^3)$. This concludes the proof of the theorem. □

The construction of the approximation in the proof of Theorem 4.6 is based on an L^1-estimate for the harmonic extension of the Abelian Coulomb gauge. This is the content of the next lemma.

Lemma 4.7. *Let $F \in \Omega^2_{L^1}(B^3)$ and let $C \subset B^3$ be an open unit cube such that the restriction $\iota_{\partial C}^* F$ is smooth. Assume also that there exists a smooth one-form a such that*

$$\begin{cases} da &= \iota_{\partial C}^* F \\ d_* a &= 0 \end{cases} \qquad \textit{on } \partial C \,. \tag{4.57}$$

Moreover, consider the smooth harmonic extension $\tilde{A}$ of a on the cube C given by

$$\begin{cases} \Delta \tilde{A} = 0 & \textit{on } C \\ \iota_{\partial C}^* \tilde{A} = a \,, \quad \iota_{\partial C}^*(d_* \tilde{A}) = 0 & \textit{on } \partial C \,. \end{cases} \tag{4.58}$$

Then, the following estimate holds:

$$\|d\tilde{A}\|_{L^1(C)} \leq C \, \|\iota_{\partial C}^* F\|_{L^1(\partial C)} \,. \tag{4.59}$$

Proof. Let $0 \le s < 1$. Then, we have that

$$\|a\|_{W^{s,p}(\partial C)} \le C \, \|\iota^*_{\partial C} F\|_{L^1(\partial C)} \,, \tag{4.60}$$

for all $p < 2/(1+s)$. This follows from (4.57), the characterization of fractional Sobolev spaces by Bessel potentials and the results on Riesz potentials in E.M. Stein [43], Chapter V. By the Sobolev embedding theorem

$$W^{s,p}(\partial C) \hookrightarrow W^{1-1/q,q}(\partial C)$$

is a continuous embedding, if q satisfies the inequalities $s > 1 - 1/q$ and

$$s - \frac{2}{p} \ge \left(1 - \frac{1}{q}\right) - \frac{2}{q} = 1 - \frac{3}{q} \,.$$

Since $p < 2/(1+s)$, the last inequality translates to $q < 3/2$. Choosing $s > 1/3$ in order for the other inequality to hold, we have thus shown that there exists an integrability exponent $\tilde{p} > 1$ such that

$$W^{s,p}(\partial C) \hookrightarrow W^{1-1/\tilde{p},\tilde{p}}(\partial C)$$

is a continuous embedding. Together with (4.60), we then obtain

$$\|a\|_{W^{1-1/\tilde{p},\tilde{p}}(\partial C)} \le C \, \|a\|_{W^{s,p}(\partial C)} \le C \, \|\iota^*_{\partial C} F\|_{L^1(\partial C)} \,. \tag{4.61}$$

The trace theorem gives the existence of an one-form A on C with $\iota^*_{\partial C} A = a$ and $\iota^*_{\partial C}(d_* A) = 0$ such that

$$\|A\|_{W^{1,\tilde{p}}(C)} \le C \, \|a\|_{W^{1-1/\tilde{p},\tilde{p}}(\partial C)} \,.$$

Using (4.61), the left hand-side can also be bounded by

$$\|A\|_{W^{1,\tilde{p}}(C)} \le C \, \|\iota^*_{\partial C} F\|_{L^1(\partial C)} \,. \tag{4.62}$$

In a next step, we define

$$\bar{A} = A - \tilde{A} \,,$$

where $\tilde{A}$ is the smooth harmonic extension of a defined by the elliptic system (4.58). For the ellipticity of the boundary value problem (4.58) we refer to G. Schwarz [38], Section 1.6. It is easy to check that

$$\begin{cases} \Delta \bar{A} = \Delta A & \text{on } C \\ \iota^*_{\partial C} \bar{A} = 0 \,, \quad \iota^*_{\partial C}(d_* \bar{A}) = 0 & \text{on } \partial C \,. \end{cases}$$

Since $A \in \Omega^1_{W^{1,\tilde{p}}}(C)$ with $\tilde{p} > 1$, the theory of Calderón-Zygmund can be applied in order to show the following estimate:

$$\|d\bar{A}\|_{L^{\tilde{p}}(C)} \le C \, \|dA\|_{L^{\tilde{p}}(C)} \,. \tag{4.63}$$

For more informations on elliptic estimates for the boundary value problem (4.58) the reader should consult G. Schwarz [38], Lemma 3.4.7. From (4.63) it follows

$$\begin{aligned} \|d\tilde{A}\|_{L^{\tilde{p}}(C)} &\le \|d\bar{A}\|_{L^{\tilde{p}}(C)} + \|dA\|_{L^{\tilde{p}}(C)} \\ &\le (C+1)\|dA\|_{L^{\tilde{p}}(C)} \overset{(4.62)}{\le} C \, \|\iota^*_{\partial C} F\|_{L^1(\partial C)} \,. \end{aligned}$$

Using Hölder's inequality, we end up with

$$\|d\tilde{A}\|_{L^1(C)} \le C\|d\tilde{A}\|_{L^{\tilde{p}}(C)} \le C\,\|\iota^*_{\partial C}F\|_{L^1(\partial C)}\,.$$

□

In order to mollify the normal components of the approximation in the proof of Theorem 4.6 we need the following lemma:

Lemma 4.8. *Let $\varepsilon > 0$ be fixed and let $F_\varepsilon \in \Omega^2_{L^1}(B^3)$ with*

$$dF_\varepsilon = 2\pi \sum_{i=1}^{N^b_\varepsilon} d^i_\varepsilon \delta_{a^i_\varepsilon}\, dx^1 \wedge dx^2 \wedge dx^3 \quad \text{in } \mathcal{D}' \tag{4.64}$$

as constructed before with the help of a cubic decomposition. Then there exists a sequence $(\tilde{F}_{\varepsilon,\delta})_{\delta>0}$ in $\mathcal{F}_R(B^3)$ such that

$$\tilde{F}_{\varepsilon,\delta} \longrightarrow F_\varepsilon \qquad \text{in } L^1 \quad (\delta \to 0)\,. \tag{4.65}$$

Proof. We consider one face $S^{i_0}_\varepsilon$ of the boundary $\partial C^{i_0}_\varepsilon$ of a fixed cube $C^{i_0}_\varepsilon$ in the cubic decomposition C_ε. Then we define for $S^{i_0}_\varepsilon$ the neighborhood

$$N_{S^{i_0}_\varepsilon} = \big\{x \in B^3 \;:\; \operatorname{dist}(x, S^{i_0}_\varepsilon) \le \varepsilon/4\big\}\,.$$

Note that $a^i_\varepsilon \notin N_{S^{i_0}_\varepsilon}$ for all $1 \le i \le N^b_\varepsilon$. From (4.64) we thus deduce the existence of a $W^{1,1}$-one-form A^N such that $F_\varepsilon = dA^N$ on $N_{S^{i_0}_\varepsilon}$.

Next, we define a function $\phi \in C^\infty_c(N_{S^{i_0}_\varepsilon})$ such that $\phi \equiv 0$ on $\{x \in N_{S^{i_0}_\varepsilon} \;:\; \operatorname{dist}(x, S^{i_0}_\varepsilon) \ge \varepsilon/8\}$ and $\phi \equiv 1$ on $\{x \in N_{S^{i_0}_\varepsilon} \;:\; \operatorname{dist}(x, S^{i_0}_\varepsilon) \le \varepsilon/16\}$ and write $A^N = (1-\phi)A^N + \phi\, A^N$. Now let $(\eta_k)_{k\in\mathbb{N}}$ be a mollifying sequence and we set

$$\tilde{A}^N_k = (1-\phi)A^N + (\phi\, A^N) \star \eta_k\,.$$

Choosing k sufficiently large there exists $\tilde{A}^N_\varepsilon$ such that

$$\|\tilde{A}^N_\varepsilon - A^N\|_{W^{1,1}(N_{S^{i_0}_\varepsilon})} \le \frac{\delta}{M(\varepsilon)}\,, \tag{4.66}$$

where $M(\varepsilon)$ denotes the finite number of faces in the cubic decomposition C_ε. Then we define a two-form $\tilde{F}^N_\varepsilon$ on B^3 with $\tilde{F}^N_\varepsilon = F_\varepsilon$ on $B^3 \setminus N_{S^{i_0}_\varepsilon}$ and $\tilde{F}^N_\varepsilon = d\tilde{A}^N_\varepsilon$ on $N_{S^{i_0}_\varepsilon}$. By construction $\tilde{F}^N_\varepsilon$ is smooth on $\{x \in N_{S^{i_0}_\varepsilon} \;:\; \operatorname{dist}(x, S^{i_0}_\varepsilon) \le \varepsilon/16\}$ – recall also the smoothness properties of F_ε – and satisfies $d\tilde{F}^N_\varepsilon = 0$ on the neighborhood $N_{S^{i_0}_\varepsilon}$ of the face $S^{i_0}_\varepsilon$. Moreover, from (4.66) we deduce that

$$\|\tilde{F}^N_\varepsilon - F_\varepsilon\|_{L^1(B^3)} \le \frac{\delta}{M(\varepsilon)}\,. \tag{4.67}$$

In a next step, we iterate the previous procedure for each of the remaining faces in the cubic decomposition. More precisely, we take an arbitrary face in the cubic decomposition and a mollification of $\tilde{F}^N_\varepsilon$ on this face. It is important to mention that this mollification does not change the smoothness of $\tilde{F}^N_\varepsilon$. Finally, using (4.67) and the triangular inequality, we end up with a sequence $(\tilde{F}_{\varepsilon,\delta})_{\delta>0}$ in $\mathcal{F}_R(B^3)$ – and hence smooth on $B^3 \setminus \bigcup_{i=1}^{N^b_\varepsilon} a^i_\varepsilon$ – which satisfies (4.65). □

Chapter 5

The Non-Abelian Case

This chapter is devoted to the non-Abelian case. In the first section, we present the various classes of weak curvatures for the non-Abelian case. In particualar, motivated by Theorem 1.1 in the Abelian case, we will introduce the class $\mathcal{F}_{\mathbb{Z}}(B^5)$. The geometry of the non-Abelian case causes some difficulties in order to show density results. Especially, we have to deal with the problems of only locally defined weak curvatures and the effect of gauge transformations. In a first step, we can solve these problems by considering a weak notion of convergence and a gauge invariant quantity associated to $\mathcal{F}_{\mathbb{Z}}(B^5)$ (see Section 5.2). In turns out, however, that this class of weak curvature is not suitable to solve our problem of finding some strong closure of $\mathcal{F}_R(B^5)$ as described in the introduction. An other class of weak curvature namely $\mathcal{F}_{\otimes}(B^5)$ will be appropriate for a strong density result in the non-Abelian case. This will be explained in detail in Section 5.3.

5.1 Weak Curvatures in the Non-Abelian Case

We explain rigorously the different classes of weak curvatures for the non-Abelian case. We emphasize that the underlying geometry is of fundamental importance for later analytic considerations.

We start with the simplest class of weak curvatures given by

$$\begin{aligned} \mathcal{F}_\infty(\bar{B}^5) \quad = \quad \Big\{ F \in \Omega^2_\infty\big(\bar{B}^5, \mathfrak{su}(2)\big) \ \ : \ \ \exists \ A \in \Omega^1_\infty\big(\bar{B}^5, \mathfrak{su}(2)\big) \\ \text{st. } F \overset{\sigma}{=} dA + A \wedge A \text{ and } d\,\mathrm{Tr}(F \wedge F) = 0 \Big\} \,. \end{aligned} \tag{5.1}$$

For $F \in \mathcal{F}_\infty(\bar{B}^5)$ there exist hence a gauge transformation $\sigma : B^5 \longrightarrow SU(2)$ and a smooth $\mathfrak{su}(2)$-valued one-form A on B^5 such that $\sigma^{-1}F\sigma = dA + A \wedge A$ is smooth. Elements in $\mathcal{F}_\infty(\bar{B}^5)$ can be interpreted as curvature forms of some smooth trivial principal $SU(2)$-bundle over B^5. More details are given in Example B.2. The equation $d\,\mathrm{Tr}(F \wedge F) = 0$ is then actually a direct consequence of Bianchi's identity (B.12). For a direct proof we refer to M. Nakahara [33], Chapter 10 and for a more general result to G.L. Naber [32], Theorem 6.3.1. Using $d\,\mathrm{Tr}(F \wedge F) = 0$ and Stokes' theorem we obtain

$$c_2(\iota^*_{S^4} P) = \frac{1}{8\pi^2} \int_{S^4} \iota^*_{S^4} \mathrm{Tr}(F \wedge F) = 0 \,, \tag{5.2}$$

showing that the second Chern number of the restricted principal $SU(2)$-bundle $\iota_{S^4}^* P$ over S^4 vanishes which is equivalent to the triviality of $\iota_{S^4}^* P$.

The curvature as $\mathfrak{su}(2)$-valued two-form has no global meaning, if we introduce isolated singularities in the underlying principal $SU(2)$-bundle over B^5. This comes from the fact that the curvature for a non-trivial principal $SU(2)$-bundle – like the present bundle with isolated singularities – is not gauge invariant. Recall that in the Abelian $U(1)$-case, however, there exists for non-trivial bundles a well-defined global curvature form for $\mathcal{F}_R(B^3)$, since the curvature is a gauge invariant object for $U(1)$ (Section 4.1). So we are led to define

$$\begin{aligned}\mathcal{F}_R(B^5) = \Big\{ F \ &= \ \{F_i\}_{i\in I} \in \Omega^2_{L^2}\big(B^5, \mathfrak{su}(2)\big) \ : \ \exists \ a_1, \dots, a_N \in B^5 \ \text{ st.} \\ & F \overset{\sigma, loc.}{=} dA + A \wedge A \ \text{ locally smooth on } \ B^5 \setminus \{a_1, \dots, a_N\}\,, \\ & d\,\mathrm{Tr}(F \wedge F) = 8\pi^2 \left(\sum_{i=1}^N d_i \delta_{a_i} \right) dx^1 \wedge \ldots \wedge dx^5 \ \text{ in } \ \mathcal{D}' \Big\}\,. \end{aligned} \tag{5.3}$$

For weak curvatures in $\mathcal{F}_R(B^5)$ there exist hence locally on $B^5 \setminus \{a_1, \dots, a_N\}$ a gauge transformation σ and a smooth $\mathfrak{su}(2)$-valued one-form A such that $\sigma^{-1} F \sigma = dA + A \wedge A$ is locally smooth. In the case of $\mathcal{F}_R(B^5)$, the fundamental geometrical object consists in a smooth principal $SU(2)$-bundle $P_{U,g}$ over $B^5 \setminus \bigcup_{i=1}^N a_i$ with $U = \{U_i\}_{i\in I}$ an open covering of $B^5 \setminus \bigcup_{i=1}^N a_i$ and $g = \{g_{i,j}\}_{i,j\in I}$ corresponding transition functions. The family $F = \{F_i\}_{i\in I}$ of locally smooth $\mathfrak{su}(2)$-valued two-forms on U_i with $F_i = dA_i + A_i \wedge A_i$, where $A = \{A_i\}_{i\in I} \in \mathcal{A}_\infty(P_{U,g})$ defines an element of $\mathcal{F}_R(B^5)$.

The assumption $d\,\mathrm{Tr}(F \wedge F) = 8\pi^2 \left(\sum_{i=1}^N d_i \delta_{a_i}\right) dx^1 \wedge \ldots \wedge dx^5$ in (5.3) implies – by a similar calculation as in Section 4.1 – that

$$c_2(\iota^*_{\partial B^5_r(a_i)} P) = \frac{1}{8\pi^2} \int_{\partial B^5_r(a_i)} \iota^*_{\partial B^5_r(a_i)} \mathrm{Tr}(F \wedge F) = d_i\,, \tag{5.4}$$

for $i = 1, \dots, N$. In words, we have that the second Chern numbers of the principal $SU(2)$-bundles over small spheres around the singularities $a_1, \dots, a_N$ of the principal $SU(2)$-bundle $P_{U,g}$ over $B^5 \setminus \bigcup_{i=1}^N a_i$ are given by the integers $d_1, \dots, d_N$. Note that $c_2(\iota^*_{\partial B^5_r(a_i)} P)$ is independent of $r > 0$ small enough as shown in T. Isobe [25], Lemma 2.1.

In a next step, we introduce a third class of weak curvatures for the non-Abelian case which can be seen as analogue to $\mathcal{F}_{\mathbb{Z}}(B^3)$ in the Abelian case. More precisely, we define

$$\begin{aligned}\mathcal{F}_{\mathbb{Z}}(B^5) = \Big\{ F \in \Omega^2_{L^2}\big(B^5, \mathfrak{su}(2)\big) \ : \ & \iota^*_{\partial B^5_r(x)} F \ \text{ is a curvature form on } \ \partial B^5_r(x) \\ & \text{and } \ \frac{1}{8\pi^2} \int_{\partial B^5_r(x)} \iota^*_{\partial B^5_r(x)} \mathrm{Tr}(F \wedge F) \in \mathbb{Z}\,, \\ & \text{for } \forall x \in B^5 \ \text{ and a.e } \ 0 < r \leq \mathrm{dist}(x, \partial B^5) \Big\}\,. \end{aligned} \tag{5.5}$$

The geometrical interpretation of the class $\mathcal{F}_{\mathbb{Z}}(B^5)$ is the following: For all $x \in B^5$ and a.e. $0 < r \leq \mathrm{dist}(x, \partial B^5)$, there exists some smooth principal $SU(2)$-bundle $P_{U,g} \in$

$\mathcal{P}_{\infty}^{SU(2)}(\partial B_r^5(x))$ and a connection form $a = \{a_i\}_{i\in I} \in \mathcal{A}_{W^{1,2}}(P_{U,g})$ whose curvature form is given by

$$\iota^*_{\partial B_r^5(x)} F = da_i + a_i \wedge a_i \qquad \text{on } U_i \, . \tag{5.6}$$

Moreover, the second Chern number characterizing principal $SU(2)$-bundles over $\partial B_r^5(x)$ up to equivalence is given by the integer

$$c_2(P_{U,g}) = \frac{1}{8\pi^2} \int_{\partial B_r^5(x)} \iota^*_{\partial B_r^5(x)} \mathrm{Tr}(F \wedge F) \in \mathbb{Z} \, . \tag{5.7}$$

5.2 Weak Density Result in the Non-Abelian Case

In this section, we give a proof of Theorem 1.2 in the introduction. This weak density result will be obtained employing the technique of a cubic decomposition already encountered in Section 4.2.1.

Cubic Decomposition with its Good and Bad Cubes

We use the same notations as in Section 4.2.1 for the grid resulting from the cubic decomposition and show that it can be chosen as described in the next lemma.

Lemma 5.1. *Let $F \in \Omega^2_{L^2}\big(B^5, \mathfrak{su}(2)\big)$ and define the piecewise constant two-form $\bar{F}_{\varepsilon,a}$ on B^5 by $\bar{F}_{\varepsilon,a} = \bar{F}^i_{\varepsilon,a}$ on $C^i_{\varepsilon,a}$, where $\bar{F}^i_{\varepsilon,a}$ denotes the average of F in the cube $C^i_{\varepsilon,a}$. Then, for every $\varepsilon > 0$, there exists $\bar{a}_\varepsilon \in [0,\varepsilon]^5$ such that*

$$\lim_{\varepsilon \to 0} \varepsilon \, \| F - \, {}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon} \|_{L^2(\partial C_{\varepsilon,\bar{a}_\varepsilon})} = 0 \, , \tag{5.8}$$

where ${}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon}$ is defined as the "left" trace of $\bar{F}_{\varepsilon,a}$ on $\partial C_{\varepsilon,\bar{a}_\varepsilon}$, and also

$$\|F\|_{L^2(\partial C_{\varepsilon,\bar{a}_\varepsilon})} \leq \frac{C}{\varepsilon} \, \|F\|_{L^2(B^5)} + \|F - \, {}^l\bar{F}_{\varepsilon,\bar{a}_\varepsilon}\|_{L^2(\partial C_{\varepsilon,\bar{a}_\varepsilon})} \, . \tag{5.9}$$

Proof. The proof is the same as for Lemma 4.2. □

Consider now $F \in \mathcal{F}_{\mathbb{Z}}(B^5)$ and divide the cubes $\{C^i_{\varepsilon,a}\}_{i=1}^{N_\varepsilon}$ into "good" and "bad" ones. More precisely, we say that a cube $C^{g_i}_{\varepsilon,a}(\delta)$ in $\{C^i_{\varepsilon,a}\}_{i=1}^{N_\varepsilon}$ is a *good* cube for the parameter $0 < \delta < 1$ if

$$\|F\|_{L^2(\partial C^{g_i}_{\varepsilon,a}(\delta))} = \left(\int_{\partial C^{g_i}_{\varepsilon,a}(\delta)} \left| \iota^*_{\partial C^{g_i}_{\varepsilon,a}(\delta)} F(x) \right|^2 d^4x \right)^{1/2} \leq \delta \, . \tag{5.10}$$

Since

$$\left| \mathrm{Tr}(F \wedge F) \right| \leq |F|^2 \, , \tag{5.11}$$

as shown in Appendix A, we obtain

$$0 \leq \left| \int_{\partial C^{g_i}_{\varepsilon,a}(\delta)} \iota^*_{\partial C^{g_i}_{\varepsilon,a}(\delta)} \mathrm{Tr}(F \wedge F) \right| \leq \int_{\partial C^{g_i}_{\varepsilon,a}(\delta)} \left| \iota^*_{\partial C^{g_i}_{\varepsilon,a}(\delta)} F(x) \right|^2 d^4x < 1 \, ,$$

and hence, using the definition (5.5) for $\mathcal{F}_{\mathbb{Z}}(B^5)$,

$$\int_{\partial C^{g_i}_{\varepsilon,a}(\delta)} \iota^*_{\partial C^{g_i}_{\varepsilon,a}(\delta)} \mathrm{Tr}(F \wedge F) = 0\,. \tag{5.12}$$

The *bad* cubes $C^{b_i}_{\varepsilon,a}(\delta)$ for the parameter δ are the remaining ones, i.e.,

$$0 \neq \frac{1}{8\pi^2} \int_{\partial C^{b_i}_{\varepsilon,a}(\delta)} \iota^*_{\partial C^{b_i}_{\varepsilon,a}(\delta)} \mathrm{Tr}(F \wedge F) \in \mathbb{Z}\,. \tag{5.13}$$

As a consequence, for every $0 < \delta < 1$, we can write

$$\bigcup_{i=1}^{N_\varepsilon} C^i_{\varepsilon,a} = \bigcup_{i=1}^{N^g_\varepsilon(\delta)} C^{g_i}_{\varepsilon,a}(\delta) \cup \bigcup_{i=1}^{N^b_\varepsilon(\delta)} C^{b_i}_{\varepsilon,a}(\delta)\,,$$

where $N_\varepsilon = N^g_\varepsilon(\delta) + N^b_\varepsilon(\delta)$ with $N^g_\varepsilon(\delta)$ and $N^b_\varepsilon(\delta)$ the number of good, respectively, bad cubes for δ in B^5. In the following, the sets $\bigcup_{i=1}^{N^g_\varepsilon(\delta)} C^{g_i}_{\varepsilon,a}(\delta)$ and $\bigcup_{i=1}^{N^b_\varepsilon(\delta)} C^{b_i}_{\varepsilon,a}(\delta)$ will be denoted by $C^g_{\varepsilon,a}(\delta)$ and $C^b_{\varepsilon,a}(\delta)$, respectively. – Moreover, with the choice as in Lemma 5.1 for the cubic decomposition, the volume of the bad cubes tends to zero for every parameter δ as the decomposition becomes smaller meaning that

$$\lim_{\varepsilon\to 0} \mu\big(C^b_{\varepsilon,\bar{a}_\varepsilon}(\delta)\big) = 0\,, \tag{5.14}$$

for every $0 < \delta < 1$. This is Lemma 4.3 in the Abelian case.

Smoothing on the Boundary of the Cubic Decomposition

The following considerations will be used later in the proof of Theorem 1.2. – In the non-Abelian case we are dealing with $W^{2,2}$-Sobolev principal $SU(2)$-bundles. They are critical in four-dimensions and hence the density results of Section 3.1 apply. For completeness, we now give a precise formulation of these results in the non-Abelian case.

Let M be a four-dimensional Riemannian manifold and $\mathcal{P}^{SU(2)}_{W^{2,2}}(M)$ the space of $W^{2,2}$-Sobolev principal $SU(2)$-bundles over M. For connections $A \in \mathcal{A}_{W^{1,2}}(P_{U,g})$ on $P_{U,g} \in \mathcal{P}^{SU(2)}_{W^{2,2}}(M)$, we have the following density result:

Proposition 5.2. *Let $P_{U,g} \in \mathcal{P}^{SU(2)}_\infty(M)$ and $A = \{A_i\}_{i\in I}$ a $W^{1,2}$-Sobolev connection on the smooth principal $SU(2)$-bundle $P_{U,g}$. Then, there exists a sequence $(\tilde{A}^\varepsilon)_{\varepsilon>0}$ of smooth connections on $P_{U,g}$ such that*

$$\lim_{\varepsilon\to 0} \|\tilde{A}^\varepsilon_i - A_i\|_{W^{1,2}(U_i)} = 0\,, \tag{5.15}$$

for every $i \in I$.

Corollary 5.3. *Let $A = \{A_i\}_{i\in I} \in \mathcal{A}_{W^{1,2}}(P_{U,g})$ and let $F(A)$ be its curvature. Assume also that there exists a sequence $(\tilde{A}^\varepsilon)_{\varepsilon>0}$ of smooth connections such that $\lim_{\varepsilon\to 0}\|\tilde{A}^\varepsilon_i - A_i\|_{W^{1,2}(U_i)} = 0$, for every $i \in I$. Then, for the sequence of smooth curvatures $\big(F(\tilde{A}^\varepsilon)\big)_{\varepsilon>0}$ associated to $\tilde{A}^\varepsilon$, we have that*

$$\lim_{\varepsilon\to 0} \|F_i(\tilde{A}^\varepsilon) - F_i(A)\|_{L^2(U_i)} = 0\,, \tag{5.16}$$

for every $i \in I$.

Weak Density Result

The weak density result in Theorem 1.2 is a first step in order to solve the main problem of the present dissertation, namely to determine the strong closure of $\mathcal{F}_R(B^5)$. Instead of considering the *locally* defined curvature forms F, we will work with the real-valued four-form $\mathrm{Tr}(F \wedge F)$ which is *globally* well-defined (see Section 2.1). Moreover, using a weak notion of convergence, we can avoid the effect of gauge transformations as explained later in Section 5.3.

We now establish that for every $F \in \mathcal{F}_{\mathbb{Z}}(B^5)$, there exists a sequence $(F_k)_{k\in\mathbb{N}}$ in $\mathcal{F}_R(B^5)$ such that the following weak convergence holds:

$$\int_{B^5} \mathrm{Tr}(F_k \wedge F_k) \wedge \omega \overset{k\to\infty}{\longrightarrow} \int_{B^5} \mathrm{Tr}(F \wedge F) \wedge \omega\,, \qquad \forall \omega \in \Omega^1_{C^0}(B^5)\,, \tag{5.17}$$

This convergence has a global meaning. Note also that as dual pairing of an integrable four-form with a (compactly supported) continuous one-form it can be interpreted as weak*-convergence for L^1, since L^1 embedds in the Banach space of Radon measures being the dual of compactly supported continuous functions (see H. Brezis [5], Comments on Chapter IV). – We now prove (5.17).

Proof. Let $F \in \mathcal{F}_{\mathbb{Z}}(B^5)$ and choose a parameter $0 < \delta < 1$. Then we devide B^5 into open cubes $\{C^i_{\varepsilon,\bar{a}_\varepsilon}\}_{i=1}^{N_\varepsilon}$ in such a way that Lemma 5.1 and (5.14) hold for the parameter δ. The chosen cubic decomposition will be simply denoted by $C_\varepsilon = C^g_\varepsilon(\delta) \cup C^b_\varepsilon(\delta)$, for every $\varepsilon > 0$.

Mollification on the Boundary of the Cubic Decomposition.

Let $\varepsilon > 0$. From the defining equation (5.5) for $\mathcal{F}_{\mathbb{Z}}(B^5)$, we have that F is a curvature form on the boundary $\partial C_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} \partial C^i_\varepsilon$ of the cubic decomposition. Denote by $a_\varepsilon = \{a_{\varepsilon,i}\}_{i\in I(\varepsilon)}$ the connection on some $P_{U_\varepsilon,g_\varepsilon} \in \mathcal{P}^{SU(2)}_{W^{2,2}}(\partial C_\varepsilon)$ whose curvature form is given by the restriction of F to the boundary of the cubic decomposition C_ε. Applying Proposition 5.2, we deduce the existence of a smooth connection $\tilde{a}^k_\varepsilon = \{\tilde{a}^k_{\varepsilon,i}\}_{i\in I(\varepsilon)}$ on $P_{U_\varepsilon,g_\varepsilon}$ such that

$$\|\tilde{a}^k_{\varepsilon,i} - a_{\varepsilon,i}\|_{W^{1,2}(U_{\varepsilon,i})} \leq \frac{1}{k}\,, \tag{5.18}$$

for every $i \in I(\varepsilon)$. Its associated *smooth* curvature $f(\tilde{a}^k_\varepsilon) = \{f_i(\tilde{a}^k_\varepsilon)\}_{i\in I(\varepsilon)}$ satisfies, as a consequence of Corollary 5.3,

$$\|f_i(\tilde{a}^k_\varepsilon) - \iota^*_{U_{\varepsilon,i}} F\|_{L^2(U_{\varepsilon,i})} \leq \frac{C}{k}\,, \tag{5.19}$$

for every $i \in I(\varepsilon)$. After summation over $i \in I(\varepsilon)$ and choosing $1/k$ – in function of $\varepsilon > 0$ – small enough, we can achieve that

$$\sum_{i\in I(\varepsilon)} \|f_i(\tilde{a}^k_\varepsilon) - \iota^*_{U_{\varepsilon,i}} F\|_{L^2(U_{\varepsilon,i})} \leq \varepsilon\,. \tag{5.20}$$

Density on the Good Cubes.

In order to establish the weak density result given in Theorem 1.2 on the good cubes, we will first show the existence of a sequence $(F^g_{\varepsilon,\delta})_{\varepsilon>0}$ in $\mathcal{F}_R(C^g_\varepsilon(\delta))$ such that (5.17) holds up to δ, i.e.,

$$\limsup_{\varepsilon\to 0} \int_{C^g_\varepsilon(\delta)} \left|\big(\mathrm{Tr}(F^g_{\varepsilon,\delta}\wedge F^g_{\varepsilon,\delta}) - \mathrm{Tr}(F\wedge F)\big)\wedge\omega\right| d^5x \le C\,\delta\,, \tag{5.21}$$

for every $\omega\in\Omega^1_{C^0}(B^5)$.

First step: For a fixed good cube $C^{g_{i_0}}_\varepsilon(\delta)$, we consider the constant $\mathfrak{su}(2)$-valued two-form $\bar F^{g_{i_0}}_{\varepsilon,\delta}$ given by the average of F in $C^{g_{i_0}}_\varepsilon(\delta)$. Since $\bar F^{g_{i_0}}_{\varepsilon,\delta}$ is constant, and hence

$$\int_{\partial C^{g_{i_0}}_\varepsilon(\delta)} \iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta} = 0\,,$$

there exists a unique $\bar a^{g_{i_0}}_{\varepsilon,\delta}\in\Omega^1\big(\partial C^{g_{i_0}}_\varepsilon(\delta),\mathfrak{su}(2)\big)$ such that

$$\begin{cases} d\bar a^{g_{i_0}}_{\varepsilon,\delta} &= \iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\\ d_*\bar a^{g_{i_0}}_{\varepsilon,\delta} &= 0\end{cases} \qquad \text{on } \partial C^{g_{i_0}}_\varepsilon(\delta)\,, \tag{5.22}$$

with the upper bound

$$\|\bar a^{g_{i_0}}_{\varepsilon,\delta}\|_{W^{1,2}(\partial C^{g_{i_0}}_\varepsilon(\delta))} \le C\,\|\iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,. \tag{5.23}$$

This is a classical Calderón-Zygmund L^2-estimate and we refer to G. Schwarz [38], Theorem 3.1.1, for more details. Together with the continuity of the multiplication $L^4\otimes L^4\longrightarrow L^2$ and the Sobolev embedding $W^{1,2}\hookrightarrow L^4$ in four dimensions, this leads to the following estimate, which will be useful in the following:

$$\begin{aligned}\|\bar a^{g_{i_0}}_{\varepsilon,\delta}\wedge\bar a^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} &\le \|\bar a^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^4(\partial C^{g_{i_0}}_\varepsilon(\delta))}\\ &\le C\,\|\bar a^{g_{i_0}}_{\varepsilon,\delta}\|^2_{W^{1,2}(\partial C^{g_{i_0}}_\varepsilon(\delta))} \overset{(5.23)}{\le} C\,\|\iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,.\end{aligned} \tag{5.24}$$

Moreover, as in the Abelian case, it is not difficult to construct an $\mathfrak{su}(2)$-valued one-form $\bar A^{g_{i_0}}_{\varepsilon,\delta}$ such that

$$\begin{cases} d\bar A^{g_{i_0}}_{\varepsilon,\delta} &= \bar F^{g_{i_0}}_{\varepsilon,\delta}\\ d_*\bar A^{g_{i_0}}_{\varepsilon,\delta} &= 0\end{cases} \qquad \text{on } C^{g_{i_0}}_\varepsilon(\delta)\,,$$

and hence

$$\begin{cases} \Delta\bar A^{g_{i_0}}_{\varepsilon,\delta} = 0 & \text{on } C^{g_{i_0}}_\varepsilon(\delta)\\ \iota^*_{\partial C}\bar A^{g_{i_0}}_{\varepsilon,\delta} = \bar a^{g_{i_0}}_{\varepsilon,\delta}\,,\quad \iota^*_{\partial C}(d_*\bar A^{g_{i_0}}_{\varepsilon,\delta}) = 0 & \text{on } \partial C^{g_{i_0}}_\varepsilon(\delta)\,.\end{cases} \tag{5.25}$$

Applying Lemma 5.4 to $\bar A^{g_{i_0}}_{\varepsilon,\delta}$, we then obtain

$$\|\bar A\|_{W^{1,5/2}(C^{g_{i_0}}_\varepsilon(\delta))} \le C\,\|\iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,. \tag{5.26}$$

Together with the continuity of the multiplication $L^4 \otimes L^4 \longrightarrow L^2$ and the Sobolev embedding $W^{1,5/2} \hookrightarrow L^5$ in five dimensions, this leads to the following estimate, which will be useful in the following:

$$\begin{aligned} \|\bar{A}^{g_{i_0}}_{\varepsilon,\delta} \wedge \bar{A}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))} &\leq \|\bar{A}^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^4(C^{g_{i_0}}_\varepsilon(\delta))} \leq C\,\|\bar{A}^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^5(C^{g_{i_0}}_\varepsilon(\delta))} \\ &\leq C\,\|\bar{A}^{g_{i_0}}_{\varepsilon,\delta}\|^2_{W^{1,5/2}(C^{g_{i_0}}_\varepsilon(\delta))} \overset{(5.26)}{\leq} C\,\|\iota^*_{\partial C}\bar{F}^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,. \end{aligned} \tag{5.27}$$

Recall that by assumption $\iota^*_{\partial C}F$ is a curvature form for some principal $SU(2)$-bundle over the boundary of the fixed good cube $C^{g_{i_0}}_\varepsilon(\delta)$. Since (5.12) holds for the good cubes, the second Chern number of this principal $SU(2)$-bundle vanishes implying that it is *trivial.* Moreover, due to the small energy assumption (5.10) on the boundary of the good cubes, we can assume that

$$\|\tilde{f}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} = \left(\int_{\partial C^{g_{i_0}}_\varepsilon(\delta)} \left|\tilde{f}^{g_{i_0}}_{\varepsilon,\delta}(x)\right|^2 d^4x\right)^{1/2} \leq \delta\,, \tag{5.28}$$

where $\tilde{f}^{g_{i_0}}_{\varepsilon,\delta}$ denotes the smooth approximation for $\iota^*_{\partial C}F$ constructed before (see (5.20)). It is important to mention that because of the triviality of the underlying principal $SU(2)$-bundle, we have the *global* representation $\tilde{f}^{g_{i_0}}_{\varepsilon,\delta} = d\tilde{a}^{g_{i_0}}_{\varepsilon,\delta} + \tilde{a}^{g_{i_0}}_{\varepsilon,\delta} \wedge \tilde{a}^{g_{i_0}}_{\varepsilon,\delta}$ on $\partial C^{g_{i_0}}_\varepsilon(\delta)$. We then define the smooth harmonic extension $\tilde{A}^{g_{i_0}}_{\varepsilon,\delta}$ of $\tilde{a}^{g_{i_0}}_{\varepsilon,\delta}$ by

$$\begin{cases} \Delta\tilde{A}^{g_{i_0}}_{\varepsilon,\delta} = 0 & \text{on } C^{g_{i_0}}_\varepsilon(\delta) \\ \iota^*_{\partial C}\tilde{A}^{g_{i_0}}_{\varepsilon,\delta} = \tilde{a}^{g_{i_0}}_{\varepsilon,\delta}\,, \quad \iota^*_{\partial C}(d_*\tilde{A}^{g_{i_0}}_{\varepsilon,\delta}) = 0 & \text{on } \partial C^{g_{i_0}}_\varepsilon(\delta)\,. \end{cases} \tag{5.29}$$

Second step: On the fixed good cube $C^{g_{i_0}}_\varepsilon(\delta)$, we define with (5.25) the "constant" curvature

$$\hat{F}^{g_{i_0}}_{\varepsilon,\delta} = \bar{F}^{g_{i_0}}_{\varepsilon,\delta} + \bar{A}^{g_{i_0}}_{\varepsilon,\delta} \wedge \bar{A}^{g_{i_0}}_{\varepsilon,\delta} = d\bar{A}^{g_{i_0}}_{\varepsilon,\delta} + \bar{A}^{g_{i_0}}_{\varepsilon,\delta} \wedge \bar{A}^{g_{i_0}}_{\varepsilon,\delta}\,, \tag{5.30}$$

and with (5.29) the smooth "harmonic" curvature

$$\tilde{F}^{g_{i_0}}_{\varepsilon,\delta} = d\tilde{A}^{g_{i_0}}_{\varepsilon,\delta} + \tilde{A}^{g_{i_0}}_{\varepsilon,\delta} \wedge \tilde{A}^{g_{i_0}}_{\varepsilon,\delta}\,. \tag{5.31}$$

Now, we consider the difference

$$\int_{C^{g_{i_0}}_\varepsilon(\delta)} \left|\Big(\mathrm{Tr}(\tilde{F}^{g_{i_0}}_{\varepsilon,\delta} \wedge \tilde{F}^{g_{i_0}}_{\varepsilon,\delta}) - \mathrm{Tr}(F \wedge F)\Big) \wedge \bar{\omega}^{g_{i_0}}_{\varepsilon,\delta}\right| d^5x\,, \tag{5.32}$$

where $\bar{\omega}^{g_{i_0}}_{\varepsilon,\delta}$ denotes the average of the continuous one-form ω in $C^{g_{i_0}}_\varepsilon(\delta)$. Using the triangular inequality this difference can be bounded by

$$\begin{aligned} &\int_{C^{g_{i_0}}_\varepsilon(\delta)} \left|\Big(\mathrm{Tr}(\tilde{F}^{g_{i_0}}_{\varepsilon,\delta} \wedge \tilde{F}^{g_{i_0}}_{\varepsilon,\delta}) - \mathrm{Tr}(\hat{F}^{g_{i_0}}_{\varepsilon,\delta} \wedge \hat{F}^{g_{i_0}}_{\varepsilon,\delta})\Big) \wedge \bar{\omega}^{g_{i_0}}_{\varepsilon,\delta}\right| d^5x \\ +\; &\int_{C^{g_{i_0}}_\varepsilon(\delta)} \left|\Big(\mathrm{Tr}(\hat{F}^{g_{i_0}}_{\varepsilon,\delta} \wedge \hat{F}^{g_{i_0}}_{\varepsilon,\delta}) - \mathrm{Tr}(F \wedge F\Big) \wedge \bar{\omega}^{g_{i_0}}_{\varepsilon,\delta}\right| d^5x\,. \end{aligned} \tag{5.33}$$

For the socond term in (5.33), we compute

$$\begin{aligned}\mathrm{Tr}(\hat{F}^{g_{i_0}}_{\varepsilon,\delta}\wedge\hat{F}^{g_{i_0}}_{\varepsilon,\delta})-\mathrm{Tr}(F\wedge F) &= \mathrm{Tr}\big((\hat{F}^{g_{i_0}}_{\varepsilon,\delta}\wedge\hat{F}^{g_{i_0}}_{\varepsilon,\delta})-(F\wedge F)\big)\\ &= \mathrm{Tr}\big((\hat{F}^{g_{i_0}}_{\varepsilon,\delta}-F)\wedge(\hat{F}^{g_{i_0}}_{\varepsilon,\delta}+F)\big)\,,\end{aligned}$$

since $\hat{F}^{g_{i_0}}_{\varepsilon,\delta}\wedge F=F\wedge\hat{F}^{g_{i_0}}_{\varepsilon,\delta}$ holds due to the commutativity property of the wedge product (see Appendix A). This implies that

$$\begin{aligned}&\int_{C^{g_{i_0}}_\varepsilon(\delta)}\Big|\Big(\mathrm{Tr}(\hat{F}^{g_{i_0}}_{\varepsilon,\delta}\wedge\hat{F}^{g_{i_0}}_{\varepsilon,\delta})-\mathrm{Tr}(F\wedge F\Big)\wedge\bar{\omega}^{g_{i_0}}_{\varepsilon,\delta}\Big|\,d^5x\\ =\;&\int_{C^{g_{i_0}}_\varepsilon(\delta)}\Big|\Big(\mathrm{Tr}\big((\hat{F}^{g_{i_0}}_{\varepsilon,\delta}-F)\wedge(\hat{F}^{g_{i_0}}_{\varepsilon,\delta}+F)\big)\Big)\wedge\bar{\omega}^{g_{i_0}}_{\varepsilon,\delta}\Big|\,d^5x\\ \leq\;& C\int_{C^{g_{i_0}}_\varepsilon(\delta)}\Big|\mathrm{Tr}\big((\hat{F}^{g_{i_0}}_{\varepsilon,\delta}-F)\wedge(\hat{F}^{g_{i_0}}_{\varepsilon,\delta}+F)\big)\Big|\,d^5x\,.\end{aligned}\tag{5.34}$$

Hölder's inequality then gives the following bound for integral on the right-hand side:

$$\|\hat{F}^{g_{i_0}}_{\varepsilon,\delta}-F\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))}\|\hat{F}^{g_{i_0}}_{\varepsilon,\delta}+F\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))}\leq C\,\|\hat{F}^{g_{i_0}}_{\varepsilon,\delta}-F\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))}\,,$$

where the constant C depends on the L^2-norm of F. From the definition (5.30) for the "constant" curvature and (5.27) together with the triangular inequality, we obtain the estimate

$$\begin{aligned}\|\hat{F}^{g_{i_0}}_{\varepsilon,\delta}-F\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))} &\leq \|\bar{F}^{g_{i_0}}_{\varepsilon,\delta}-F\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))}+\|\bar{A}^{g_{i_0}}_{\varepsilon,\delta}\wedge\bar{A}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))}\\ &\leq \|\bar{F}^{g_{i_0}}_{\varepsilon,\delta}-F\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))}+C\,\|\iota^*_{\partial C}\bar{F}^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,.\end{aligned}$$

Inserting this into (5.34), it follows for the second term in (5.33) that

$$\begin{aligned}&\int_{C^{g_{i_0}}_\varepsilon(\delta)}\Big|\Big(\mathrm{Tr}(\hat{F}^{g_{i_0}}_{\varepsilon,\delta}\wedge\hat{F}^{g_{i_0}}_{\varepsilon,\delta})-\mathrm{Tr}(F\wedge F\Big)\wedge\bar{\omega}^{g_{i_0}}_{\varepsilon,\delta}\Big|\,d^5x\\ \leq\;& C\,\|\bar{F}^{g_{i_0}}_{\varepsilon,\delta}-F\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))}+C\,\|\iota^*_{\partial C}\bar{F}^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,.\end{aligned}\tag{5.35}$$

In order to get an estimate for the first term in (5.33), note that

$$d\tilde{\Omega}^{g_{i_0}}_{\varepsilon,\delta}=d\big(\mathrm{Tr}(\tilde{F}^{g_{i_0}}_{\varepsilon,\delta}\wedge\tilde{F}^{g_{i_0}}_{\varepsilon,\delta})\big)=0\,,\tag{5.36}$$

and

$$d\hat{\Omega}^{g_{i_0}}_{\varepsilon,\delta}=d\big(\mathrm{Tr}(\hat{F}^{g_{i_0}}_{\varepsilon,\delta}\wedge\hat{F}^{g_{i_0}}_{\varepsilon,\delta})\big)=0\,,\tag{5.37}$$

where we used Bianchi's identity (B.12) and the notation $\tilde{\Omega}^{g_{i_0}}_{\varepsilon,\delta}=\mathrm{Tr}(\tilde{F}^{g_{i_0}}_{\varepsilon,\delta}\wedge\tilde{F}^{g_{i_0}}_{\varepsilon,\delta})$, $\hat{\Omega}^{g_{i_0}}_{\varepsilon,\delta}=\mathrm{Tr}(\hat{F}^{g_{i_0}}_{\varepsilon,\delta}\wedge\hat{F}^{g_{i_0}}_{\varepsilon,\delta})$. Moreover, since obviously $d\bar{\omega}^{g_{i_0}}_{\varepsilon,\delta}=0$, there exists due to Poincaré's lemma a function $\phi^{g_{i_0}}_{\varepsilon,\delta}$ on the fixed good cube such that $d\phi^{g_{i_0}}_{\varepsilon,\delta}=\bar{\omega}^{g_{i_0}}_{\varepsilon,\delta}$. It is not difficult to check $\|\phi^{g_{i_0}}_{\varepsilon,\delta}\|_{C^0}\leq C$, where the constant depends on the average of ω on $C^{g_{i_0}}_\varepsilon(\delta)$ and on the side length ε of the good cube. Stokes' theorem then gives

$$\begin{aligned}\int_{C^{g_{i_0}}_\varepsilon(\delta)}(\tilde{\Omega}^{g_{i_0}}_{\varepsilon,\delta}-\hat{\Omega}^{g_{i_0}}_{\varepsilon,\delta})\wedge\bar{\omega}^{g_{i_0}}_{\varepsilon,\delta}\,d^5x &= \int_{C^{g_{i_0}}_\varepsilon(\delta)}(\tilde{\Omega}^{g_{i_0}}_{\varepsilon,\delta}-\hat{\Omega}^{g_{i_0}}_{\varepsilon,\delta})\wedge d\phi^{g_{i_0}}_{\varepsilon,\delta}\,d^5x\\ &\overset{(5.36),(5.37)}{=} \int_{C^{g_{i_0}}_\varepsilon(\delta)} d\big((\tilde{\Omega}^{g_{i_0}}_{\varepsilon,\delta}-\hat{\Omega}^{g_{i_0}}_{\varepsilon,\delta})\phi^{g_{i_0}}_{\varepsilon,\delta}\big)\,d^5x\\ &= \int_{\partial C^{g_{i_0}}_\varepsilon(\delta)}\iota^*_{\partial C}\big((\tilde{\Omega}^{g_{i_0}}_{\varepsilon,\delta}-\hat{\Omega}^{g_{i_0}}_{\varepsilon,\delta})\phi^{g_{i_0}}_{\varepsilon,\delta}\big)\,d^4x\,.\end{aligned}$$

Thus, using $\|\phi^{g_{i_0}}_{\varepsilon,\delta}\|_{C^0} \le C$, we obtain for the first term in (5.33) that

$$\int_{C^{g_{i_0}}_\varepsilon(\delta)} \big|(\tilde\Omega^{g_{i_0}}_{\varepsilon,\delta} - \hat\Omega^{g_{i_0}}_{\varepsilon,\delta}) \wedge \bar\omega^{g_{i_0}}_{\varepsilon,\delta}\big|\, d^5x \le C \int_{\partial C^{g_{i_0}}_\varepsilon(\delta)} \big|\iota^*_{\partial C}\big((\tilde\Omega^{g_{i_0}}_{\varepsilon,\delta} - \hat\Omega^{g_{i_0}}_{\varepsilon,\delta})\phi^{g_{i_0}}_{\varepsilon,\delta}\big)\big|\, d^4x\,. \tag{5.38}$$

With a standard computation – as in the estimate for the second term in (5.33) – the integral on the right-hand side can be bounded by

$$\begin{aligned} &\|\iota^*_{\partial C}\tilde F^{g_{i_0}}_{\varepsilon,\delta} - \iota^*_{\partial C}\hat F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\ \|\iota^*_{\partial C}\tilde F^{g_{i_0}}_{\varepsilon,\delta} + \iota^*_{\partial C}\hat F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ \le\ & C\,\|\iota^*_{\partial C}\tilde F^{g_{i_0}}_{\varepsilon,\delta} - \iota^*_{\partial C}\hat F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,. \end{aligned}$$

From (5.24) and (5.30) together with the triangular inequality, it then follows

$$\begin{aligned} \|\iota^*_{\partial C}\tilde F^{g_{i_0}}_{\varepsilon,\delta} - \iota^*_{\partial C}\hat F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \ \le\ & \|\tilde f^{g_{i_0}}_{\varepsilon,\delta} - \iota^*_{\partial C}F\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + \|\iota^*_{\partial C}F - \iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + \|\bar a^{g_{i_0}}_{\varepsilon,\delta} \wedge \bar a^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ \le\ & \|\tilde f^{g_{i_0}}_{\varepsilon,\delta} - \iota^*_{\partial C}F\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + \|\iota^*_{\partial C}F - \iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + C\,\|\iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,. \end{aligned}$$

Inserting this into (5.38), we deduce for the first term in (5.33) that

$$\begin{aligned} \int_{C^{g_{i_0}}_\varepsilon(\delta)} \big|(\tilde\Omega^{g_{i_0}}_{\varepsilon,\delta} - \hat\Omega^{g_{i_0}}_{\varepsilon,\delta}) \wedge \bar\omega^{g_{i_0}}_{\varepsilon,\delta}\big|\, d^5x \ \le\ & C\,\|\tilde f^{g_{i_0}}_{\varepsilon,\delta} - \iota^*_{\partial C}F\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + C\,\|\iota^*_{\partial C}F - \iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + C\,\|\iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,. \end{aligned} \tag{5.39}$$

Putting (5.35) and (5.39) together, we end up with the following estimate for (5.32):

$$\begin{aligned} \int_{C^{g_{i_0}}_\varepsilon(\delta)} \big|(\tilde\Omega^{g_{i_0}}_{\varepsilon,\delta} - \Omega) \wedge \bar\omega^{g_{i_0}}_{\varepsilon,\delta}\big|\, d^5x \ \le\ & C\,\|\tilde f^{g_{i_0}}_{\varepsilon,\delta} - \iota^*_{\partial C}F\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + C\,\|\iota^*_{\partial C}F - \iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + C\,\|\iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + C\,\|\bar F^{g_{i_0}}_{\varepsilon,\delta} - F\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))}\,. \end{aligned}$$

With a simple scaling argument and the small energy assumption (5.10) on the boundary of the good cubes which also holds for $\iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}$ – as a direct consequence of the choice (5.8) for the cubic decomposition – we can rewrite the last inequality as

$$\begin{aligned} \int_{C^{g_{i_0}}_\varepsilon(\delta)} \big|(\tilde\Omega^{g_{i_0}}_{\varepsilon,\delta} - \Omega) \wedge \bar\omega^{g_{i_0}}_{\varepsilon,\delta}\big|\, d^5x \ \le\ & C\varepsilon\,\|\tilde f^{g_{i_0}}_{\varepsilon,\delta} - \iota^*_{\partial C}F\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + C\varepsilon\,\|\iota^*_{\partial C}F - \iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + C\varepsilon\delta\,\|\iota^*_{\partial C}\bar F^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \\ & + C\,\|\bar F^{g_{i_0}}_{\varepsilon,\delta} - F\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))}\,, \end{aligned} \tag{5.40}$$

where we used the notation $\Omega = \mathrm{Tr}(F \wedge F)$. Here the constants C are independent of the size of the good cubes.

Third step: At this stage, we are ready to pass from one fixed good cube to the union $C^g_\varepsilon(\delta) = \bigcup_{i=1}^{N^g_\varepsilon(\delta)} C^{g_i}_\varepsilon(\delta)$ of good cubes. We define the *piecewise* smooth $\mathfrak{su}(2)$-valued two-form $\tilde{F}^g_{\varepsilon,\delta}$ on $C^g_\varepsilon(\delta)$ by

$$\tilde{F}^g_{\varepsilon,\delta} = \tilde{F}^{g_i}_{\varepsilon,\delta} \qquad \text{on } \; C^{g_i}_\varepsilon(\delta)\,, \tag{5.41}$$

which – because of its local representation (5.31) – belongs to $\mathcal{F}_R(C^g_\varepsilon(\delta))$ (see also (5.36)). It is important to note that the local curvature forms $\tilde{F}^g_{\varepsilon,\delta}$ piece together to give a *globally* smooth real-valued four-form

$$\tilde{\Omega}^g_{\varepsilon,\delta} = \mathrm{Tr}(\tilde{F}^g_{\varepsilon,\delta} \wedge \tilde{F}^g_{\varepsilon,\delta})\,, \tag{5.42}$$

which is a consequence of the gauge invariance of the second Chern class in the non-Abelian $SU(2)$-case. Moreover, as in (5.41), we can define the piecewise constant two-form $\bar{F}^g_{\varepsilon,\delta}$ and piecewise constant one-form $\bar{\omega}^g_{\varepsilon,\delta}$ on $C^g_\varepsilon(\delta)$.

Summation over the good cubes yields

$$\int_{C^g_\varepsilon(\delta)} \big|(\tilde{\Omega}^g_{\varepsilon,\delta} - \Omega) \wedge \bar{\omega}^g_{\varepsilon,\delta}\big|\, d^5x \le \sum_{i=1}^{N^g_\varepsilon(\delta)} \int_{C^{g_{i_0}}_\varepsilon(\delta)} \big|(\tilde{\Omega}^{g_{i_0}}_{\varepsilon,\delta} - \Omega) \wedge \bar{\omega}^{g_{i_0}}_{\varepsilon,\delta}\big|\, d^5x\,.$$

Applying (5.40) to each good cube the right-hand side can be bounded by

$$\begin{aligned}
& C\varepsilon \sum_{i=1}^{N^g_\varepsilon(\delta)} \|\tilde{f}^{g_i}_{\varepsilon,\delta} - \iota^*_{\partial C} F\|_{L^2(\partial C^{g_i}_\varepsilon(\delta))} \\
+ \quad & 2C\varepsilon \Big(\|F - {}^l\bar{F}^g_{\varepsilon,\delta}\|_{L^2(\partial C^g_\varepsilon(\delta))} + \|F - {}^r\bar{F}^g_{\varepsilon,\delta}\|_{L^2(\partial C^g_\varepsilon(\delta))} \Big) \\
+ \quad & 2C\varepsilon\, \delta\, \|F\|_{L^2(\partial C^g_\varepsilon(\delta))} + C\, \|\bar{F}^g_{\varepsilon,\delta} - F\|_{L^2(C^g_\varepsilon(\delta))}\,,
\end{aligned} \tag{5.43}$$

where ${}^l\bar{F}^g_{\varepsilon,\delta}$, respectively ${}^r\bar{F}^g_{\varepsilon,\delta}$ denote the "left" respectively "right" trace of $\bar{F}^g_{\varepsilon,\delta}$ on $\partial C^g_\varepsilon(\delta)$.

Forth Step: In a next step, we take the limit $\varepsilon \to 0$ in (5.43). – The smooth approximation $\tilde{f}^{g_i}_{\varepsilon,\delta}$ of $\iota^*_{\partial C} F$ on each good cube given by (5.20) satisfies

$$\lim_{\varepsilon\to 0} \sum_{i=1}^{N^g_\varepsilon(\delta)} \|\tilde{f}^{g_i}_{\varepsilon,\delta} - \iota^*_{\partial C} F\|_{L^2(\partial C^{g_i}_\varepsilon(\delta))} = 0\,.$$

From the choice (5.8) of the cubic decomposition, we deduce that the second term on the right-hand side of (5.43) converges to zero in the limit $\varepsilon \to 0$. Moerover, due to (5.9) for the cubic decomposition, we have that

$$C\varepsilon\, \delta\, \|F\|_{L^2(\partial C^g_\varepsilon(\delta))} \le C\delta\,,$$

for $\varepsilon \to 0$. The forth term on the right-hand side of (5.43) also tends to zero, because of Lebesgue's theorem. In summary, we arrive at

$$\limsup_{\varepsilon\to 0} \int_{C^g_\varepsilon(\delta)} \big|(\tilde{\Omega}^g_{\varepsilon,\delta} - \Omega) \wedge \bar{\omega}^g_{\varepsilon,\delta}\big|\, d^5x \le C\,\varepsilon\,, \tag{5.44}$$

for every $\omega \in \Omega^1_{C^0}(B^5)$, showing (5.21).

Fifth step: Observe that

$$\begin{aligned}\int_{C^g_\varepsilon(\delta)} \big|(\tilde{\Omega}^g_{\varepsilon,\delta} - \Omega) \wedge \omega\big|\, d^5x \;\; &\le \;\; \int_{C^g_\varepsilon(\delta)} \big|(\tilde{\Omega}^g_{\varepsilon,\delta} - \Omega) \wedge (\omega - \bar{\omega}^g_{\varepsilon,\delta})\big|\, d^5x \\ &\quad + \int_{C^g_\varepsilon(\delta)} \big|(\tilde{\Omega}^g_{\varepsilon,\delta} - \Omega) \wedge \bar{\omega}^g_{\varepsilon,\delta}\big|\, d^5x\,.\end{aligned}$$

From (5.44) and Lebesgue's theorem, it then follows that

$$\limsup_{\varepsilon\to 0} \int_{C^g_\varepsilon(\delta)} \big|(\tilde{\Omega}^g_{\varepsilon,\delta} - \Omega) \wedge \omega\big|\, d^5x \le C\,\varepsilon\,, \tag{5.45}$$

for every $\omega \in \Omega^1_{C^0}(B^5)$.

A standard diagonal argument finally leads to the weak convergence (5.17) on the good cubes. More precisely, we let the parameter δ now depend on $k \in \mathbb{N}$ and set $\delta_k = 1/(k_0+k)$ with $k_0 > 0$ large enough. Then we choose $\varepsilon(k) > 0$ in (5.44) sufficiently small such that

$$\int_{C^g_{\varepsilon(k)}(\delta_k)} \big|(\tilde{\Omega}^g_{\varepsilon(k),\delta_k} - \Omega) \wedge \omega\big| \le \frac{2\,C}{k_0+k}\,.$$

Hence, for the diagonal sequence – which we will simply denote by $(\tilde{\Omega}^g_k)_{k\in\mathbb{N}} = \big(\mathrm{Tr}(\tilde{F}^g_k \wedge \tilde{F}^g_k)\big)_{k\in\mathbb{N}}$ with $(\tilde{F}^g_k)_{k\in\mathbb{N}}$ a sequence in $\mathcal{F}_R(C^g_\varepsilon(\delta))$ – we obtain in the limit $k \to \infty$ the desired result

$$\lim_{k\to\infty} \int_{C^g_k} \Big|\Big(\mathrm{Tr}(\tilde{F}^g_k \wedge \tilde{F}^g_k) - \mathrm{Tr}(F\wedge F)\Big) \wedge \omega\Big| = 0\,, \tag{5.46}$$

for all $\omega \in \Omega^1_{C^0}(B^5)$.

Density on the Bad Cubes.

In order to establish the density result (5.17) on the bad cubes, we will show the *strong* L^1-convergence

$$\int_{C^b_\varepsilon} \big|\mathrm{Tr}(F^b_\varepsilon \wedge F^b_\varepsilon) - \mathrm{Tr}(F\wedge F)\big|\, d^5x \longrightarrow 0 \qquad (\varepsilon \to 0)\,, \tag{5.47}$$

for some sequence $(F^b_\varepsilon)_{\varepsilon>0}$ in $\mathcal{F}_R(C^b_\varepsilon)$, which obviously implies the weak convergence (5.17) on the bad cubes. The proof for the density on the bad cubes is based on the fact that the volume of the bad cubes vanishes in the limit $\varepsilon \to 0$. As mentioned in (5.14), this holds for every parameter $0 < \delta < 1$. Hence, we can omit δ for the convergence on the bad cubes.

The triangular inequality and (5.11) give

$$\begin{aligned}\int_{C^b_\varepsilon} \big|\mathrm{Tr}(F^b_\varepsilon \wedge F^b_\varepsilon) - \mathrm{Tr}(F\wedge F)\big|\, d^5x \;\; &\le \;\; \int_{C^b_\varepsilon} \big|\Omega^b_\varepsilon(x)\big|\, d^5x + \int_{C^b_\varepsilon} \big|\Omega(x)\big|\, d^5x \\ &\le \;\; \int_{C^b_\varepsilon} \big|F^b_\varepsilon(x)\big|^2\, d^5x + \int_{C^b_\varepsilon} \big|F(x)\big|^2\, d^5x\,,\end{aligned} \tag{5.48}$$

where we used as before the notations $\Omega_\varepsilon^b = \mathrm{Tr}(F_\varepsilon^b \wedge F_\varepsilon^b)$ and $\Omega = \mathrm{Tr}(F \wedge F)$. As a consequence of the vanishing volume of the bad cubes (see (5.14)) and dominated convergence, the second term on the right-hand side tends to zero in the limit $\varepsilon \to 0$. Hence, for the density result (5.47), it remains to show that the first term on the right-hand side of (5.48) satisfies

$$\int_{C_\varepsilon^b} \left|F_\varepsilon^b(x)\right|^2 d^5x \longrightarrow 0 \qquad (\varepsilon \to 0)\,. \tag{5.49}$$

We thus have to construct a sequence $(F_\varepsilon^b)_{\varepsilon>0}$ in $\mathcal{F}_R(C_\varepsilon^b)$ such that (5.49) holds.

For every bad cube $C_\varepsilon^{b_i}$, we assume that (see (5.13))

$$\frac{1}{8\pi^2}\int_{\partial C_\varepsilon^{b_i}} \iota^*_{\partial C_\varepsilon^{b_i}} \mathrm{Tr}(F \wedge F) = d_\varepsilon^{b_i}\,, \tag{5.50}$$

where $0 \neq d_\varepsilon^{b_i} \in \mathbb{Z}$. This implies that the principal $SU(2)$-bundles over the boundaries of the bad cubes are *not* trivial. However, the smooth approximation $\tilde{f}_\varepsilon^b$ (see (5.20)) for the restriction of $F \in \mathcal{F}_{\mathbb{Z}}(B^5)$ to the boundaries of the bad cubes piece together to give the *globally* well-defined smooth four-form $\mathrm{Tr}(\tilde{f}_\varepsilon^{b_i} \wedge \tilde{f}_\varepsilon^{b_i})$. On the bad cubes, we can thus take the radial extension of this smooth four-form.

More precisely, we define the radial extension

$$\tilde{\Omega}_\varepsilon^{b_i} = (\pi_\varepsilon^{b_i})^*\big(\mathrm{Tr}(\tilde{f}_\varepsilon^{b_i} \wedge \tilde{f}_\varepsilon^{b_i})\big)\,, \tag{5.51}$$

where $\pi_\varepsilon^{b_i}$ denotes – as in the Abelian $U(1)$-case – the radial projection on the boundary of $C_\varepsilon^{b_i}$. Since $\pi_\varepsilon^{b_i}$ has only one point singularity at the center a_ε^i of $C_\varepsilon^{b_i}$, we deduce that $\tilde{\Omega}_\varepsilon^{b_i}$ is smooth on all of $C_\varepsilon^{b_i}$ except at the point a_ε^i. Moreover, using (5.50), a straightforward computation yields

$$d\tilde{\Omega}_\varepsilon^{b_i} = 8\pi^2 d_\varepsilon^{b_i}\delta_{a_\varepsilon^i}\, dx^1 \wedge \ldots \wedge dx^5 \quad \text{in } \mathcal{D}'\,, \tag{5.52}$$

and we have the representation

$$\tilde{\Omega}_\varepsilon^{b_i} = \mathrm{Tr}(\tilde{F}_\varepsilon^{b_i} \wedge \tilde{F}_\varepsilon^{b_i})\,, \tag{5.53}$$

where $\tilde{F}_\varepsilon^{b_i}$ is a *locally* defined smooth curvature form for some principal $SU(2)$-bundle over $C_\varepsilon^{b_i} \setminus \{a_\varepsilon^i\}$.

On the union $C_\varepsilon^b = \bigcup_{i=1}^{N_\varepsilon^b} C_\varepsilon^{b_i}$ of bad cubes, we define the four-form Ω_ε^b by

$$\tilde{\Omega}_\varepsilon^b = \tilde{\Omega}_\varepsilon^{b_i} \qquad \text{on } C_\varepsilon^{b_i}\,. \tag{5.54}$$

Because of the gauge invariance in the $SU(2)$-case, we have that $\tilde{\Omega}_\varepsilon^b \in \Omega_\infty^4\left(C_\varepsilon^b \setminus \bigcup_{i=1}^{N_\varepsilon^b} a_\varepsilon^i\right)$ and, due to (5.52) and (5.53), we then also get that the corresponding local smooth curvature form $\tilde{F}_\varepsilon^b$ belongs to $\mathcal{F}_R(C_\varepsilon^b)$. Proceeding step by step as in the proof of the Abelian case Theorem 4.6 for the bad cubes – but using L^2-norms instead of L^1-norms – we obtain (5.49) and thus the strong L^1-convergence (5.47) on the bad cubes.

Conclusion.

In summary, using (5.41) and (5.54) for the good respectively bad cubes, we define

$$F_k = \begin{cases} \tilde{F}^g_{1/k} & \text{on } C^g_{1/k} \\ \tilde{F}^b_{1/k} & \text{on } C^b_{1/k}\,, \end{cases} \tag{5.55}$$

for every $k \in \mathbb{N}$. The resulting sequence $(F_k)_{k\in\mathbb{N}}$ approximates weakly in the sense of (5.17) the given two-form $F \in \mathcal{F}_{\mathbb{Z}}(B^5)$. □

Lemma 5.4. *Let $F \in \Omega^2_{L^2}\big(B^5, \mathfrak{su}(2)\big)$ and let $C \subset B^5$ be an open unit cube such that $\iota^*_{\partial C} F = da + a \wedge a$ is smooth, where the $\mathfrak{su}(2)$-valued one-form a satisfies*

$$\|a\|_{W^{1,2}(\partial C)} \leq C\,\|\iota^*_{\partial C} F\|_{L^2(\partial C)}\,. \tag{5.56}$$

Moreover, consider the smooth harmonic extension $\tilde{A}$ of a on C given by

$$\begin{cases} \Delta\tilde{A} = 0 & \text{on } C \\ \iota^*_{\partial C}\tilde{A} = a\,, \quad \iota^*_{\partial C}(d_*\tilde{A}) = 0 & \text{on } \partial C\,. \end{cases} \tag{5.57}$$

Then, the following estimate holds:

$$\|\tilde{A}\|_{W^{1,5/2}(C)} \leq C\,\|\iota^*_{\partial C} F\|_{L^2(\partial C)}\,. \tag{5.58}$$

Proof. Observe that by the Sobolev embedding theorem in four dimensions

$$W^{1,2}(\partial C) \hookrightarrow W^{1-1/q,q}(\partial C)$$

is a continuous embedding, if $1 > 1 - 1/q$ and if

$$1 - \frac{4}{2} \geq \left(1 - \frac{1}{q}\right) - \frac{4}{q} = 1 - \frac{5}{q}\,.$$

Using (5.56), we then have

$$\|a\|_{W^{1-1/q,q}(\partial C)} \leq C\,\|a\|_{W^{1,2}(\partial C)} \leq C\,\|\iota^*_{\partial C} F\|_{L^2(\partial C)}\,, \tag{5.59}$$

for $q \leq 5/2$.

In a next step[1], we deduce from the trace theorem the existence of an $\mathfrak{su}(2)$-valued one-form A on C with $\iota^*_{\partial C} A = a$ and $\iota^*_{\partial C}(d_* A) = 0$ such that

$$\|A\|_{W^{1,q}(C)} \leq C\,\|a\|_{W^{1-1/q,q}(\partial C)}\,.$$

Combining this with (5.59), the left hand-side can also be bounded by

$$\|A\|_{W^{1,q}(C)} \leq C\,\|\iota^*_{\partial C} F\|_{L^2(\partial C)}\,, \tag{5.60}$$

for $q \leq 5/2$.

[1] We now follow step by step the proof of Lemma 4.7 for the Abelian case.

Now, we define $\bar{A} = A - \tilde{A}$, where $\tilde{A}$ is the smooth harmonic extension of a defined by the elliptic system (5.57). Obviously, we have

$$\begin{cases} \Delta \bar{A} = \Delta A & \text{on } C \\ \iota^*_{\partial C}\bar{A} = 0 \quad \iota^*_{\partial C}(d_*\bar{A}) = 0 & \text{on } \partial C\,. \end{cases}$$

According to the theory of Calderón-Zygmund, it then follows for $A \in \Omega^1_{W^{1,q}}\big(C, \mathfrak{su}(2)\big)$ that

$$\|\bar{A}\|_{W^{1,q}(C)} \leq C\|A\|_{W^{1,q}(C)}\,. \tag{5.61}$$

This directly implies, choosing $q = 5/2$,

$$\begin{aligned} \|\tilde{A}\|_{W^{1,5/2}(C)} &\leq \|\bar{A}\|_{W^{1,5/2}(C)} + \|A\|_{W^{1,5/2}(C)} \\ &\leq (C+1)\|A\|_{W^{1,5/2}(C)} \overset{(5.60)}{\leq} C\,\|\iota^*_{\partial C}F\|_{L^2(\partial C)}\,. \end{aligned}$$

□

5.3 Strong Density Result in the Non-Abelian Case

Recall from the introduction that we want to produce a suitable framework for the existence of minimizers for the Yang-Mills functional in five dimensions. Motivated by the results for harmonic maps from B^3 into S^2, we know that the strong closure of $\mathcal{F}_R(B^5)$ is the best candidate for the existence of minimizers. In this section, we now investigate the strong closure of $\mathcal{F}_R(B^5)$.

A first attempt in order to determine the strong closure of $\mathcal{F}_R(B^5)$ in the non-Abelian case is to show that $\mathcal{F}_R(B^5)$ is strongly dense in $\mathcal{F}_{\mathbb{Z}}(B^5)$ for the L^2-norm by transfering directly the ideas of the proof for the Abelian Theorem 4.6 to this non-Abelian setting. Doing this it turns out that the non-linearities, i.e., quadratic terms of the form $A \wedge A$ in the curvature, are simple to handle. However, we have to face the following two problems:

a) By definition we know that elements in $\mathcal{F}_R(B^5)$ are smooth up to gauge transformations and a finite number of points. Approximating $\mathcal{F}_{\mathbb{Z}}(B^5)$ strongly by weak curvatures in $\mathcal{F}_R(B^5)$ gives hence no approximation by smooth objects. There is always the restriction of smoothness "up to gauge transformations".

b) Thanks to K. Uhlenbeck's Theorem 3.2 we can choose the Coulomb gauge on the boundary of the good cubes in the cubic decomposition. This induces a change in curvature on the boundary. More precisely, the curvature transforms with the adjoint action of the Lie group $SU(2)$ under gauge transformations. Recall that in the Abelian case with the Abelian Lie group $U(1)$ there is no adjoint action. As a consequence of the change in curvature after Coulomb gauge transformations, the choice (5.8) of the decomposition is no longer sufficient for convergence. We only obtain a kind of convergence "modulo gauge transformations".

Remark 5.1. We mention that the weak density result[2] of the previous section is a possibility to deal with problem a), since $\mathrm{Tr}(F \wedge F)$ represents a gauge invariant global object. Moreover, considering only weak convergence, we can exclude problem b).

[2] We put the calculations making more precise how the strong L^2-convergence of $\mathcal{F}_R(B^5)$ to $\mathcal{F}_{\mathbb{Z}}(B^5)$ fails in Appendix D.

From these reflections we get an understanding of how to treat the strong density in the non-Abelian case and of how to define the notion of non-Abelian "singular bundles". Namely, for the latter we have to look for an object which is on the one hand gauge invariant and on the other hand able to characterize the curvature – modulo the adjoint action of $SU(2)$. The object $\mathrm{Tr}(F \wedge F)$ being gauge invariant encodes not enough information for characterizing the curvature. This lead us to consider the tensor product $\mathrm{Tr}(F \otimes F)$ which is still gauge invariant and determines uniquely the curvature – modulo the adjoint action of $SU(2)$ (see Lemma 3.7). Combining the defining equation (5.5) of $\mathcal{F}_{\mathbb{Z}}(B^5)$ with the tensor product we end up with the next definition describing "singular bundles" for the non-Abelian case – or, more precisely – everywhere singular $SU(2)$-bundles over B^5 with bounded L^2-curvature.

Definition 5.1 (L^2-curvature of a singular $SU(2)$-bundle). *An L^2-curvature of a singular principal $SU(2)$-bundle over B^5 is defined as an element Ω in $L^1\big(B^5, \bigwedge^2(TB^5) \otimes \bigwedge^2(TB^5)\big)$ with the following properties:*

a) There exists $F \in \Omega^2_{L^2}(B^5, \mathfrak{su}(2))$ such that $\Omega = \mathrm{Tr}(F \otimes F)$.

b) For every $x \in B^5$ and a.e. $0 < r \leq dist(x, \partial B^5)$ there exists a curvature f of a $W^{1,2}$-connection on smooth principal $SU(2)$-bundle over $\partial B^5_r(x)$ such that $\iota^*_{\partial B^5_r(x)}\Omega = \mathrm{Tr}(f \otimes f)$.

The class of L^2-curvatures of singular principal $SU(2)$-bundles over B^5 is denoted by $\mathcal{F}_{\otimes}(B^5)$.

In order to solve the problem of strong density in the non-Abelian case, we then have to show that for every $\Omega \in \mathcal{F}_{\otimes}(B^5)$ there exists a sequence $(F_k)_{k\in\mathbb{N}}$ in $\mathcal{F}_R(B^5)$ such that

$$\mathrm{Tr}(F_k \otimes F_k) \longrightarrow \Omega \qquad \text{in } L^1 . \tag{5.62}$$

This is Open Problem 1 in the introduction. We emphasize again that with the help of the tensor product there is no longer any kind of convergence "modulo gauge transformation" and that the approximating sequence $\big(\mathrm{Tr}(F_k \otimes F_k)\big)_{k\in\mathbb{N}}$ is really smooth except at a finite number of points. Moreover, note that in (5.62) we consider on the space of singular $SU(2)$-bundles with bounded L^2-curvatures the metric given by

$$\delta(F_1, F_2) = \int_{B^5} \big|\mathrm{Tr}(F_1 \otimes F_1) - \mathrm{Tr}(F_2 \otimes F_2)\big| \, d^5x . \tag{5.63}$$

There is an alternative notion for non-Abelian "singular bundles" given in Definition 1.3 of the introduction. Lemma 3.7 – which obviously also holds on B^5 – then states that Definition 5.1 and 1.3 are equivalent. Together with Definition 1.3 we can adjust the topology on the space of L^2-curvatures of singular principal $SU(2)$-bundles using the metric given by (see T. Kessel and T. Rivière, [27])

$$d([F_1], [F_2]) = \inf_{\sigma \in L^\infty(B^5, SU(2))} \left(\int_{B^5} |F_1 - \sigma^{-1} F_2 \sigma|^2 \, d^5x \right)^{1/2} . \tag{5.64}$$

From Proposition 3.5 we then deduce that the two metrics d and δ generate equivalent topologies. Note that the metric δ is more explicit and thus simpler to handle.

Strategy for Solving Open Problem 1

In order to show (5.62), we propose a strategy which resembles the one for the strong density in the Abelian case. From now on, we require the reader to be familiar with the notations of the previous chapters. The strategy is then the following: We divide B^5 into small open cubes of side-length $\varepsilon > 0$. Let good and bad cubes as in Section 5.2 with respect to $F \in \Omega^2_{L^2}(B^5, \mathfrak{su}(2))$ given by the definition of $\Omega \in \mathcal{F}_\otimes(B^5)$. On each good cube of the cubic decomposition we take the average $\bar{F}_\varepsilon$ of F. Then we define the piecewise constant four-form $\bar{\Omega}_\varepsilon = \mathrm{Tr}(\bar{F}_\varepsilon \otimes \bar{F}_\varepsilon)$ and choose the grid in such a way that on the boundary

$$\lim_{\varepsilon \to 0} \|\iota^*_{\partial C}\Omega - \iota^*_{\partial C}\bar{\Omega}_\varepsilon\|_{L^1(\partial C_\varepsilon)} = 0\,. \tag{5.65}$$

This can be rewritten as

$$\lim_{\varepsilon \to 0} \|\mathrm{Tr}(f_\varepsilon \otimes f_\varepsilon) - \mathrm{Tr}(\bar{f}_\varepsilon \otimes \bar{f}_\varepsilon)\|_{L^1(\partial C_\varepsilon)} = 0\,, \tag{5.66}$$

where $\iota^*_{\partial C}F = f_\varepsilon = da_\varepsilon + a_\varepsilon \wedge a_\varepsilon$ and $\iota^*_{\partial C}\bar{F} = \bar{f}_\varepsilon = d\bar{a}_\varepsilon$ with $d_*\bar{a}_\varepsilon = 0$. Note that, using the smooth approximability of Section 3.1, we can assume that $\iota^*_{\partial C}F$ is smooth and, moreover, that the curvature $\bar{f}_\varepsilon$ agrees except for the small quadratic term $\bar{a}_\varepsilon \wedge \bar{a}_\varepsilon$ with the curvature $\hat{f}_\varepsilon := d\bar{a}_\varepsilon + \bar{a}_\varepsilon \wedge \bar{a}_\varepsilon$ being already in Coulomb gauge. We are allowed to pass to the Coulomb gauge $a_{Coulomb,\varepsilon} = \sigma_{Coulomb}(a_\varepsilon)$ on each boundary of the good cubes due to the small energy assumption. It is important to mention that owing to Proposition 3.4 the Coulomb gauge $a_{Coulomb,\varepsilon}$ is still smooth.

Next, for the good cubes we want to show that the fact of "being closed to a constant" on the boundary for the original curvature – see (5.66) – remains also true for the gauge transformed curvature. More precisely, this can be formulated in the following way:

Open Problem 5. *Assuming that*

$$\lim_{\varepsilon \to 0} \|\mathrm{Tr}(f_\varepsilon \otimes f_\varepsilon) - \mathrm{Tr}(\hat{f}_\varepsilon \otimes \hat{f}_\varepsilon)\|_{L^1(\partial C^g_\varepsilon)} = 0\,, \tag{5.67}$$

with curvatures of small L^2-energy, can we conclude on the boundary of the good cubes that

$$\lim_{\varepsilon \to 0} \Big\| f(a_{Coulomb,\varepsilon}) = da_{Coulomb,\varepsilon} + a_{Coulomb,\varepsilon} \wedge a_{Coulomb,\varepsilon} - \hat{f}_\varepsilon \Big\|_{L^2(\partial C^g_\varepsilon)} = 0 \ \ ? \tag{5.68}$$

In a next step, we take the harmonic extension of $a_{Coulomb,\varepsilon}$ on each good cube, which we will denote by $\tilde{A}_\varepsilon$ with corresponding smooth curvature $\tilde{F}_\varepsilon = d\tilde{A}_\varepsilon + \tilde{A}_\varepsilon \wedge \tilde{A}_\varepsilon$. Then, as a consequence of (5.68), we want to establish that

$$\mathrm{Tr}(\tilde{F}_\varepsilon \otimes \tilde{F}_\varepsilon) \longrightarrow \mathrm{Tr}(\bar{F} \otimes \bar{F}) \qquad \text{in } \ L^1\,. \tag{5.69}$$

This is much like in the proof for the Abelian Theorem 4.6, where the fact of "being closed to a constant" on the boundary of a cube implies "being closed to a constant" in the interior of a cube by using the harmonic extension. Finally, the strong convergence (5.62) follows from (5.69), since by Lebesgue's theorem $\mathrm{Tr}(\bar{F} \otimes \bar{F})$ converges to Ω.

We already encounter in our direct approach to strong convergence in the non-Abelian case (see Appendix D) the question if the choice of the cubic decomposition also holds for the gauge transformed quantities. To be more precise, under the assumption (see (5.8))

$$\lim_{\varepsilon\to 0} \|f_\varepsilon - \bar{f}_\varepsilon\|_{L^2(\partial C^g_\varepsilon)} = 0\,, \tag{5.70}$$

with curvatures of small L^2-energy, we want to conclude on the boundary of the good cubes that

$$\lim_{\varepsilon\to 0} \|f(a_{Coulomb,\varepsilon}) - \bar{f}_\varepsilon\|_{L^2(\partial C^g_\varepsilon)} = 0\,. \tag{5.71}$$

One first possibility for showing this is to use an estimate as in T. Tao and G. Tian [45], Section 9. We write the initial gauge a_ε as the sum of the Coulomb gauge $\bar{a}_\varepsilon$ and some perturbation term λ_ε. Then we have the estimate

$$\begin{aligned} \|f(a_{Coulomb,\varepsilon}) - \bar{f}_\varepsilon\|_{L^2(\partial C^g_\varepsilon)} &\leq C\,\|a_{Coulomb,\varepsilon} - \bar{a}_\varepsilon\|_{W^{1,2}(\partial C^g_\varepsilon)} \\ &= C\,\|\sigma_{Coulomb}(\bar{a}_\varepsilon + \lambda_\varepsilon) - \bar{a}_\varepsilon\|_{W^{1,2}(\partial C^g_\varepsilon)} \\ &\leq C\,\|(\bar{a}_\varepsilon + \lambda_\varepsilon) - \bar{a}_\varepsilon\|_{W^{1,2}(\partial C^g_\varepsilon)} = C\,\|\lambda_\varepsilon\|_{W^{1,2}(\partial C^g_\varepsilon)}\,. \end{aligned} \tag{5.72}$$

However, it turns out that the right-hand side can not be controlled by (5.71).

In a second approach, we consider an auxilliary problem which is directly related to Open Problem 5. – For a fixed good cube C^g with boundary ∂C^g and average $\bar{f}$ let $(f_k)_{k\in\mathbb{N}}$ be a sequence of small L^2-energy curvatures such that

$$\lim_{k\to\infty} \|f_k - \bar{f}\|_{L^2(\partial C^g)} = 0\,. \tag{5.73}$$

In order to show that this remains true for gauge transformed curvatures, i.e.

$$\lim_{k\to\infty} \|f(a_{Coulomb,k}) - \bar{f}\|_{L^2(\partial C^g)} = 0\,, \tag{5.74}$$

we first choose the Coulomb gauge $a_{Coulomb,k} = \sigma_{Coulomb,k}(a_k)$. The estimate (3.26) for Coulomb gauges then gives

$$\|a_{Coulomb,k}\|_{W^{1,2}(\partial C^g)} \leq C_{Uh}\,\|f_k\|_{L^2(\partial C^g)} \leq C < \infty\,,$$

where we used also the convergence assumption (5.73). Due to this uniform boundedness, it then follows from weak compactness for a subsequence

$$a_{Coulomb,k'} \rightharpoonup a_\infty \qquad \text{weakly in } W^{1,2}\,. \tag{5.75}$$

Note that we have $d_* a_\infty = 0$, since $d_* a_{Coulomb,k'} = 0$ passes to the limit. From (5.75), we deduce that

$$f(a_{Coulomb,k'}) \rightharpoonup f(a_\infty) \qquad \text{weakly in } L^2\,. \tag{5.76}$$

Moreover, in Coulomb gauge the convergence assumption (5.73) reads as

$$\lim_{k\to\infty} \|f(a_{Coulomb,k}) - \sigma^{-1}_{Coulomb,k}\bar{f}\sigma_{Coulomb,k}\|_{L^2(\partial C^g)} = 0\,. \tag{5.77}$$

At this stage, we don't know wether there exists a limiting gauge transformation σ_∞ such that

$$f(a_\infty) = \sigma_\infty^{-1} \bar{f} \sigma_\infty \,. \tag{5.78}$$

Since as seen before $\bar{f}$ is nearly in Coulomb gauge, we then deduce from the uniqueness of Coulomb gauges in the small L^2-energy regime given in Proposition 3.3 that σ_∞ must be the identity in $SU(2)$. Hence, combining the weak convergence (5.76) with the norm convergence (5.77), we end up with the strong convergence (5.74) for curvatures in Coulomb gauge.

In terms of the metrics introduced in Section 3.3, we get a better understanding of the open question (5.78). – Note that the convergence assumption (5.77) can be rewritten with the help of the metric d as

$$\lim_{k\to\infty} d\big(f(a_{Coulomb,k}), \bar{f}\big) = 0 \,. \tag{5.79}$$

If this would imply that

$$\lim_{k\to\infty} \gamma\big(f(a_{Coulomb,k}), \bar{f}\big) = 0 \,, \tag{5.80}$$

where the metric γ for the Coulomb gauges is defined in (3.58), then it is not difficult to check that we could directly deduce the strong convergence (5.74) for curvatures in Coulomb gauge. However, we have already shown in Proposition 3.5 that d and γ do not induce the same topology. This can thus be seen as the reason for the open question (5.78). In order to solve (5.78), we then have to find the curvatures $\bar{f}$ for which the convergence (5.79) with respect to the metric d also implies the convergence (5.80) with respect to the metric γ.

Chapter 6

Curvature Currents

In this second part of the thesis, we introduce currents associated to the weak curvatures of Chapter 4 and Chapter 5. With the help of these currents the problem of connecting the singularities of the weak curvatures translates to the existence of an i.m. rectifiable current which completes the initial current to a boundaryless current. Such currents will be called curvature currents.

The problem of connecting singularities is strongly related to the notion of minimal connection. So we will give definitions of minimal connections for weak curvatures in the Abelian as well as in the non-Abelian case. It is again useful to review briefly results in this direction in the case of maps from B^3 into S^2.

6.1 Preliminaries

We collect results on the minimal connection for maps from B^3 into S^2 and on Cartesian currents. The reader should be familiar with Section 2.3.

Minimal Connection for Maps from B^3 into S^2

The notion of minimal connection first appeared in H. Brezis, J.-M. Coron and E. H. Lieb, [2] for the calculation of the minimum Dirichlet energy of maps from a domain in $\mathbb{R}^3$ into S^2 with prescribed location and topological degree of their point singularities. At the beginning of this section, we explain the idea of a minimal connection and give a formula for its length (see also H. Brezis, [3]).

Let $H^1_\varphi(B^3, S^2)$ denote the class of $W^{1,2}$-Sobolev maps from B^3 into S^2 with given smooth boundary data $\varphi : \partial B^3 \longrightarrow S^2$. Given $u \in H^1_\varphi(B^3, S^2)$, we introduce a vector field $\vec{D}(u)$ on B^3 defined by

$$\vec{D}(u) = \left(u \cdot u_{x_2} \times u_{x_3}, u \cdot u_{x_3} \times u_{x_1}, u \cdot u_{x_1} \times u_{x_2}\right) . \tag{6.1}$$

Obviously, we have that $\vec{D}(u) \in L^1(B^3, \mathbb{R}^3)$ by Hölder's inequality. In the special case of u belonging the the class $R^\infty_\varphi(B^3, S^2)$ of smooth maps except at a finite number of points $a_1, \ldots, a_N \in B^3$, we have that (see H. Brezis, J.-M. Coron and E. H. Lieb [2], Appendix

B)

$$\operatorname{div} \vec{D}(u) = 4\pi \sum_{i=1}^{N} \deg(u, a_i)\, \delta_{a_i} \quad \text{in } \mathcal{D}' , \tag{6.2}$$

where δ_{a_i} is the Dirac measure at $a_i \in B^3$ and $\deg(u, a_i) \in \mathbb{Z}$ denotes the Brouwer degree of u restricted to any small sphere $S^2_r(a_i)$ centered at a_i, called the *degree* of u at a_i. Note that roughly speaking the divergence of the vector field $\vec{D}(u)$ detects the singularities of $u \in R^\infty_\varphi(B^3, S^2)$.

Let $u \in R^\infty_\varphi(B^3, S^2)$ and assume that

$$\deg(u|_{\partial B^3}) = \deg(\varphi) = 0 . \tag{6.3}$$

Moreover, we suppose that the degrees $\deg(u, a_i)$ of u at $a_i \in B^3$, for $i = 1, \ldots, N$, are prescribed and we denote them by $d_i \in \mathbb{Z}$. The assumption (6.3) then reads as

$$\sum_{i=1}^{N} d_i = 0 . \tag{6.4}$$

Next, we define the *total positive degree* K of u by

$$K = \sum_{d_i > 0} d_i \overset{(6.4)}{=} - \sum_{d_i < 0} d_i ,$$

and denote by $A^+ = \{a_1^+, \ldots, a_K^+\}$ the list of singularities with positive degree, each singularity a_i repeated according to their multiplicity $d_i > 0$. Similarly, we write $A^- = \{a_1^-, \ldots, a_K^-\}$ for the list of singularities with negative degree. In other words, the map $u \in R^\infty_\varphi(B^3, S^2)$ can be seen as a smooth map except at the $2K$ – not necessarily different – point singularities $a_1^+, \ldots, a_K^+, a_1^-, \ldots, a_K^-$ having all degree ± 1. Then the length of a *minimal connection* for u is defined by

$$L(u) = \min_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} \| a_i^+ - a_{\sigma(i)}^- \| , \tag{6.5}$$

where $\sigma \in \mathcal{S}_K$ is a permutation of $\{1, \ldots, K\}$. Note that by a *connection* for u we mean a pairing of the points in A^+ with the points in A^- and the length of this connection is given by the sum of the distances between the paired points. Clearly, for a minimal connection the singularities of u are connected in such a way that the sum of distances between the paired points is minimal.

From H. Brezis, J.-M. Coron and E. H. Lieb [2], Section IV, we know that the length of a minimal connection can also be expressed as

$$L(u) = \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|\nabla \xi\|_\infty \le 1}} \left\{ \sum_{i=1}^{K} \xi(a_i^+) - \sum_{i=1}^{K} \xi(a_i^-) \right\} . \tag{6.6}$$

We observe that

$$\sum_{i=1}^{K} \big(\xi(a_i^+) - \xi(a_i^-) \big) = \sum_{i=1}^{N} d_i\, \xi(a_i) = \int_{B^3} \left(\sum_{i=1}^{N} d_i\, \delta_{a_i} \right) \xi \, d^3x .$$

Using (6.2) and a standard density argument, we then obtain

$$\sum_{i=1}^{K}\big(\xi(a_i^+)-\xi(a_i^-)\big)=\frac{1}{4\pi}\int_{B^3}\operatorname{div}\vec{D}(u)\xi\,d^3x\,. \tag{6.7}$$

With Stokes' theorem, the last equation becomes

$$\sum_{i=1}^{K}\big(\xi(a_i^+)-\xi(a_i^-)\big)=\frac{1}{4\pi}\int_{\partial B^3}\big(\vec{D}(u)\cdot\nu\big)\xi\,dA_{\partial B^3}-\frac{1}{4\pi}\int_{B^3}\vec{D}(u)\cdot\nabla\xi\,d^3x\,, \tag{6.8}$$

where ν denotes the unit outward normal to ∂B^3. Thus, for $u\in R^\infty_\varphi(B^3,S^2)$ such that $\deg(\varphi)=0$, we end up with the following formula for the length of a minimal connection[1]:

$$L(u)=\frac{1}{4\pi}\sup_{\substack{\xi:B^3\to\mathbb{R}\\ \|\nabla\xi\|_\infty\le 1}}\left\{\int_{B^3}\vec{D}(u)\cdot\nabla\xi\,d^3x-\int_{\partial B^3}\big(\vec{D}(u)\cdot\nu\big)\xi\,dA_{\partial B^3}\right\}. \tag{6.9}$$

Minimal Connection in Terms of Differential Forms

We construct differential forms with the help of maps from B^3 into S^2 and then rewrite the classical formula (6.9) for the length of a minimal connection in terms of these differential forms.

Let $\omega_{S^2}\in\Omega^2(S^2)$ be the volume two-form on S^2 given by $\omega_{S^2}=\iota^*_{S^2}\Omega$, where $\iota_{S^2}: S^2\hookrightarrow\mathbb{R}^3$ is the canonical inclusion and

$$\Omega=x_1\,dx^2\wedge dx^3+x_2\,dx^3\wedge dx^1+x_3\,dx^1\wedge dx^2\in\Omega^2(\mathbb{R}^3).$$

Then a straightforward computation shows that the pull-back under $u\in H^1_\varphi(B^3,S^2)$ of ω_{S^2} reads as

$$u^*\omega_{S^2}=D_1\,dx^2\wedge dx^3+D_2\,dx^3\wedge dx^1+D_3\,dx^1\wedge dx^2\,,$$

where the components $D_1,D_2,D_3\in L^1(B^3)$ are exactly the components of the vector field $\vec{D}(u)$ defined in (6.1). This leads to the following notation:

$$D(u)=u^*\omega_{S^2}\in\Omega^2_{L^1}(B^3)\,. \tag{6.10}$$

For the exterior derivative of $D(u)$, we obtain

$$dD(u)=d(u^*\omega_{S^2})=\operatorname{div}\vec{D}(u)\,dx^1\wedge dx^2\wedge dx^3\quad\text{in }\mathcal{D}'\,. \tag{6.11}$$

We consider the following two particular cases:

a) If u is *smooth*, the pull-back and the exterior derivation commute implying that

$$dD(u)=d(u^*\omega_{S^2})=u^*(d\omega_{S^2})=0\,. \tag{6.12}$$

[1] Note that there is a difference in sign compared to the right-hand side of (6.8). However, because of the linearity in ξ, the supremum remains unchanged.

b) If $u \in R^\infty_\varphi(B^3, S^2)$, it follows from (6.2) that

$$dD(u) = \left(4\pi \sum_{i=1}^{N} \deg(u, a_i)\, \delta_{a_i} \right) dx^1 \wedge dx^2 \wedge dx^3 \quad \text{in } \mathcal{D}'(B^3)\,. \tag{6.13}$$

Next, we express the length of a minimal connection for $u \in R^\infty_\varphi(B^3, S^2)$ with $\deg(\varphi) = 0$ in terms of differential forms. – Together with (6.11) we can rewrite the length of a minimal connection (6.7) as

$$\sum_{i=1}^{K} \big(\xi(a_i^+) - \xi(a_i^-) \big) = \frac{1}{4\pi} \int_{B^3} dD(u)\xi\,, \tag{6.14}$$

where the function $\xi : B^3 \longrightarrow \mathbb{R}$ satisfies $\|d\xi\|_{L^\infty} \leq 1$. Stokes' theorem then gives

$$\sum_{i=1}^{K} \big(\xi(a_i^+) - \xi(a_i^-) \big) = \frac{1}{4\pi} \int_{\partial B^3} \iota^*_{\partial B^3}(D(u)\,\xi) - \frac{1}{4\pi} \int_{B^3} D(u) \wedge d\xi\,.$$

Thus, using (6.6), we end up with the following formula for the length of a minimal connection in terms of differential forms being equivalent to (6.9):

$$L(u) = \frac{1}{4\pi} \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|_{L^\infty} \leq 1}} \left\{ \int_{B^3} D(u) \wedge d\xi - \int_{\partial B^3} \iota^*_{\partial B^3}(D(u)\,\xi) \right\}. \tag{6.15}$$

Note that the right-hand side of (6.15) is also well-defined for *any* $u \in H^1_\varphi(B^3, S^2)$ and, moreover, finite since $\deg(\varphi) = 0$ by assumption. More precisely, the first term being clearly finite, the finiteness of the second term follows from

$$\deg(\varphi) = \frac{1}{4\pi} \int_{\partial B^3} \varphi^* \omega_{S^2} = 0\,.$$

Thus, we take (6.15) as definition for $L(u)$ in the case of $u \in H^1_\varphi(B^3, S^2)$.

Cartesian Currents

The study of Dirichlet energy minimizing maps from a domain of $\mathbb{R}^3$ into S^2 in the context of Cartesian currents goes back to M. Giaquinta, G. Modica and J. Souček. At this place, we explain how a Cartesian current can be associated to a map from B^3 into S^2. Motivated by the analysis of the boundary of this Cartesian current, we introduce a particular class of Cartesian currents and the notion of minimal connection in the language of Cartesian currents.

Let $u : B^3 \longrightarrow S^2$ be a *smooth* map with its graph given by

$$\mathcal{G}_u = \big\{ (x, y) \in B^3 \times S^2 \ : \ y = u(x) \big\}\,.$$

We can associate to u the three-dimensional Cartesian current $T_u \in \mathcal{D}_3(B^3 \times S^2)$ defined by

$$T_u(\omega) = \int_{\mathcal{G}_u} \omega = [\![G_u]\!](\omega)\,, \qquad \forall \omega \in \mathcal{D}^3(B^3 \times S^2)\,. \tag{6.16}$$

Considering the map

$$\begin{array}{rcl} id \bowtie u : B^3 & \longrightarrow & B^3 \times S^2 \\ x & \longmapsto & \big(x, u(x)\big)\,, \end{array}$$

we have

$$T_u(\omega) = \int_{B^3} (id \bowtie u)^* \omega\,. \tag{6.17}$$

For the boundary $\partial T_u \in \mathcal{D}_2(B^3 \times S^2)$ of T_u, we have by definition

$$\partial T_u(\eta) = T_u(d\eta) = \int_{B^3} (id \bowtie u)^* d\eta\,, \tag{6.18}$$

for every $\eta \in \mathcal{D}^2(B^3 \times S^2)$. Using the fact that the pull-back and the exterior derivative commute and Stokes' theorem, the last equation becomes

$$\partial T_u(\eta) = \int_{B^3} d\big((id \bowtie u)^* \eta\big) = \int_{\partial B^3} (id \bowtie u)^* \eta\,.$$

Since $(id \bowtie u)^* \eta$ is a compactly supported two-form on ∂B^3, the right hand side vanishes and hence T_u is boundaryless in $B^3 \times S^2$, i.e.,

$$\partial T_u \llcorner B^3 \times S^2 = 0\,. \tag{6.19}$$

For $u \in H^1(B^3, S^2)$ the Cartesian current $T_u \in \mathcal{D}_3(B^3 \times S^2)$ carried by the graph of u is defined as before. Its boundary, however, does *not* vanish in general. For example, in the particular case of $u \in R^\infty(B^3, S^2)$ being smooth except at a finite number of singular points $a_1, \ldots, a_N$ in B^3 with degrees $d_1, \ldots, d_N$, we have that (see M. Giaquinta, G. Modica and J. Souček [17], Sect. 4.2.1, Proposition 1)

$$\partial T_u \llcorner B^3 \times S^2 = -\sum_{i=1}^N d_i \delta_{a_i} \times [\![S^2]\!]\,. \tag{6.20}$$

From this we conclude that T_u can be completed to a *boundaryless* Cartesian current by defining

$$T = T_u + L \times [\![S^2]\!]\,,$$

where L is a one-dimensional i.m. rectifiable current on B^3 such that $\partial L = -\sum_{i=1}^N d_i \delta_{a_i}$.

The class $\mathrm{cart}^{2,1}(B^3, S^2)$ of Cartesian currents is defined in M. Giaquinta, G. Modica and J. Souček, [16] and [19], and can be characterized (see [17], Sect. 4.2.4, Theorem 1 and also [19], Sect. 5, Theorem 1) as the class of $T \in \mathcal{D}_3(B^3 \times S^2)$ without boundary in $B^3 \times S^2$ for which there exist a unique function $u_T \in H^1(B^3, S^2)$ and a unique one-dimensional i.m. rectifiable current L_T such that

$$T = T_{u_T} + L_T \times [\![S^2]\!]\,, \tag{6.21}$$

where T_{u_T} denotes the current integration over the graph of u_T. There are the following two particular types of Cartesian currents belonging to $\mathrm{cart}^{2,1}(B^3, S^2)$:

a) For $u \in C^\infty(B^3, S^2)$ the Cartesian current T_u is an element of $\operatorname{cart}^{2,1}(B^3, S^2)$, since (6.19) holds.

b) For $u \in R^\infty(B^3, S^2)$ the Cartesian current T_u can be completed to an element in $\operatorname{cart}^{2,1}(B^3, S^2)$ as explained before.

There is also a notion of minimal connection in the context of the Cartesian currents $T_u \in \mathcal{D}_3(B^3 \times S^2)$. – A minimal connection is defined as one-dimensional i.m. rectifiable current L of minimal mass with support $\operatorname{spt} L \subset \bar{B}^3$ such that

$$\partial T_u \llcorner B^3 \times S^2 = -\partial L \times [\![S^2]\!] \,. \tag{6.22}$$

This current is not necessarily unique and the mass $M(L)$ of L permits to recover formula (6.15) for the length of a minimal connection. To be more precise, it is possible to establish (6.6) with the help of $M(L)$ and the concept of calibration (see Section 6.2) in a more elegant way than the original proof of H. Brezis, J.-M. Coron and E. H. Lieb [2], Section IV. For more details we refer to M. Giaquinta, G. Modica and J. Souček [17], Sect. 4.2 and [18], Theorem 1.

6.2 Curvature Currents and Minimal Connection in the Abelian Case

In this section, we consider the Abelian case with its three classes $\mathcal{F}_\infty(B^3)$, $\mathcal{F}_R(B^3)$ and $\mathcal{F}_\mathbb{Z}(B^3)$ of weak curvatures (see Section 4.1). We will follow the methods described in the previous section.

Curvature Currents

The first aim of this section will be – roughly speaking – to construct "Cartesian currents" for weak curvatures in the Abelian case. More precisely, starting with two-forms on B^3 – instead of maps from B^3 into S^2 in the case of Cartesian currents – it is possible to construct a class of one-dimensional currents on B^3 with properties being very similar to the class of Cartesian currents introduced at the end of Section 6.1.

To every given $F \in \Omega^2_{L^1}(B^3)$, we can associate a one-dimensional current $T_F \in \mathcal{D}_1(B^3)$ defined by

$$T_F(\omega) = \frac{1}{2\pi} \int_{B^3} F \wedge \omega \,, \qquad \forall \omega \in \mathcal{D}^1(B^3) \,. \tag{6.23}$$

For the boundary $\partial T_F \in \mathcal{D}_0(B^3)$ of T_F, we have by definition

$$\partial T_F(\phi) = T_F(d\phi) \overset{(6.23)}{=} \frac{1}{2\pi} \int_{B^3} F \wedge d\phi \,, \tag{6.24}$$

for every $\phi \in \mathcal{D}^0(B^3) = C_c^\infty(B^3, \mathbb{R})$. Interpreting the right-hand side of the last equation as weak derivative of F and using the notation S_F for the boundary $\partial T_F \in \mathcal{D}_0(B^3)$ of T_F, we can rewrite (6.24) as

$$S_F(\phi) = \frac{1}{2\pi} \int_{B^3} F \wedge d\phi = -\frac{1}{2\pi} \int_{B^3} dF \, \phi \,. \tag{6.25}$$

We determine S_F explicitly for the following two particular cases:

a) If $F \in \mathcal{F}_\infty(B^3)$, the right-hand side of (6.25) vanishes and hence T_F is boundaryless, i.e.,
$$S_F = 0\,. \tag{6.26}$$

b) If $F \in \mathcal{F}_R(B^3)$, the right-hand side of (6.25) becomes
$$-\int_{B^3} dF\,\phi = -2\pi \int_{B^3} \left(\sum_{i=1}^N d_i \delta_{a_i} \right) \phi\, d^3x = -2\pi \sum_{i=1}^N d_i\, \phi(a_i)\,.$$
Thus, we arrive for the boundary of T_F at
$$S_F(\phi) = -\sum_{i=1}^N d_i \delta_{a_i}(\phi)\,, \qquad \forall \phi \in \mathcal{D}^0(B^3)\,,$$
which can be written as
$$S_F = -\sum_{i=1}^N d_i\, [\![a_i]\!]\,. \tag{6.27}$$
This shows that $S_F \in \mathcal{R}_0(B^3)$, i.e., in the case of $F \in \mathcal{F}_R^2(B^3)$ the boundary of T_F is an zero-dimensional i.m. rectifiable current. For later use, the following observation will be important: Since S_F is given in (6.27) as integration over a finite number of points with $\sum_{i=1}^N d_i = 0$ (this follows from appropriate boundary conditions), we deduce that S_F bounds an one-dimensional i.m. rectifiable current. More precisely, there exists $L \in \mathcal{R}_1(B^3)$ such that $S_F = -\partial L$.

Next, we want to show that for $F \in \Omega^2_{L^1}(B^3)$ the Euclidean mass $\underline{M}(T_F)$ of $T_F \in \mathcal{D}_1(B^3)$ in B^3 given by
$$\underline{M}(T_F) = \sup_{\substack{\omega \in \mathcal{D}^1(B^3) \\ \|\omega\|_{L^\infty} \le 1}} T_F(\omega) = \sup_{\substack{\omega \in \mathcal{D}^1(B^3) \\ \|\omega\|_{L^\infty} \le 1}} \int_{B^3} F \wedge \omega \tag{6.28}$$
is finite. For this purpose, we compute
$$\begin{aligned} \underline{M}(T_F) &= \sup_{\substack{\omega \in \mathcal{D}^1(B^3) \\ \|\omega\|_{L^\infty} \le 1}} \int_{B^3} F \wedge \omega \le \sup_{\substack{\omega \in \mathcal{D}^1(B^3) \\ \|\omega\|_{L^\infty} \le 1}} \int_{B^3} |F \wedge \omega(x)|\, d^3x \\ &\le \int_{B^3} |F(x)|\, d^3x \le C < \infty\,, \end{aligned} \tag{6.29}$$
where we used that the two-form F is integrable by assumption. Since the mass $M(T_F)$ of $T_F \in \mathcal{D}_1(B^3)$ in B^3 given by
$$M(T_F) = \sup_{\substack{\omega \in \mathcal{D}^1(B^3) \\ \|\omega\|^{co}_{L^\infty} \le 1}} T_F(\omega) = \sup_{\substack{\omega \in \mathcal{D}^1(B^3) \\ \|\omega\|^{co}_{L^\infty} \le 1}} \int_{B^3} F \wedge \omega \tag{6.30}$$

satisfies – as a direct consequence of $\|\omega\|^{co}_{L^\infty} \leq \|\omega\|_{L^\infty}$ –

$$\underline{M}(T_F) \leq M(T_F)\,, \tag{6.31}$$

it follows that $M(T_F)$ is also finite.

In summary, comparing (6.26) with (6.19) and (6.27) with (6.20), we conclude that the current (6.23) has some analogies with the Cartesian current (6.16). Motivated also by (6.21), it seems therefore natural to introduce the following class of one-dimensional currents on B^3:

$$\begin{aligned}\mathrm{Curv}(B^3) = \Big\{ T \in \mathcal{D}_1(B^3) \quad : \quad & \exists\, F_T \in \Omega^2_{L^1}(B^3) \text{ and } \exists\, L_T \in \mathcal{R}_1(B^3) \\ & \text{s.t. } T = T_{F_T} + L_T\,,\ \ \partial T = 0 \text{ on } B^3 \Big\}\,.\end{aligned} \tag{6.32}$$

In the following, these currents will be called *curvature currents*. There are two particular types of currents belonging to $\mathrm{Curv}(B^3)$:

a) For $F \in \mathcal{F}_\infty(B^3)$ the current T_F is an element of $\mathrm{Curv}(B^3)$, since it already satisfies $\partial T_F = 0$.

b) As explained before, we know that for the one-dimensional current T_F associated to $F \in \mathcal{F}_R(B^3)$, there exists $L \in \mathcal{R}_1(B^3)$ such that $S_F = -\partial L$. Thus the completed current $T = T_F + L$ belongs to $\mathrm{Curv}(B^3)$.

Smooth boundary Data

At this stage, we introduce some *smooth* boundary data $\varphi \in \Omega^2_\infty(S^2)$ which additionally satisfy

$$\int_{S^2} \varphi = 0\,. \tag{6.33}$$

Since

$$\int_{S^2} \varphi = \int_{\partial B^3} \iota^*_{\partial B^3} F = \int_{B^3} dF = \int_{B^3} d(dA) = 0\,,$$

we conclude that (6.33) is obviously verified for $F \in \mathcal{F}_\infty(B^3)$. In the case of $F \in \mathcal{F}_R(B^3)$, however, the assumption (6.33) translates to

$$\sum_{i=1}^N d_i = 0\,. \tag{6.34}$$

Moreover, we set

$$\mathcal{F}_{\infty,\varphi}(B^3) = \big\{ F \in \mathcal{F}_\infty(\bar{B}^3) \ : \ \iota^*_{\partial B^3} F = \varphi \big\}\,, \tag{6.35}$$

and similarly for $\mathcal{F}_{R,\varphi}(B^3)$.

We can also think of φ as being the restriction of a smooth two-form – still denoted by φ – defined on some open set $\tilde{B}^3 \supset\supset B^3$. We then define

$$\mathrm{Curv}_\varphi(\tilde{B}^3) = \big\{ T \in \mathrm{Curv}(\tilde{B}^3) \ : \ (T - T_\varphi) \llcorner \tilde{B}^3 \setminus \bar{B}^3 = 0 \big\}\,, \tag{6.36}$$

where T_φ is defined as in (6.23) and $\llcorner$ denotes the restriction of $T - T_\varphi \in \mathcal{D}_1(\tilde{B}^3)$ to $\tilde{B}^3 \setminus \bar{B}^3$.

Minimal Connection

In the context of curvature currents, there is a notion of minimal connection. – Starting from $F \in \mathcal{F}_{R,\varphi}(\tilde{B}^3)$, which can be interpreted as an element in $\mathcal{F}_R(B^3)$ extended by the smooth boundary data φ on $\tilde{B}^3 \setminus B^3$, we consider T_F with boundary $S_F \in \mathcal{R}_0(\tilde{B}^3)$. Then we define the *minimal connection* L_F of F as a one-dimensional i.m. rectifiable current supported in $\bar{B}^3$ of minimal mass whose boundary satisfies $\partial L_F = -S_F$. In other words, setting

$$m_{integer}(S_F) = \inf_{\substack{L \in \mathcal{R}_1(\tilde{B}^3) \\ spt\, L \subset \bar{B}^3\,,\ \partial L = -S_F}} M(L)\,, \tag{6.37}$$

we can write $m_{integer}(S_F) = M(L_F)$.

In order to give a formula for $m_{integer}(S_F)$, we first need some preparing results on minimal currents.

Theorem 6.1. *Let S be a zero-dimensional rectifiable current on $\tilde{B}^3$ with $spt\, S \subset \bar{B}^3$ which is the boundary of an one-dimensional rectifiable current on $\tilde{B}^3$ supported in $\bar{B}^3$. Then, we have*

$$m_{integer}(S) = m_{real}(S)\,, \tag{6.38}$$

where

$$m_{real}(S) = \inf_{\substack{T \in \mathcal{D}_1(\tilde{B}^3) \\ spt\, T \subset \bar{B}^3\,,\ \partial T = -S}} M(T)\,. \tag{6.39}$$

Proof. From the definitions (6.37) and (6.39), we obtain directly

$$m_{real}(S) \leq m_{integer}(S)\,. \tag{6.40}$$

For the converse inequality, we refer to M. Giaquinta, G. Modica and J. Souček [17], Sect. 1.3.4, especially Theorem 8 and references therein. – It is important to note that for a one-dimensional rectifiable current in four dimensions the inequality in (6.40) is strict (see also R.M. Pakzad, [35]). □

Proposition 6.2. *Let S be a zero-dimensional current on $\tilde{B}^3$ with $spt\, S \subset \bar{B}^3$ which is the boundary of an one-dimensional current on $\tilde{B}^3$ supported in $\bar{B}^3$. Then, we have*

$$m_{real}(S) = \sup_{\substack{\xi : B^3 \to \mathbb{R}^3 \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} S(\xi)\,. \tag{6.41}$$

Proof. Let T denote an one-dimensional *minimal* current on $\tilde{B}^3$ supported in $\bar{B}^3$ which satisfies $\partial T = -S$, i.e. $M(T) = m_{real}(S)$. From R.M. Pakzad [35], Chap.1, Proposition 2.3, we know that this is equivalent to the existence of an exact one-form $d\xi_{cal}$ – called calibration – such that $M(T) = T(d\xi_{cal})$ (see also Definition 6.1 and Proposition 6.4 below). This implies that

$$M(T) = T(d\xi_{cal}) = -S(\xi_{cal})\,,$$

and taking the supremum

$$M(T) \leq \sup_{\substack{\xi : B^3 \to \mathbb{R}^3 \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} S(\xi)\,.$$

On the other hand, it is evident from the definition of the mass of a current

$$M(T) \geq \sup_{\substack{\xi : B^3 \to \mathbb{R}^3 \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} T(d\xi) = \sup_{\substack{\xi : B^3 \to \mathbb{R}^3 \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} S(\xi) \,.$$

Thus, we arrive at

$$m_{real}(S) = M(T) = \sup_{\substack{\xi : B^3 \to \mathbb{R}^3 \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} S(\xi) \,.$$

□

At this stage, we can combine Theorem 6.1 with Proposition 6.2, in order to obtain a formula for $m_{integer}(S_F)$ being very similar to (6.15). In the following, we will thus refer to $m_i(S_F)$ as the *length of a minimal connection* for $F \in \mathcal{F}_{R,\varphi}(\tilde{B}^3)$ and denote it by $L(F)$.

Proposition 6.3. *Let $F \in \mathcal{F}_{R,\varphi}(\tilde{B}^3)$ and $S_F \in \mathcal{R}_0(\tilde{B}^3)$ the boundary of T_F. Then, we have*

$$L(F) = m_{integer}(S_F) = m_{real}(S_F) = \sup_{\substack{\xi : B^3 \to \mathbb{R}^3 \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} S_F(\xi) \,. \tag{6.42}$$

Moreover, the following formula holds for the length of a minimal connection:

$$L(F) = \frac{1}{2\pi} \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} \left\{ \int_{B^3} F \wedge d\xi - \int_{\partial B^3} \iota^*_{\partial B^3}(F\,\xi) \right\} . \tag{6.43}$$

Proof. Recall from (6.27) that if $F \in \mathcal{F}_{R,\varphi}(\tilde{B}^3)$, then S_F is a zero-dimensional i.m. rectifiable current which bounds an one-dimensional i.m. rectifiable current. Thus Theorem 6.1 gives directly the second equality in (6.42). The third equality in (6.42) is a direct consequence of Proposition 6.2. – For completeness note that the first equality in (6.42) is just the definition of the length of a minimal connection.

In order to show the formula (6.43) for the length of a minimal connection, we have using Stokes' theorem

$$-\int_{B^3} dF\,\xi = -\int_{\partial B^3} \iota^*_{\partial B^3}(F\,\xi) + \int_{B^3} F \wedge d\xi \,,$$

for every function $\xi : B^3 \longrightarrow \mathbb{R}$. Hence, we obtain from (6.25) that

$$\begin{aligned} L(F) &= \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} S_F(\xi) = \frac{1}{2\pi} \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} \left\{ -\int_{B^3} dF\,\xi \right\} \\ &= \frac{1}{2\pi} \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} \left\{ \int_{B^3} F \wedge d\xi - \int_{\partial B^3} \iota^*_{\partial B^3}(F\,\xi) \right\} , \end{aligned}$$

showing the formula (6.43) for $L(F)$. □

Note that the right-hand side of (6.43) is also well-defined for any F in the L^1-closure $\mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3)$ of $\mathcal{F}_{R,\varphi}(\tilde{B}^3)$ and, moreover, finite since by assumption (see (6.33))

$$\int_{\partial B^3} \iota^*_{\partial B^3} F = \int_{\partial B^3} \varphi = 0\,.$$

Thus, we take (6.43) as definition for $L(F)$ in the case of $F \in \mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3)$. We emphasize that it is not clear wether $L(F)$ can also be defined by $m_{integer}(S_F)$, since there exists a priori no i.m. rectifiable current L such that $\partial L = -S_F$. It will turn out that such a definition for the length of a minimal connection can be extended to the class $\mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3)$ (see Corollary 6.6 below). At the end of this section, we will also show that $\mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3) \cap \mathcal{F}_{\mathbb{Z},\eta}(\tilde{B}^3) \neq \emptyset$ with η some other smooth boundary data implies $\varphi = \eta$.

Calibration and Minimal Connection

Now, we want to study the length of a minimal connection in the context of calibrated currents.

Definition 6.1. *Let $L \in \mathcal{R}_1(\tilde{B}^3)$ such that $spt\, L \subset \bar{B}^3$. The exact measurable one-form α with $\|\alpha\|^{co}_{L^\infty} \leq 1$ is called a calibration for L in $\bar{B}^3$ if*

$$M(L) = L(\alpha)\,. \tag{6.44}$$

Moreover, we then say that L is calibrated in $\bar{B}^3$.

There is an interesting connection between calibrated and minimal currents stated in the next proposition.

Proposition 6.4. *Let S be as before and assume that $L \in \mathcal{R}_1(\tilde{B}^3)$ is minimal, i.e., $M(L) = m_{integer}(S)$. Then L is calibrated in $\bar{B}^3$ if and only if $m_{integer}(S) = m_{real}(S)$.*

Proof. We already know that $L \in \mathcal{R}_1(\tilde{B}^3)$ is calibrated if and only if $m_{real}(S) = M(L)$ (see R.M. Pakzad [35], Chap.1, Proposition 2.3). This result suffices for the proof of the proposition.

In addition, we give a direct proof for one direction of the equivalence in the proposition. – Assume that $L \in \mathcal{R}_1(\tilde{B}^3)$ verifying $M(L) = m_{integer}(S)$ is calibrated. By Definition 6.1, there exists $\alpha = d\xi_{cal}$ with $\|\alpha\|^{co}_{L^\infty} \leq 1$ such that $M(L) = L(\alpha)$. This leads to

$$M(L) = L(\alpha) = L(d\xi_{cal}) = \partial L(\xi_{cal}) = -S(\xi_{cal})\,. \tag{6.45}$$

Now, let $T \in \mathcal{D}_1(\tilde{B}^3)$ with $\operatorname{spt} T \subset \bar{B}^3$ such that $\partial T = -S$. We then have

$$M(T) \geq T(\alpha) = T(d\xi_{cal}) = -S(\xi_{cal}) \overset{(6.45)}{=} M(L) = m_{integer}(S)\,,$$

showing by taking the infimum that $m_{real}(S) \geq m_{integer}(S)$. Since the converse inequality is clear (see (6.40)), we have thus shown that

$$m_{real}(S) = m_{integer}(S)\,.$$

$\square$

At this stage, we are ready to recover again a well-known result for the length of a minimal connection. For this purpose, we consider $F \in \mathcal{F}_{R,\varphi}(\tilde{B}^3)$ with $S_F \in \mathcal{R}_0(\tilde{B}^3)$ given by (see (6.27))

$$S_F = -\sum_{i=1}^{N} d_i \, [\![a_i]\!] = -\sum_{i=1}^{K} \big([\![a_i^+]\!] - [\![a_i^-]\!]\big) \, .$$

Denote by $L_F \in \mathcal{R}_1(\tilde{B}^3)$ a minimal connection for F. Since $m_{integer}(S_F) = m_{real}(S_F)$ holds, Proposition 6.4 implies that L_F is calibrated in $\bar{B}^3$. Thus, there exists $d\xi_{cal}$ with $\|d\xi_{cal}\|^{co}_{L^\infty} \leq 1$ such that

$$M(L_F) = L(d\xi_{cal}) = -S(\xi_{cal}) = \sum_{i=1}^{K} \big(\xi_{cal}(a_i^+) - \xi_{cal}(a_i^-)\big) \, .$$

Hence, taking the supremum, we have

$$M(L_F) \leq \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} \left\{ \sum_{i=1}^{K} \big(\xi(a_i^+) - \xi(a_i^-)\big) \right\} .$$

On the other hand, it is evident from the definition of the mass of a current that

$$M(L_F) \geq \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} L_F(d\xi) = \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} \left\{ \sum_{i=1}^{K} \big(\xi(a_i^+) - \xi(a_i^-)\big) \right\} .$$

Thus, we end up with the following result for the length $L(F) = M(L_F)$ of a minimal connection:

$$L(F) = \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} \left\{ \sum_{i=1}^{K} \big(\xi(a_i^+) - \xi(a_i^-)\big) \right\} . \tag{6.46}$$

This was already obtained in (6.6) for the case of maps from B^3 into S^2 without using the language of calibrations.

Example of a Dipole

Consider the dipole $F_d \in \mathcal{F}_R(B^3)$ defined by

$$dF_d = 2\pi(\delta_{a_+} - \delta_{a_-}) \, dx^1 \wedge dx^2 \wedge dx^3 \quad \text{in } \mathcal{D}' \, . \tag{6.47}$$

Its boundary then reads as

$$S_{F_d}(\phi) = -\frac{1}{2\pi} \int_{B^3} dF_d \, \phi = \phi(a_-) - \phi(a_+) \, ,$$

where $\phi \in C_c^\infty(B^3, \mathbb{R})$. We claim that the minimal connection $L_{F_d} \in \mathcal{R}_1(B^3)$ for the dipole is given by integration over the segment joining a_- and $a_+ \in B^3$, i.e.,

$$L_{F_d} = [\![a_-, a_+]\!] \, . \tag{6.48}$$

First, we observe that

$$\phi(a_+) - \phi(a_-) = \int_{[a_-,a_+]} d\phi = L_{F_d}(d\phi) = \partial L_{F_d}(\phi)\,,$$

implying that $\partial L_{F_d} = -S_{F_d}$. Moreover, it is not difficult to check that

$$L(F_d) \overset{(6.46)}{=} \sup_{\substack{\xi : B^3 \to \mathbb{R} \\ \|d\xi\|^{co}_{L^\infty} \leq 1}} \big\{\xi(a_+) - \xi(a_-)\big\} = \|a_+ - a_-\|\,.$$

Since $M(L_{F_d}) = \|a_+ - a_-\|$, the claim is proved.

For the dipole, we go a step further. – Choose coordinates on B^3 such that $a_+ = (0,0,\frac{1}{2})$ and $a_- = (0,0,-\frac{1}{2})$. Then, for every $\phi \in C_c^\infty(B^3,\mathbb{R})$, the following equation holds:

$$\int_{B^3} \chi_{[a_-,a_+]}\, d\phi \wedge dx^1 \wedge dx^2 = \int_{[a_-,a_+]} d\phi\,,$$

where $\chi_{[a_-,a_+]}$ is the characteristic function of the segment $[a_-,a_+]$. This shows that in this coordinates the minimal connection $L_{F_d} \in \mathcal{R}_1(B^3)$ defined in (6.48) has the coordinate representation

$$L_{F_d}(\omega) = \int_{B^3} \chi_{[a_-,a_+]}\, dx^1 \wedge dx^2 \wedge \omega\,, \qquad \forall \omega \in \mathcal{D}^1(B^3)\,. \tag{6.49}$$

We then obtain that $S_{F_d} + \partial L_{F_d} = 0$ translates to

$$\int_{B^3} F_d \wedge d\phi + \int_{B^3} \chi_{[a_-,a_+]}\, d\phi \wedge dx^1 \wedge dx^2 = 0\,,$$

or, equivalently,

$$\int_{B^3} \big(F_d + \chi_{[a_-,a_+]}\, dx^1 \wedge dx^2\big) \wedge d\phi = 0\,.$$

Using Stokes' theorem, we deduce

$$\int_{B^3} d\big(F_d + \chi_{[a_-,a_+]}\, dx^1 \wedge dx^2\big)\phi = 0\,,$$

for all $\phi \in C_c^\infty(B^3,\mathbb{R})$, and hence

$$d\big(F_d + \chi_{[a_-,a_+]}\, dx^1 \wedge dx^2\big) = 0 \quad \text{in } \mathcal{D}'\,. \tag{6.50}$$

This shows that for the dipole $F_d \in \mathcal{F}_R^2(B^3)$ – which is clearly not exact on B^3 – we have, however, that $F_d + \chi_{[a_-,a_+]}\, dx^1 \wedge dx^2$ is exact. In other words, the two-form F_d completed by $\chi_{[a_-,a_+]}\, dx^1 \wedge dx^2$ is exact.

Closure of the Class of Curvature Currents

At this place, we want to answer the question, if the current associated to an element in $\mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3)$ can be completed to a boundaryless current by an i.m. rectifiable current. This is Theorem 1.8 in the introduction. – The next proposition shows that the class $\mathrm{Curv}_\varphi(\tilde{B}^3)$ of curvature currents is closed with respect to L^1-convergence of the associated two-forms. The proof follows M. Giaquinta, G. Modica and J. Souček [17], Section 4.2.5, Proposition 4.

Theorem 6.5. *Let $(F_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}_{R,\varphi}(\tilde{B}^3)$ such that $F_k \stackrel{k\to\infty}{\longrightarrow} F$ in L^1. Then, there exists $L_T \in \mathcal{R}_1(\tilde{B}^3)$ with $\operatorname{spt} L_T \subset \bar{B}^3$ such that $\partial L_T = -S_F$. In other words, we have that*

$$T = T_F + L_T \in \mathit{Curv}_\varphi(\tilde{B}^3)\,.$$

Proof. Recall that to the given $F \in \Omega^2_{L^1,\varphi}(\tilde{B}^3)$ and $F_k \in \mathcal{F}_{R,\varphi}(\tilde{B}^3)$ we can associate one-dimensional currents T_F and T_{F_k} on $\tilde{B}^3$ with finite mass via (6.23). For $\omega \in \mathcal{D}^1(\tilde{B}^3)$ with $\|\omega\|^{co}_{L^\infty} \le 1$, we then get

$$\begin{aligned}
\big|T_{F_k}(\omega) - T_F(\omega)\big| &= \frac{1}{2\pi}\left|\int_{\tilde{B}^3} F_k \wedge \omega - \int_{\tilde{B}^3} F \wedge \omega\right| \\
&\le \frac{1}{2\pi}\int_{\tilde{B}^3}\big|F_k - F(x)\wedge\omega(x)\big|\,d^3x \\
&\le \frac{1}{2\pi}\int_{\tilde{B}^3}\big|F_k - F(x)\big|\big|\omega(x)\big|\,d^3x \\
&\le \frac{1}{2\pi}\int_{\tilde{B}^3}\big|F_k - F(x)\big|\,d^3x \longrightarrow 0 \qquad (k\to\infty)\,,
\end{aligned}$$

where we used that $F_k \stackrel{k\to\infty}{\longrightarrow} F$ in L^1 by assumption. From this we deduce the convergence in mass $M(T_{F_k} - T_F) \stackrel{k\to\infty}{\longrightarrow} 0$. Since

$$M^*(S_{F_k}) := \sup_{\substack{\phi\in C^\infty_c(\tilde{B}^3,\mathbb{R})\\ \|d\phi\|^{co}_{L^\infty}\le 1}} S_{F_k}(\phi) = \sup_{\substack{\phi\in C^\infty_c(\tilde{B}^3,\mathbb{R})\\ \|d\phi\|^{co}_{L^\infty}\le 1}} T_{F_k}(d\phi) \le M(T_{F_k})\,, \tag{6.51}$$

the convergence in mass also gives $M^*(S_{F_k} - S_F) \stackrel{k\to\infty}{\longrightarrow} 0$. Hence, there exists a subsequence of $(F_k)_{k\in\mathbb{N}}$ for which (without changing the notation)

$$M^*(S_{F_k} - S_{F_{k+1}}) \le \frac{1}{2^k}\,. \tag{6.52}$$

We already know that for every $F_k \in \mathcal{F}_{R,\varphi}(\tilde{B}^3)$, there exists $L_{F_k} \in \mathcal{R}_1(\tilde{B}^3)$ with $\operatorname{spt} L_{F_k} \subset \bar{B}^3$ such that the completed current $T_k = T_{F_k} + L_{F_k}$ belongs to $\mathrm{Curv}_\varphi(\tilde{B}^3)$. Moreover, considering the difference $S_{F_k} - S_{F_{k+1}} \in \mathcal{R}_0(\tilde{B}^3)$, we denote its minimal connection by $L_{F_k,F_{k+1}} \in \mathcal{R}_1(\tilde{B}^3)$. This means that the one-dimensional i.m. rectifiable current $L_{F_k,F_{k+1}}$ supported in $\bar{B}^3$ satisfies

$$\partial L_{F_k,F_{k+1}} = -(S_{F_k} - S_{F_{k+1}})\,, \tag{6.53}$$

and

$$M(L_{F_k,F_{k+1}}) = m_{integer}(S_{F_k} - S_{F_{k+1}})\,. \tag{6.54}$$

Then, using Theorem 6.1 and Proposition 6.2, we obtain that

$$\begin{aligned} M(L_{F_k,F_{k+1}}) &\overset{(6.54)}{=} m_{integer}(S_{F_k} - S_{F_{k+1}}) = m_{real}(S_{F_k} - S_{F_{k+1}}) \\ &= M^*(S_{F_k} - S_{F_{k+1}}) \overset{(6.52)}{\leq} \frac{1}{2^k}\,. \end{aligned} \tag{6.55}$$

In a next step, we consider the current $L_N = L_{F_0} - \sum_{k=0}^{N-1} L_{F_k,F_{k+1}}$. Its boundary reads as

$$\begin{aligned} \partial\left(L_{F_0} - \sum_{k=0}^{N-1} L_{F_k,F_{k+1}}\right) &= \partial L_{F_0} - \big(\partial L_{F_0,F_1} + \partial L_{F_1,F_2} + \ldots + L_{F_{N-1},F_N}\big) \\ &\overset{(6.53)}{=} -S_{F_0} + \Big((S_{F_0} - S_{F_1}) + \ldots + (S_{F_{N-1}} - S_{F_N})\Big) \\ &= -S_{F_N}\,. \end{aligned} \tag{6.56}$$

Defining $L_T = \lim_{N\to\infty} L_N$, we deduce from (6.55) that its mass is finite and from (6.56) together with the assumption $F_k \overset{k\to\infty}{\longrightarrow} F$ in L^1 it follows that

$$\partial L_T = \lim_{N\to\infty} -S_{F_N} = -S_F\,,$$

with convergence in the weak sense. In summary, we deduce that

$$L_T = L_0 - \sum_{k=0}^{\infty} L_{F_k,F_{k+1}} \tag{6.57}$$

is an i.m. rectifiable current satisfying $\partial L_T = -S_F$. This proves the proposition. $\square$

Remark 6.1. a) Remark that due to the existence of $L_T \in \mathcal{R}_1(\tilde{B}^3)$ such that $\partial L_T = -S_F$, for $F \in \Omega^2_{L^1,\varphi}(\tilde{B}^3)$, the current S_F is in fact summation over an *infinite* number of points in B^3, i.e.,

$$S_F = -\sum_{i=1}^{\infty}\big([\![a_i^+]\!] - [\![a_i^-]\!]\big)\,.$$

However, for the points the following must hold:

$$\sum_{i=1}^{\infty} \|a_i^+ - a_i^-\| \leq M(L_T) < \infty\,.$$

Recall that $L_T \in \mathcal{R}_1(\tilde{B}^3)$ was defined in (6.57).

b) Since $F_k \overset{k\to\infty}{\longrightarrow} F$ in L^1 by assumption, we deduce that $M(T_{F_k})$ is uniformly bounded (see (6.29)), i.e., for all $k \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$M(T_{F_k}) \leq C < \infty\,.$$

Since

$$L(F_k) \overset{(6.42)}{=} \sup_{\substack{\xi:B^3\to\mathbb{R}^3\\ \|d\xi\|^{co}_{L^\infty}\le 1}} S_{F_k}(\xi) = \sup_{\substack{\xi:B^3\to\mathbb{R}^3\\ \|d\xi\|^{co}_{L^\infty}\le 1}} T_{F_k}(d\xi) \le M(T_{F_k})\,,$$

the sequence $(L_{F_k})_{k\in\mathbb{N}}$ in $\mathcal{R}_1(\tilde{B}^3)$ of minimal connections for F_k is also uniformly bounded and thus converges weakly

$$L_{F_k} \rightharpoonup L \qquad (k\to\infty)$$

to some $L\in\mathcal{D}_1(B^3)$. However, there is *no* uniform bound for $M(\partial L_{F_k}) = M(S_{F_k})$, $k\in\mathbb{N}$, which would imply due to the closure theorem for i.m. rectifiable currents (see H. Federer [12], 4.2.16) that L is in fact an i.m. rectifiable current. – This would be a direct proof of Theorem 6.5.

Recalling that $\mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3)$ denotes the L^1-closure of $\mathcal{F}_{R,\varphi}(\tilde{B}^3)$ we deduce from Theorem 6.5 that elements in $\mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3)$ generate currents which can be completed in such a way that they become curvature currents. This gives a proof of Theorem 1.8. As a direct consequence, we obtain the following corollary:

Corollary 6.6. *Let $F\in\mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3)$. Then, we have that*

$$L(F) = m_{integer}(S_F) = m_{real}(S_F) = \sup_{\substack{\xi:B^3\to\mathbb{R}^3\\ \|d\xi\|^{co}_{L^\infty}\le 1}} S_F(\xi)\,. \tag{6.58}$$

Moreover, the length of a minimal connection is continuous with respect to L^1-convergence, i.e.,

$$L(F_k) \longrightarrow L(F) \qquad (k\to\infty)\,. \tag{6.59}$$

Proof. Because of Theorem 1.8, the proof of (6.42) also apply to $F\in\mathcal{F}_{\mathbb{Z},\varphi}(\tilde{B}^3)$. Thus (6.58) is shown.

The continuity of L is then an immediate consequence of (6.58) and the trivial estimate

$$\begin{aligned}\big|L(F_k)-L(F)\big| &= \big|M^*(S_{F_k})-M^*(S_F)\big| \\ &\le M^*(S_{F_k}-S_F) \longrightarrow 0 \qquad (k\to\infty)\,,\end{aligned}$$

where we used also the assumption of L^1-convergence. □

Proposition 6.7. *Let $F\in\mathcal{F}_{\mathbb{Z},\varphi}(B^3)\cap\mathcal{F}_{\mathbb{Z},\eta}(B^3)$ for smooth boundary data $\varphi,\eta\in\Omega^2_\infty(\partial B^3)$. Then we have that $\varphi=\eta$.*

Proof. By assumption there exists a sequence $(F_k)_{k\in\mathbb{N}}$ in $\mathcal{F}_{R,\varphi}(B^3)$ such that

$$F_k \longrightarrow F \qquad \text{in } L^1\,. \tag{6.60}$$

We already know that there exists a sequence $(L_{F_k})_{k\in\mathbb{N}}$ in $\mathcal{R}_1(B^3)$ such that $\partial(T_{F_k}+L_{F_k}) = S_{F_k}+\partial L_{F_k} = 0$ in B^3. Moreover, Theorem 6.5 implies that F can also be completed by an one-dimensional i.m. rectifiable current L such that $\partial(T_F+L) = S_F+\partial L = 0$ in B^3.

Next, we deduce from (6.60) that $T_{F_k}(\omega) \longrightarrow T_F(\omega)$ for every $\omega \in \mathcal{D}^1(B^3)$, and the proof of Theorem 6.5 gives $L_{F_k}(d\phi) \longrightarrow L(d\phi)$ for every $\phi \in C_c^\infty(B^3)$. Hence, we deduce

$$T_{F_k}(d\phi) + L_{F_k}(d\phi) \longrightarrow T_F(d\phi) + L(d\phi)\,, \tag{6.61}$$

for every $\phi \in C_c^\infty(B^3)$. On the other hand, we have by definition that

$$T_{F_k}(d\phi) + L_{F_k}(d\phi) = \int_{\partial B^3} \varphi\,\phi\,.$$

The convergence (6.61) then yields

$$T_F(d\phi) + L(d\phi) = \int_{\partial B^3} \varphi\,\phi\,. \tag{6.62}$$

By assumption there exists another sequence $(F_k')_{k\in\mathbb{N}}$ in $\mathcal{F}_{R,\eta}(B^3)$ such that $F_k \longrightarrow F$ in L^1. Proceeding as before, we deduce the existence of an one-dimensional i.m. rectifiable current J such that

$$T_F(d\phi) + J(d\phi) = \int_{\partial B^3} \eta\,\phi\,. \tag{6.63}$$

Since $\partial I = \partial J$, it follows that $\partial(T_F + L) = \partial(T_F + J)$ in B^3. Hence, comparing (6.62) with (6.63) we arrive at

$$\int_{\partial B^3} \varphi\,\phi = \int_{\partial B^3} \eta\,\phi\,,$$

for every $\phi \in C_c^\infty(B^3)$, showing that $\varphi = \eta$ as desired. □

6.3 Curvature Currents and Minimal Connection in the Non-Abelian Case

In this section, we pass from the Abelian to the non-Abelian case by transfering the notion of curvature currents to the classes of weak curvatures in the non-Abelian case. Since the arguments and calculations are very similar to the previous Section 6.2, we often only quote the results.

To every $F \in \Omega^2_{L^2}\big(B^5, \mathfrak{su}(2)\big)$ we can associate the four-form $\hat\Omega = \mathrm{Tr}(F \wedge F)$ belonging to $\Omega^4_{L^1}(B^5)$ and then a one-dimensional current $T_{\hat\Omega} \in \mathcal{D}_1(B^5)$ defined by

$$T_{\hat\Omega}(\omega) = \frac{1}{8\pi^2}\int_{B^5} \hat\Omega \wedge \omega = \frac{1}{8\pi^2}\int_{B^5} \mathrm{Tr}(F\wedge F)\wedge\omega\,, \qquad \forall \omega \in \mathcal{D}^1(B^5)\,. \tag{6.64}$$

Repeating the computations leading to (6.31), we deduce that its mass $M(T_{\hat\Omega})$ is finite. The boundary $S_{\hat\Omega} \in \mathcal{D}_0(B^5)$ of $T_{\hat\Omega}$ reads as

$$S_{\hat\Omega}(\phi) = \frac{1}{8\pi^2}\int_{B^5} \mathrm{Tr}(F\wedge F)\wedge d\phi = -\frac{1}{8\pi^2}\int_{B^5} d\,\mathrm{Tr}(F\wedge F)\phi \qquad \forall \phi \in C^\infty(B^5,\mathbb{R})\,. \tag{6.65}$$

As in the Abelian case, we introduce the following class of one-dimensional curvature currents on B^5:

$$\mathrm{Curv}(B^5) = \Big\{ T \in \mathcal{D}_1(B^5) \quad : \quad \exists\, \hat{\Omega}_T \in \Omega^4_{L^1}(B^5) \ \text{and} \ \exists\, L_T \in \mathcal{R}_1(B^5) \\ \text{s.t.}\ T = T_{\hat{\Omega}_T} + L_T\,,\ \ \partial T = 0 \ \text{ on } \ B^5 \Big\}\,. \tag{6.66}$$

There are two particular types of curvature currents.

a) If $F \in \mathcal{F}_\infty(B^5)$, we obtain that

$$S_{\hat{\Omega}} = 0\,. \tag{6.67}$$

Hence the current $T_{\hat{\Omega}}$ in the case of $F \in \mathcal{F}_\infty(B^5)$ belongs to $\mathrm{Curv}(B^5)$.

b) If $F \in \mathcal{F}_R(B^5)$, we obtain that

$$S_{\hat{\Omega}} = -\sum_{i=1}^{N} d_i\, [\![a_i]\!]\,. \tag{6.68}$$

Thus there exists an one-dimensional i.m. rectifiable current L_T such that $\partial L_T = -S_{\hat{\Omega}}$. It follows also that the completed current $T = T_{\hat{\Omega}} + L_T$ belongs to $\mathrm{Curv}(B^5)$.

The notion of minimal connection in the non-Abelian case can be defined exactly in the same way as in the Abelian case with the help of minimal currents. A formula for the length of a minimal connection for elements of $\mathcal{F}_{R,\varphi}(\tilde{B}^5)$ has already appeared in T. Isobe [25], Theorem 1.6, where the approach of H. Brezis, J.-M. Coron and E. H. Lieb, [2] without the language of currents is used.

Proposition 6.8. *Let $F \in \mathcal{F}_{R,\varphi}(\tilde{B}^5)$ and $S_{\hat{\Omega}} \in \mathcal{R}_0(\tilde{B}^5)$ the boundary of $T_{\hat{\Omega}}$. Then, we have*

$$L(F) = m_{integer}(S_{\hat{\Omega}}) = m_{real}(S_{\hat{\Omega}}) = \sup_{\substack{\xi : B^5 \to \mathbb{R}^3 \\ \|d\xi\|^{co}_{L^\infty} \le 1}} S_{\hat{\Omega}}(\xi)\,. \tag{6.69}$$

Moreover, the following formula holds for the length of a minimal connection:

$$L(F) = \frac{1}{8\pi^2} \sup_{\substack{\xi : B^5 \to \mathbb{R}^3 \\ \|d\xi\|^{co}_{L^\infty} \le 1}} \left\{ \int_{B^5} \mathrm{Tr}(F \wedge F) \wedge d\xi - \int_{\partial B^5} \iota^*_{\partial B^5}\big(\mathrm{Tr}(F \wedge F)\xi\big) \right\}\,. \tag{6.70}$$

Proof. The only difficulty in the proof is two show the second equality in (6.69), namely that

$$m_{integer}(S_{\hat{\Omega}}) = m_{real}(S_{\hat{\Omega}})\,.$$

But this holds, since the rectifiable current $S_{\hat{\Omega}}$ is zero-dimensional (see R.M. Pakzad [35], Section 2.2 and references therein). – The proof of Proposition 6.3 gives the remaining details. □

In a next step, we want to show that the current $T_{\hat{\Omega}}$ associated to an element in $\mathcal{F}_{\otimes,\varphi}(\tilde{B}^5)$ can be completed to a boundaryless current. This is Theorem 1.9 in the introduction. For the proof, we first establish Theorem 6.5 for the non-Abelian case replacing the strong L^1-convergence by the weak convergence in Theorem 1.2. This weak density result then directly implies Theorem 1.9 and we are done.

Appendix A

Differential Forms and Scalar Products

In this first appendix, we introduce differential forms on a Riemannian manifold which play a crucial role in the present thesis, since all results are formulated with the help of such forms. For the geometry of principal bundles in Appendix B the notion of vector-valued differential forms is of particular interest. In order to define later in Appendix C Sobolev spaces, we also need the definition of scalar products for differential forms. For a more detailed introduction the reader should read the textbooks by S. Kobayashi and K. Nomizu [28], Chapter I, by G.L. Naber [31], [32] and by G. Walschap [48], Chapter 1.

Alternated Forms

Let E be an n-dimensional real vector space and denote by $\bigwedge^s(E)$ the alternating s-forms on E. In a basis $\{e_i\}_{i=1,\dots,n}$ of E with corresponding dual bases $\{e_i^*\}_{i=1,\dots,n}$ of $E^* = \bigwedge^1(E)$ every $\alpha \in \bigwedge^s(E)$ can uniquely be written as

$$\alpha = \sum_I \alpha_I \, e_{i_1}^* \wedge \ldots \wedge e_{i_s}^* \,, \tag{A.1}$$

where $\alpha_I = \alpha(e_{i_1}, \dots, e_{i_s})$ and $I = \{(i_1, \dots, i_s) \ : \ 1 \leq i_1 < \ldots < i_s \leq n\}$ is an ordered multi-index. Recall that the wedge product for alternating forms on E is denoted by $\wedge : \bigwedge^s(E) \times \bigwedge^k(E) \longrightarrow \bigwedge^{s+k}(E)$. Note that (A.1) can also be written as

$$\alpha = \frac{1}{s!} \sum_{i_1,\dots,i_s=1}^{n} \alpha_{i_1 \dots i_s} \, e_{i_1}^* \wedge \ldots \wedge e_{i_s}^* = \frac{1}{s!} \alpha_{i_1 \dots i_s} \, e^{i_1} \wedge \ldots \wedge e^{i_s} \,,$$

with the usual Einstein summation convention.

Now, let $g : E \times E \longrightarrow \mathbb{R}$ be a positive definite scalar product on E and define $g_{ij} = g(e_i, e_j)$, for $i, j = 1, \dots, n$. For $\alpha \in \bigwedge^s(E)$, we then set

$$\alpha^{j_1 \dots j_s} = g^{i_1 j_1} \ldots g^{i_s j_s} \alpha_{i_1 \dots i_s} \,,$$

where (g^{ij}) denotes the inverse matrix to (g_{ij}). Moreover, the induced scalar product $\langle \cdot, \cdot \rangle_{\bigwedge^s} : \bigwedge^s(E) \times \bigwedge^s(E) \longrightarrow \mathbb{R}$ on $\bigwedge^s(E)$ reads as

$$\langle \alpha, \beta \rangle_{\bigwedge^s} = \frac{1}{s!} \, \alpha^{i_1 \dots i_s} \beta_{i_1 \dots i_s} \,, \tag{A.2}$$

for $\alpha, \beta \in \bigwedge^s(E)$. Form the definition of the wedge product we then observe that

$$\langle \alpha \wedge \beta, \alpha \wedge \beta \rangle_{\bigwedge^{s+k}} \leq \langle \alpha, \alpha \rangle_{\bigwedge^s} \langle \beta, \beta \rangle_{\bigwedge^k} \,, \tag{A.3}$$

for $\alpha \in \bigwedge^s(E)$ and $\beta \in \bigwedge^k(E)$.

Example A.1. Consider $E = \mathbb{R}^n$ with the standard scalar product and canonical orthonormal basis. We then have

$$|\alpha|^2_{\bigwedge^s} = \langle \alpha, \alpha \rangle_{\bigwedge^s} = \frac{1}{s!} \sum_{i_1, \ldots, i_s = 1}^{n} (\alpha_{i_1 \ldots i_s})^2 = \sum_I (\alpha_I)^2 \,, \tag{A.4}$$

for $\alpha \in \bigwedge^s(\mathbb{R}^n)$.

Theorem A.1 ([32]). *Let E be an oriented n-dimensional real vector space with scalar product g and $\eta \in \bigwedge^n(E)$ the corresponding volume form for E. Then, for $0 \leq s \leq n$, there exists a unique isomorphism, the so-called Hodge star operation,*

$$\star : \bigwedge^s(E) \longrightarrow \bigwedge^{n-s}(E)$$

such that

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle_{\bigwedge^s} \eta \,, \tag{A.5}$$

for $\alpha, \beta \in \bigwedge^s(E)$. In particular, we have

$$\alpha \wedge \star\alpha = |\alpha|^2_{\bigwedge^s} \eta \,. \tag{A.6}$$

Explicitely, the components of the Hodge dual $\star\alpha$ of $\alpha \in \bigwedge^s(E)$ are given by

$$\star\alpha_{j_1 \ldots j_{n-s}} = \frac{\sqrt{\det(g_{ij})}}{s!} \, \varepsilon_{i_1 \ldots i_s j_1 \ldots j_{n-s}} \alpha^{i_1 \ldots i_s} \,,$$

where $\varepsilon_{i_1 \ldots i_s j_1 \ldots j_{n-s}}$ denotes the Levi-Cività symbol. Note that as a consequence of (A.6) together with $\star(\star\alpha) = (-1)^s(n-s)\alpha$, we have that

$$|\alpha|^2_{\bigwedge^s} = |\star\alpha|^2_{\bigwedge^s} \,. \tag{A.7}$$

In a next step, we generalize the previous results to vector-valued s-forms. – Let V be a real m-dimensional vector space with basis $\{E_i\}_{i=1,\ldots,m}$ and denote alternating V-valued s-forms on E by $\bigwedge^s(E, V)$. We then write $\alpha \in \bigwedge^s(E, V)$ as

$$\alpha = \alpha^1 E_1 + \ldots + \alpha^m E_m \,,$$

where $\alpha^i \in \bigwedge^s(E)$, for $i = 1, \ldots, m$.

At this stage for later use, we recall some basic facts on tensor calculus. – The covariant tensors $T_s(E)$ of degree s on E are defined by s-times the tensor product of the dual E^*, i.e., $T_s(E) = E^* \otimes \ldots \otimes E^*$. The following is well-known:

a) Alternating s-forms $\bigwedge^s(E)$ on E can be interpreted as alternating covariant tensors of degree s on E.

b) The vector space of s-multilinear maps from $E \times \ldots \times E$ into V is isomorphic to $T_s(E) \otimes V$.

Both facts then imply the existence of an isomorphism between $\bigwedge^s(E, V)$ and $\bigwedge^s(E) \otimes V$. For more details on tensor calculus we refer to S. Kobayashi and K. Nomizu [28], Chapter I.2.

If we assume that the vector space V also has a scalar product $h : V \times V \longrightarrow \mathbb{R}$, we can define a scalar product $\langle \cdot, \cdot \rangle_{\bigwedge^s, h} : \bigwedge^s(E, V) \times \bigwedge^s(E, V) \longrightarrow \mathbb{R}$ on $\bigwedge^s(E, V)$ as follows: Let $\alpha = \alpha^i E_i$, $\beta = \beta^j E_j$ be two elements in $\bigwedge^s(E, V)$, set $h_{ij} = h(E_i, E_j)$, for $i, j = 1, \ldots, m$, and then define

$$\langle \alpha, \beta \rangle_{\bigwedge^s, h} = h_{ij} \, \langle \alpha^i, \beta^j \rangle_{\bigwedge^s} . \tag{A.8}$$

For an h-orthonormal basis $\{E_i\}_{i=1,\ldots,m}$ of V, this becomes

$$\langle \alpha, \beta \rangle_{\bigwedge^s, h} = \langle \alpha^1, \beta^1 \rangle_{\bigwedge^s} + \ldots + \langle \alpha^m, \beta^m \rangle_{\bigwedge^s} . \tag{A.9}$$

In particular, we have

$$|\alpha|^2_{\bigwedge^s, h} = \langle \alpha, \alpha \rangle_{\bigwedge^s, h} = |\alpha^1|^2_{\bigwedge^s} + \ldots + |\alpha^m|^2_{\bigwedge^s} .$$

Note also that for V-valued s-forms the Hodge star operation

$$\star : \bigwedge^s(E, V) \longrightarrow \bigwedge^{n-s}(E, V)$$

is defined componentwise by Theorem A.1 meaning that the Hodge dual of $\alpha \in \bigwedge^s(E, V)$ is given by

$$\begin{aligned} \star\alpha &= \star(\alpha^1 E_1 + \ldots + \alpha^m E_m) \\ &= \star\alpha^1 E_1 + \ldots + \star\alpha^m E_m . \end{aligned} \tag{A.10}$$

From now on, we assume that V is the Lie algebra $\mathfrak{g}$ of some matrix Lie group G which we regard as subset of $\mathbb{R}^{l \times l} = \mathbb{R}^m$. Identifying $\mathfrak{g}$ with a vector space of (possibly complex) matrices, the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is given, for $A, B \in \mathfrak{g}$, by the following usual matrix commutator:

$$[A, B] = AB - BA .$$

The wedge product $\wedge_{[\cdot,\cdot]} : \bigwedge^s(E, \mathfrak{g}) \times \bigwedge^k(E, \mathfrak{g}) \longrightarrow \bigwedge^{s+k}(E, \mathfrak{g})$ is then defined by

$$\alpha \wedge_{[\cdot,\cdot]} \beta = \sum_{i,j=1}^m \alpha^i \wedge \beta^j \, [E_i, E_j] , \tag{A.11}$$

for all $\alpha = \alpha^i E_i \in \bigwedge^s(E, \mathfrak{g})$ and $\beta = \beta^j E_j \in \bigwedge^k(E, \mathfrak{g})$. In particular, we have, for $\alpha \in \bigwedge^1(E, \mathfrak{g})$ and $v, w \in E$,

$$\begin{aligned} \alpha \wedge_{[\cdot,\cdot]} \alpha(v, w) &= \sum_{i,j=1}^m \alpha^i \wedge \alpha^j(v, w) \, [E_i, E_j] \\ &= \sum_{i,j=1}^m \big(\alpha^i(v)\alpha^j(w) - \alpha^i(w)\alpha^j(v)\big) \, [E_i, E_j] \\ &= 2 \big[\alpha(v), \alpha(w)\big] . \end{aligned} \tag{A.12}$$

For computational purposes, it is convenient to introduce another wedge product on Lie algebra-valued forms from which $\wedge_{[\cdot,\cdot]}$ can be easily obtained. Namely, we simply replace the Lie bracket in (A.11) by matrix multiplication and denote the wedge product by $\wedge$. It is then not difficult to check that

$$\alpha \wedge_{[\cdot,\cdot]} \beta = \alpha \wedge \beta - (-1)^{sk} \beta \wedge \alpha \,, \tag{A.13}$$

for $\alpha \in \bigwedge^s(E, \mathfrak{g})$ and $\beta \in \bigwedge^k(E, \mathfrak{g})$. Moreover, the commutation relations are given by

$$\alpha \wedge_{[\cdot,\cdot]} \beta = (-1)^{sk+1} \beta \wedge_{[\cdot,\cdot]} \alpha \qquad \text{and} \qquad \alpha \wedge \beta = (-1)^{sk} \beta \wedge \alpha \,. \tag{A.14}$$

Example A.2. Consider the special unitary group (note that $A^\dagger = \bar{A}^T$)

$$SU(2) = \{ A \in GL_2(\mathbb{C}) \ : \ A^\dagger A = A A^\dagger = \mathbb{1} \ \text{ and } \ \det A = 1 \}$$

with its Lie algebra

$$\mathfrak{su}(2) = \{ A \in M_2(\mathbb{C}) \ : \ A^\dagger = -A \ \text{ and } \ \operatorname{Tr}(A) = 0 \}$$

under matrix commutation. On $\mathfrak{su}(2)$, we then define the scalar product[1]

$$h(A, B) = -2 \operatorname{Tr}(AB) \,. \tag{A.15}$$

The basis $\{E_1, E_2, E_3\}$ for $\mathfrak{su}(2)$ consisting of

$$E_1 = -\frac{i}{2}\, \sigma_1 \,, \qquad E_2 = -\frac{i}{2}\, \sigma_2 \,, \qquad E_3 = -\frac{i}{2}\, \sigma_3 \,,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli spin matrices, is h-orthonormal, i.e., $h_{ij} = h(E_i, E_j) = \delta_{ij}$, for $i, j = 1, \ldots, 3$. Now, we claim that for any oriented n-dimensional vector space E with scalar product g and corresponding volume form η, the following holds:

$$-2 \operatorname{Tr}(\alpha \wedge \star\beta) = \langle \alpha, \beta \rangle_{\bigwedge^s, h}\, \eta \,, \tag{A.16}$$

where $\alpha, \beta \in \bigwedge^s(E, \mathfrak{su}(2))$. – In order to show the claim, we first note that $\alpha \in \bigwedge^s(E, \mathfrak{su}(2))$ can be expressed as

$$\alpha = \alpha^1 E_1 + \alpha^2 E_2 + \alpha^3 E_3 = -\frac{1}{2} \begin{pmatrix} i\,\alpha^3 & \alpha^2 + i\,\alpha^1 \\ -\alpha^2 + i\,\alpha^1 & -i\,\alpha^3 \end{pmatrix} ,$$

and similarly, since the Hodge dual $\star\beta$ of $\beta \in \bigwedge^s(E, \mathfrak{su}(2))$ is defined componentwise (see (A.10)),

$$\star\beta = \star\beta^1 E_1 + \star\beta^2 E_2 + \star\beta^3 E_3 = -\frac{1}{2} \begin{pmatrix} i \star \beta^3 & \star\beta^2 + i \star \beta^1 \\ - \star \beta^2 + i \star \beta^1 & -i \star \beta^3 \end{pmatrix} .$$

[1] This is the so-called Killing form.

It is not difficult to see that the computation of $\alpha \wedge \star\beta$ reduces to usual matrix multiplication of the matrices α and $\star\beta$ given above but with the entries multiplied by the wedge product. Thus, we obtain that

$$\mathrm{Tr}(\alpha \wedge \star\beta) = -\frac{1}{2}(\alpha^1 \wedge \star\beta^1 + \alpha^2 \wedge \star\beta^2 + \alpha^3 \wedge \star\beta^3)\,.$$

From (A.5), we then deduce that

$$\mathrm{Tr}(\alpha \wedge \star\beta) = -\frac{1}{2}\big(\langle \alpha^1, \beta^1\rangle_{\bigwedge^s} + \langle \alpha^2, \beta^2\rangle_{\bigwedge^s} + \langle \alpha^3, \beta^3\rangle_{\bigwedge^s}\big)\,\eta\,.$$

Using (A.9) the claim (A.16) follows. – Note that, in particular, we have shown that

$$-2\,\mathrm{Tr}(\alpha \wedge \star\alpha) = |\alpha|^2_{\bigwedge^2,h}\,\eta\,. \tag{A.17}$$

Differential Forms

Smooth differential s-forms on an n-dimensional manifold M – which will be denoted by $\Omega^s(M)$ – assign in a smooth way to every point $p \in M$ an s-form on the n-dimensional vector space T_pM. It is well known that elements of $\Omega^s(M)$ can be seen as smooth sections of the exterior s-bundle $\pi : \bigwedge^s(TM) \longrightarrow M$. In a local chart (U, x) of M, we can write $\omega \in \Omega^s(M)$ as

$$\omega = \sum_I \omega_I \, dx^{i_1} \wedge \ldots \wedge dx^{i_s}\,, \tag{A.18}$$

where $\omega_I \in C^\infty(M)$ is defined by $\omega_I(p) = \omega_p\left(\frac{\partial}{\partial x_{i_1}}(p), \ldots, \frac{\partial}{\partial x_{i_s}}(p)\right)$, for $p \in M$, with $\left\{\frac{\partial}{\partial x_i}(p)\right\}_{i=1,\ldots,n}$ a basis for T_pM.

If, in addition, the manifold M is supposed to be oriented and equipped with an Riemannian metric g, then each tangent space T_pM of M gets an orientation and a scalar product. From this, for each point $p \in M$ and $0 \le s \le n$, we obtain a Hodge star operation

$$\star : \bigwedge^s(T_pM) \longrightarrow \bigwedge^{n-s}(T_pM)\,.$$

Thus, the Hodge dual $\star\omega \in \Omega^{n-s}(M)$ of $\omega \in \Omega^s(M)$ can be defined pointwise in the following way:

$$\big(\star\omega\big)_p(X_1, \ldots, X_{n-s}) = \star\big(\omega_p\big)(X_1, \ldots, X_{n-s})\,,$$

where $p \in M$ and $X_1, \ldots, X_{n-s} \in T_pM$.

Using pointwise definitions all the previous results for alternated forms still hold in the setting of differential forms on a manifold. More precisely, if η now denotes the volume form of M and $\omega, \tilde\omega \in \Omega^s(M)$, we have

$$\omega \wedge \star\tilde\omega = \langle \omega, \tilde\omega\rangle\,\eta\,, \tag{A.19}$$

where

$$\langle\cdot,\cdot\rangle : \Omega^s(M) \times \Omega^s(M) \longrightarrow C^\infty(M) \tag{A.20}$$

is the pointwise scalar product defined by (A.2). Moreover, we obtain from (A.3) that

$$\langle \omega \wedge \tilde\omega, \omega \wedge \tilde\omega\rangle \le \langle \omega, \omega\rangle\,\langle \tilde\omega, \tilde\omega\rangle\,. \tag{A.21}$$

Example A.3. Let U be an open subset of $\mathbb{R}^n$ endowed with the standard Euclidean metric and $\omega \in \Omega^s(U)$. Then the (Euclidean) norm $|\omega|$ of the differential s-form ω on U is defined by

$$|\omega|^2 = \langle \omega, \omega \rangle = \sum_I \omega_I^2 \,, \tag{A.22}$$

or, comparing with (A.4), for each point $p \in U$,

$$|\omega|^2(p) = |\omega(p)|^2_{\bigwedge^s} = \big\langle \omega(p), \omega(p) \big\rangle_{\bigwedge^s} = \sum_I \omega_I(p)^2 \,.$$

We conclude this paragraph, by observing that smooth V-valued differential s-forms on a manifold M for an m-dimensional real vector space V – denoted by $\Omega^s(M, V)$ – can be seen as smooth sections of $\bigwedge^s(TM) \otimes V$. For $\omega \in \Omega^s(M, V)$, we have the representation

$$\omega = \omega^1 \, E_1 + \ldots + \omega^m \, E_m \,, \tag{A.23}$$

where $\omega^i \in \Omega^s(M)$, for $i = 1, \ldots, m$, and $\{E_i\}_{i=1,\ldots,m}$ is a basis for V. Doing everything componentwise all the previous results extends at once to this more general context.

Example A.4. In the setting of Example A.2, the following holds for $\omega, \tilde{\omega} \in \Omega^s(M, \mathfrak{su}(2))$:

$$-2\mathrm{Tr}(\omega \wedge \star\tilde{\omega}) = \langle \omega, \tilde{\omega} \rangle \, \eta \,, \tag{A.24}$$

where $\langle \cdot, \cdot \rangle$ now denotes the pointwise scalar product defined by (A.8).

Exterior Derivative and Co-differential

The exterior differentiation $d : \Omega^s(M) \longrightarrow \Omega^{s+1}(M)$ is given in a local chart by

$$d\omega = \sum_I \sum_{i=1}^n \frac{\partial \omega_I}{\partial x^i} \, dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_s} \,, \tag{A.25}$$

where we used the local representation (A.18) for $\omega \in \Omega^s(M)$. Denoting smooth vector fields on M by $\mathcal{X}(M)$ with Lie bracket $[\cdot, \cdot]$ we obtain the following coordinate-free characterization for the exterior derivative (see S. Kobayashi and K. Nomizu [28], Proposition 3.11):

$$\begin{aligned} d\omega(X_0, X_1, \ldots, X_s) \;&=\; \sum_{i=0}^s (-1)^i d\big(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_s)\big) \cdot X_i \\ &\quad + \sum_{0 \le i < j \le s} (-1)^{i+j} \omega\big([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_s\big) \,, \end{aligned} \tag{A.26}$$

where $X_0 \in \mathcal{X}(M)$ and $X_1, \ldots, X_s \in \mathcal{X}(M)$. Note that the "hat" over a vector field means that the latter is omitted. In particular, for $\omega \in \Omega^1(M)$ we have

$$d\omega(X_0, X_1) = d\big(\omega(X_1)\big) \cdot X_0 - d\big(\omega(X_0)\big) \cdot X_1 - \omega\big([X_0, X_1]\big) \,. \tag{A.27}$$

Now, we give three important properties of the exterior derivative.

a) For functions $f \in \Omega^0(M)$ on M the exterior derivative coincides with the tangent map df, i.e., $df(X) = df \cdot X$ for $X \in \mathcal{X}(M)$.

b) The exterior derivative satisfies the Leibniz-rule

$$d(\omega \wedge \tilde{\omega}) = d\omega \wedge \tilde{\omega} + (-1)^s \omega \wedge d\tilde{\omega}\,, \tag{A.28}$$

for $\omega \in \Omega^s(M)$ and $\tilde{\omega} \in \Omega^k(M)$.

c) We have that $d(d\omega) = 0$ for all $\omega \in \Omega^s(M)$.

Next, we define the co-differential $d_* : \Omega^s(M) \longrightarrow \Omega^{s-1}(M)$ by

$$d_*\omega = (-1)^{n(s+1)+1} \star d(\star\omega)\,. \tag{A.29}$$

For a differential one-form ω the "divergence" $d_*\omega$ reads in local coordinates as

$$d_*\omega = \frac{1}{\sqrt{\det(g)}} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g)}\, \omega^i \right)\,. \tag{A.30}$$

Note that the co-differential has the property $d_*(d_*\omega) = 0$ for all $\omega \in \Omega^s(M)$.

Remark A.1. In the case of vector-valued $\omega \in \Omega^s(M, V)$ the exterior derivative and the co-differential are defined componentwise. For example, using (A.23) we have

$$d\omega = d\omega^1\, E_1 + \ldots + d\omega^m\, E_m\,.$$

Appendix B

Connections and Curvatures on Principal Bundles

In this appendix, we give an overview on connections and curvatures of principal bundles and introduce the Yang-Mills functional. At the end of this appendix, we then describe the Abelian and the non-Abelian case of the previous chapters. Because of the focus on the geometry, we simply assume that all objects are smooth. More informations on the concepts of differential geometry used in this dissertation can be found in the textbooks by S. Kobayashi and K. Nomizu [28], Chapter I and II and by G. Walschap [48], Chapter 2 and 4. For a very readable exposition we also refer to R. Friedman and J.W. Morgan, [14]. A simple approach to the geometry of principal bundles is given in the textbook by M. Nakahara [33], Chapter 9 and 10. – Instead of principal bundles we can also consider the equivalent formulation of connections and curvatures on vector bundles for which we refer to the textbooks by S.K. Donaldson and P.B. Kronheimer [11], Chapter 2 and by D. Freed and K. Uhlenbeck [13], Chapter 2.

Principal Bundle

Definition B.1. *Let P, M be two manifolds and let G be a Lie group acting freely on the right on P. A principal G-bundle over the base space M with total space P and structure group G is a submersion $\pi : P \longrightarrow M$ called the bundle projection, together with a bundle atlas $\big\{\big(\pi^{-1}(U_i), (\pi, \varphi_i)\big)\big\}_{i\in I}$ on P. This means the following:*

(i) The family $\{U_i\}_{i\in I}$ is an open covering of M.

(ii) The map

$$(\pi, \varphi_i) : \pi^{-1}(U_i) \longrightarrow U_i \times G$$

is a diffeomorphism and G-equivariant, i.e.,

$$\big(\pi, \varphi_i\big)(\xi g) = \big(\pi(\xi), \varphi_i(\xi)g\big)\,, \tag{B.1}$$

for $\xi \in \pi^{-1}(U_i)$ and all $g \in G$.

Moreover, the set of all principal G-bundles over M will be denoted by $\mathcal{P}^G(M)$ and an element in $\mathcal{P}^G(M)$ with total space P by $\pi : P \longrightarrow M$ or simply by P.

Given a principal G-bundle there exist, for $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, well-defined *transition functions*

$$g_{ij} : U_i \cap U_j \longrightarrow G\,,$$

given by

$$g_{ij}\big(\pi(\xi)\big) = \varphi_i(\xi)\varphi_j(\xi)^{-1}\,. \tag{B.2}$$

It is easy to check that $g_{ii} \equiv e$, where e denotes the identity in G, and that the *cocycle condition*

$$g_{ij}(p)g_{jl}(p) = g_{il}(p)\,, \tag{B.3}$$

for all $p \in U_i \cap U_j \cap U_l \neq \emptyset$, is verified. Conversely, we have

Proposition B.1 ([28]). *Let $\{U_i\}_{i\in I}$ be an open covering of a manifold M and G a Lie group. Moreover, assume that there exists, for all $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, maps $g_{ij} : U_i \cap U_j \longrightarrow G$ such that*

$$g_{il}(p) = g_{ij}(p)g_{jl}(p)\,, \tag{B.4}$$

for every $p \in U_i \cap U_j \cap U_l \neq \emptyset$. Then, we can construct a principal G-bundle in $\mathcal{P}^G(M)$ whose transition functions are given by g_{ij}.

Associated Bundle

We can associate the so-called *adjoint bundle* to a principal G-bundle in the following way: Denote by $\mathfrak{g}$ the Lie algebra of the Lie group G with left action of G on $\mathfrak{g}$ given by the adjoint action $Ad : G \times \mathfrak{g} \longrightarrow \mathfrak{g}$. On $P \times \mathfrak{g}$, we then define the equivalence relation $(\xi, A) \sim (\xi', A')$ if and only if there exists $g \in G$ such that $(\xi', A') = (\xi g, Ad_{g^{-1}}A)$. The quotient space $(P \times \mathfrak{g})/\sim$ is denoted by $P \times_{Ad} \mathfrak{g}$ and there is a well-defined map

$$\begin{aligned} \pi_{Ad} : P \times_{Ad} \mathfrak{g} &\longrightarrow M\,, \\ [\xi, A] &\longmapsto \pi(\xi)\,. \end{aligned}$$

If $\big(\pi^{-1}(U), (\pi, \varphi)\big)$ is a local chart for the given principal G-bundle $\pi : P \longrightarrow M$, then the map

$$\begin{aligned} (\pi_{Ad}, \varphi_{Ad}) : \pi_{Ad}^{-1}(U) &\longrightarrow U \times \mathfrak{g}\,, \\ [\xi, A] &\longmapsto \big(\pi(\xi), Ad_{\varphi(\xi)}A\big) \end{aligned} \tag{B.5}$$

is well-posed and defines a local chart for the adjoint vector bundle $\pi_{Ad} : P \times_{Ad} \mathfrak{g} \longrightarrow M$. In the following, we will simply denote this bundle by $\mathrm{Ad}(P)$.

With the left action of G on itself given by conjugation $c : G \times G \longrightarrow G$, we can similarly associate to a given principal G-bundle $\pi : P \longrightarrow M$ the fiber bundle $\pi_c : P \times_c G \longrightarrow M$ whose local charts are given by

$$\begin{aligned} (\pi_c, \varphi_c) : \pi_c^{-1}(U) &\longrightarrow U \times G\,, \\ [\xi, g] &\longmapsto \big(\pi(\xi), c_{\varphi(\xi)}g\big)\,. \end{aligned} \tag{B.6}$$

Later this bundle will be called automorphism bundle and denoted by $\mathrm{Aut}(P)$.

Curvature and Connection

Let $\mathcal{H}$ be a n-dimensional distribution defining a connection on $P \in \mathcal{P}^G(M)$ with n-dimensional manifold M. Its corresponding Lie algebra-valued connection one-form on P is denoted by $\omega \in \Omega^1(P, \mathfrak{g})$. Then, by definition, the curvature form Ω is given by the exterior covariant derivative $D_{\mathcal{H}}\omega \in \Omega^2(P, \mathfrak{g})$ of ω with respect to $\mathcal{H}$. Recall that the exterior covariant derivative of ω with respect to $\mathcal{H}$ reads as

$$D_{\mathcal{H}}\omega(X_1, X_2) = d\omega(X_1^H, X_2^H)\,, \tag{B.7}$$

where $X_1^H, X_2^H \in \mathcal{H}$ are the horizontal components of the vector fields $X_1, X_2 \in \mathcal{X}(P)$ and d denotes the exterior differentiation of Appendix A. Note that the curvature form Ω is (*right*) *equivariant*, i.e., $(R_g)^*\Omega = Ad_{g^{-1}} \circ \Omega$, for all $g \in G$, and also *horizontal*, since $\Omega(X_1, X_2) = 0$ whenever one of the vector fields $X_1, X_2 \in \mathcal{X}(P)$ is vertical. This will be written as $\Omega \in \Omega^2_{Ad}(P, \mathfrak{g})$. The two previous properties of the curvature form imply that it can be seen as differential two-form $\bar{F}$ on M with values in the adjoint bundle $\mathrm{Ad}(P)$ or, in other words, as section $\bar{F}$ of $\bigwedge^2(TM) \otimes \mathrm{Ad}(P) \longrightarrow M$. More precisely, for $X_1, X_2 \in T_\xi P$, we define

$$\bar{F}_{\pi(\xi)}(d\pi_\xi \cdot X_1, d\pi_\xi \cdot X_2) = \big[\xi, \Omega_\xi(X_1, X_2)\big] \in \mathrm{Ad}(P)\,. \tag{B.8}$$

It is not difficult to check that this definition is independent of $\xi \in P$ and X_1, X_2.

Since a connection form ω is only equivariant and *not* horizontal, there is no interpretation as a section of $\bigwedge^1(TM) \otimes \mathrm{Ad}(P) \longrightarrow M$. However, if we fix a reference connection form ω_0, then the difference $\alpha := \omega - \omega_0$ – which is horizontal – can be regarded as section of $\bigwedge^1(TM) \otimes \mathrm{Ad}(P)$. Conversely, given a connection ω_0 and α a section of $\bigwedge^1(TM) \otimes \mathrm{Ad}(P)$, then $\omega := \omega_0 + \alpha$ is again a connection form on P. This shows that the space of connections on P is an affine space given by

$$\mathcal{A}(P) = \big\{\omega = \omega_0 + \alpha \;:\; \alpha \in \Omega^1(M, \mathrm{Ad}(P))\big\}\,. \tag{B.9}$$

Next, we quote two important results.

Theorem B.2 (Cartan's Structure Equation, [28]). *Let ω be a connection form on $P \in \mathcal{P}^G(M)$ and $\Omega \in \Omega^2(P, \mathfrak{g})$ its curvature form. Then, for all $X_\xi, Y_\xi \in T_\xi P$, we have*

$$\Omega_\xi(X_\xi, Y_\xi) = (d\omega)_\xi\big(X_\xi, Y_\xi\big) + \big[\omega_\xi(X_\xi), \omega_\xi(Y_\xi)\big]\,, \tag{B.10}$$

where $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ denotes the Lie bracket for $\mathfrak{g}$.

Remark B.1. Using (A.12) and then (A.13), Cartan's structure equation (B.10) can also be written as

$$\Omega = d\omega + \frac{1}{2}\,[\omega, \omega] = d\omega + \omega \wedge \omega\,, \tag{B.11}$$

where the notation $[\omega, \omega]$ is used for the wedge product $\omega \wedge_{[\cdot,\cdot]} \omega$ introduced in (A.11).

Theorem B.3 (Bianchi's Identity, [28]). *Let $\mathcal{H}$ be a connection on $P \in \mathcal{P}^G(M)$ with curvature form $\Omega \in \Omega^2(P, \mathfrak{g})$. Then the exterior covariant derivative of Ω with respect to $\mathcal{H}$ vanishes identically on P, i.e., we have*

$$D_{\mathcal{H}}\Omega = 0\,. \tag{B.12}$$

At this stage, we note that the exterior covariant derivative of an equivariant and horizontal Lie algebra-valued s-form $\alpha \in \Omega^s_{Ad}(P, \mathfrak{g})$ on P is explicitly given by[1] (see Theorem C.1 below)

$$D_{\mathcal{H}}\alpha = d\alpha + [\omega, \alpha]\,. \tag{B.13}$$

Thus, for the curvature $\Omega \in \Omega^2_{Ad}(P, \mathfrak{g})$ we have

$$D_{\mathcal{H}}\Omega = d\Omega + [\omega, \Omega]\,. \tag{B.14}$$

Note that inserting Cartan's structure equation (B.11) into (B.14), the proof of Bianchi's identity $D_{\mathcal{H}}\Omega = 0$ is straightforward.

Local Expressions for the Connection and Curvature Form

To a given connection and curvature form of a principal bundle over M, we now associate a family of $\mathfrak{g}$-valued forms defined on open subsets of the base space M. – Let $\big\{\big(\pi^{-1}(U_i), (\pi, \varphi_i)\big)\big\}_{i\in I}$ be an atlas for $P \in \mathcal{P}^G(M)$ with corresponding transition functions $g_{ij} : U_i \cap U_j \longrightarrow G$ given by (B.2). Moreover, for all $i \in I$, let $s_i : U_i \longrightarrow \pi^{-1}(U_i)$ be the (local) section *associated* to the local bundle chart $\big(\pi^{-1}(U_i), (\pi, \varphi_i)\big)$, i.e.,

$$s_i(p) = \big(\pi, \varphi_i\big)^{-1}(p, e)\,.$$

Considering two local bundle charts $\big(\pi^{-1}(U_i), (\pi, \varphi_i)\big)$ and $\big(\pi^{-1}(U_j), (\pi, \varphi_j)\big)$ with $U_i \cap U_j \neq \emptyset$, we observe that their associated local sections satisfy

$$s_j = s_i\, g_{ij} \qquad \text{on } \; U_i \cap U_j\,. \tag{B.15}$$

Such a change of the local sections will be called *local gauge transformations.* We may define a map $u : \pi^{-1}(U_i \cap U_j) \longrightarrow \pi^{-1}(U_i \cap U_j)$ by

$$u\big(s_j(p)g\big) = s_i(p)g_{ij}(p)g = s_i^{g_{ij}}(p)g\,, \tag{B.16}$$

where $s_i^{g_{ij}} = s_i\, g_{ij}$ and $g \in G$. It is not difficult to check that u is a bundle automorphism of the G-principal bundle $\pi^{-1}(U_i \cap U_j) \longrightarrow U_i \cap U_j$. This shows that a local gauge transformation induces a (local) bundle automorphism.

Now, we define a $\mathfrak{g}$-valued one-form A_i on U_i by

$$A_i = s_i^*\omega\,. \tag{B.17}$$

Note that in the physics literature the (local) section $s_i : U_i \longrightarrow \pi^{-1}(U_i)$ and the $\mathfrak{g}$-valued one-form A_i are often called *local gauge*, respectively, (local) *gauge potential* in the local gauge s_i. Similarly, we define

$$F_i = s_i^*\Omega\,, \tag{B.18}$$

which is often called the (local) *field strength.* From Cartan's structure (B.11), we then directly get

$$F_i = dA_i + A_i \wedge A_i\,. \tag{B.19}$$

[1] Obviously, this formula for the covariant derivative does not apply to the connection form (compare with (B.10)).

In a local chart (U_i, x) for the n-dimensional base manifold M, we can write $A_i = A_{i,\alpha}\, dx^\alpha$ and $F_i = \frac{1}{2}\, F_{i,\alpha\beta}\, dx^\alpha \wedge dx^\beta$, where $A_{i,\alpha}$ and $F_{i,\alpha\beta}$, for $\alpha, \beta = 1, \ldots, n$, are $\mathfrak{g}$-valued functions on U_i. From (B.19), it then follows

$$F_{i,\alpha\beta} = \frac{\partial A_{i,\beta}}{\partial x_\alpha} - \frac{\partial A_{i,\alpha}}{\partial x_\beta} + [A_{i,\alpha}, A_{i,\beta}]\,, \tag{B.20}$$

where the partial derivatives are computed componentwise in $\mathfrak{g}$.

Compatibility Conditions

For simplicity, we assume from now on that G is a matrix Lie group, for which the adjoint action reads as

$$Ad_g(A) = g\, A\, g^{-1}\,, \tag{B.21}$$

with $g \in G$ and $A \in \mathfrak{g}$. Gauge potentials must satisfy the *compatibility condition*

$$A_j = g_{ij}^{-1}\, A_i\, g_{ij} + g_{ij}^{-1} dg_{ij} \qquad \text{on } \; U_i \cap U_j\,, \tag{B.22}$$

where dg_{ij} denotes the entrywise differential of the transition function g_{ij} and the products are matrix products. Conversely, we have

Proposition B.4 ([28]). *Let $P \in \mathcal{P}^G(M)$ and $\{U_i\}_{i\in I}$ an open covering of M. Moreover, let $\{A_i\}_{i\in I}$ be a family of $\mathfrak{g}$-valued one-forms each defined on U_i and satisfying the compatibility condition (B.22). Then there exists a unique connection form ω on P which gives rise to the family $\{A_i\}_{i\in I}$ in the above described manner.*

Thus a connection on a principal bundle can be seen as family of locally defined $\mathfrak{g}$-valued one-forms satisfying the compatibility condition and we get the following alternative definition to (B.9) for the space of connections on P:

$$\mathcal{A}(P) = \big\{ A = \{A_i\}_{i\in I} \;:\; A_i \in \Omega^1(U_i, \mathfrak{g}) \;\text{ s.t. (B.22) holds} \big\}\,. \tag{B.23}$$

The compatibility condition for the local field strengths is given by

$$F_j = g_{ij}^{-1}\, F_i\, g_{ij} \qquad \text{on } \; U_i \cap U_j\,. \tag{B.24}$$

On the other hand, we already know that the curvature form can be interpreted as $\bar{F} \in \Omega^2(M, \mathrm{Ad}(P))$ given by (B.8). Consider a local bundle chart $\big(\pi^{-1}(U_i), (\pi, \varphi_i)\big)$ and note that $F_i \in \Omega^2(U_i, \mathfrak{g})$ in (B.18) can alternatively be defined by

$$F_i = \varphi_{Ad,i} \circ \bar{F}\,. \tag{B.25}$$

Then, we compute, using (B.5),

$$\begin{aligned}
(F_i)_{\pi(\xi)}(d\pi_\xi \cdot X_1, d\pi_\xi \cdot X_2) &= \varphi_{Ad,i} \circ \bar{F}_{\pi(\xi)}(d\pi_\xi \cdot X_1, d\pi_\xi \cdot X_2) \\
&= \varphi_{Ad,i}\big(\big[\xi, \Omega_\xi(X_1, X_2)\big]\big) \\
&= \varphi_i(\xi)\, \Omega_\xi(X_1, X_2)\, \varphi_i(\xi)^{-1}\,.
\end{aligned} \tag{B.26}$$

For another bundle chart $\big(\pi^{-1}(U_j), (\pi, \varphi_j)\big)$ with $U_i \cap U_j \neq \emptyset$, we then have

$$\begin{aligned} & (F_j)_{\pi(\xi)}(d\pi_\xi \cdot X_1, d\pi_\xi \cdot X_2) \\ = \quad & \varphi_j(\xi)\, \Omega_\xi(X_1, X_2)\, \varphi_j(\xi)^{-1} \\ \overset{(B.26)}{=} \quad & \varphi_j(\xi)\varphi_i(\xi)^{-1}\, (F_i)_{\pi(\xi)}(d\pi_\xi \cdot X_1, d\pi_\xi \cdot X_2)\, \varphi_i(\xi)\varphi_j(\xi)^{-1} \\ = \quad & g_{ij}(p)^{-1}\, (F_i)_{\pi(\xi)}(d\pi_\xi \cdot X_1, d\pi_\xi \cdot X_2)\, g_{ij}(p)\,, \end{aligned}$$

which is precisely (B.24).

Gauge Transformations

Let $P \in \mathcal{P}^G(M)$ and G a matrix Lie group. The bundle automorphisms $u : P \longrightarrow P$, i.e., G-equivariant maps inducing the identity on M, define a group $\mathcal{G}(P)$ under composition – the so-called *group of gauge transformations*. We observe that every gauge transformation $u : P \longrightarrow P$ can be represented for all $\xi \in P$ as

$$u(\xi) = \xi\, \sigma(\xi)\,, \tag{B.27}$$

where $\sigma : P \longrightarrow G$ is a map that must satisfy

$$\sigma(\xi g) = g^{-1}\, \sigma(\xi)\, g\,, \tag{B.28}$$

since u is G-equivariant. – For the local expression of a gauge transformation, we now proceed similarly to the case of the curvature form.

Let $s_i : U_i \longrightarrow \pi^{-1}(U_i)$ denote the (local) section associated to the bundle chart $\big(\pi^{-1}(U_i), (\pi, \varphi_i)\big)$. Then, we define the map $\sigma_i : U_i \longrightarrow G$ by

$$\sigma_i = s_i^*\sigma = \sigma \circ s_i\,. \tag{B.29}$$

Noting that $s_i\big(\pi(\xi)\big) = \xi\varphi_i(\xi)^{-1}$, we obtain

$$\begin{aligned} \sigma_i(p) \quad &= \quad \big(s_i^*\sigma\big)(p) \\ &= \quad \sigma \circ s_i(p) = \sigma\big(\xi\varphi_i(\xi)^{-1}\big) \\ &\overset{(B.28)}{=} \quad \varphi_i(\xi)\, \sigma(\xi)\, \varphi_i(\xi)^{-1}\,, \end{aligned} \tag{B.30}$$

for all $p = \pi(\xi) \in U_i$. For another bundle chart $\big(\pi^{-1}(U_j), (\pi, \varphi_j)\big)$ with $U_i \cap U_j \neq \emptyset$, we then have

$$\begin{aligned} \sigma_j(p) \quad &= \quad \varphi_j(\xi)\, \sigma(\xi)\, \varphi_j(\xi)^{-1} \\ &\overset{(B.30)}{=} \quad \varphi_j(\xi)\varphi_i(\xi)^{-1}\, \sigma_i(p)\, \varphi_i(\xi)\varphi_j(\xi)^{-1} \\ &= \quad g_{ij}(p)^{-1}\, \sigma_i(p)\, g_{ij}(p)\,. \end{aligned} \tag{B.31}$$

The last equation can be rewritten as

$$g_{ij}(p) = \sigma_i(p)\, g_{ij}(p)\, \sigma_j^{-1}(p)\,. \tag{B.32}$$

This implies that for a bundle automorphism the tansition functions remain unchanged (compare with (2.7)).

On the other hand, the map $\sigma : P \longrightarrow G$ can also be seen as map on M with values in the automorphism bundle $\mathrm{Aut}(P)$. More precisely, we define the map (compare with (B.8))

$$\bar{\sigma}\big(\pi(\xi)\big) = \big[\xi, \sigma(\xi)\big] \in \mathrm{Aut}(P)\,. \tag{B.33}$$

Using the bundle chart $\big(\pi_c^{-1}(U_i), (\pi_c, \varphi_{c,i})\big)$ for the associated automorphism bundle $\mathrm{Aut}(P)$ defined in (B.6), we obtain directly that the map $\varphi_{c,i} \circ \bar{\sigma} : U_i \longrightarrow G$ agrees with (B.29).

Note that (B.27) can now be written locally on U_i as follows:

$$u\big(s_i(p)g\big) = s_i(p)\sigma\big(s_i(p)\big)g = s_i(p)\sigma_i(p)g = s_i^{\sigma_i}(p)g\,, \tag{B.34}$$

where $s_i^{\sigma_i} = s_i\,\sigma_i$ and $g \in G$. Comparing with (B.16), this shows that we are dealing with a local gauge transformation of the form $s_i\,\sigma_i$.

In a next step, we want to determine how connection and curvature forms transform under a gauge transformation. – Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection form on $P \in \mathcal{P}^G(M)$. The action of a gauge transformation $u : P \longrightarrow P$ on ω is then defined by $u^*\omega$, being again a connection form on P. For $X_\xi \in T_\xi P$, we calculate

$$\big(u^*\omega\big)_\xi(X_\xi) = \omega_{u(\xi)}(du_\xi \cdot X_\xi)\,,$$

using (B.27) together with Leibniz's formula for the differential du and obtain

$$\big(u^*\omega\big)_\xi(X_\xi) = \sigma(\xi)^{-1}\,\omega_\xi(X_\xi)\,\sigma(\xi) + \sigma(\xi)^{-1}\,d\sigma_\xi \cdot X_\xi\,.$$

Applying the pull-back by s_i on both sides of the last equation, we then arrive at

$$s_i^*\big(u^*\omega\big) = \sigma_i^{-1}\,A_i\,\sigma_i + \sigma_i^{-1}\,d\sigma_i\,,$$

where $A_i = s_i^*\omega$ denotes the gauge potentials introduced in (B.17). In the following, it will be useful to rewrite the last equation as

$$\sigma_i(A) = \sigma_i^{-1}\,A_i\,\sigma_i + \sigma_i^{-1}\,d\sigma_i\,. \tag{B.35}$$

Since the curvature form corresponding to the gauge transformed connection form $u^*\omega$ is given by $u^*\Omega$ with $\Omega \in \Omega^2(P, \mathfrak{g})$ denoting as usual the curvature form for ω, we compute in a similar way, for $X_1, X_2 \in T_\xi P$,

$$\begin{aligned}\big(u^*\Omega\big)_\xi(X_1, X_2) &= \Omega_{u(\xi)}(du_\xi \cdot X_1, dh_\xi \cdot X_1)\\ &= \sigma(\xi)^{-1}\,\Omega_\xi(X_1, X_2)\,\sigma(\xi)\,,\end{aligned} \tag{B.36}$$

and thus

$$s_i^*(u^*\Omega) = \sigma_i^{-1}\,F_i\,\sigma_i\,,$$

where $F_i = s_i^*\Omega$ denotes the field strengths introduced in (B.18). Emphasizing the dependence of the curvature on the connection, we rewrite the last equation as

$$F_i\big(\sigma(A)\big) = \sigma_i^{-1}\,F_i(A)\,\sigma_i\,, \tag{B.37}$$

with $F_i(A) = dA_i + A_i \wedge A_i$.

From (B.35) and (B.37), it follows that the compatibility conditions (B.22) and (B.24) can be seen as the effect of a local gauge transformation (B.15). In other words, these formula can be interpreted equivalently as the effect of a local gauge transformation on the gauge potentials representing a *fixed* connection form, or as effect of a gauge transformation on a connection form, viewed in a *fixed* local gauge.

Yang-Mills Functional

Let M be an n-dimensional oriented manifold with Riemannian metric g and corresponding volume form $\eta \in \Omega^n(M)$. Moreover, let $P \in \mathcal{P}^G(M)$ with connection form $\omega \in \Omega^1(P, \mathfrak{g})$ and corresponding curvature form $\Omega \in \Omega^2(P, \mathfrak{g})$. Assume also that on the Lie algebra $\mathfrak{g}$ of the matrix Lie group G there exists a scalar product $h : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ which is invariant with respect to the adjoint action, i.e.,

$$h\big(Ad_g(A), Ad_g(B)\big) \overset{(B.21)}{=} h(g\,A\,g^{-1}, g\,A\,g^{-1}) = h(A, B)\,, \tag{B.38}$$

for all $A, B \in \mathfrak{g}$ and $g \in G$.

Recall that to the equivariant and horizontal curvature form Ω we can uniquely associate by (B.8) a differentiable two-form on M with values in the adjoint bundle Ad(P) which we will from now on simply denote by F. Since h is assumed to be invariant with respect to the adjoint action, the pointwise scalar product $\langle \cdot, \cdot \rangle$ on $\Omega^s(P, \mathfrak{g})$ introduced in Appendix A defines – in an obvious way – a pointwise scalar product on $\Omega^s(M, \mathrm{Ad(P)})$ which we will therefore also denote by $\langle \cdot, \cdot \rangle$. For $F \in \Omega^2(M, \mathrm{Ad(P)})$ and $p \in M$, we thus get

$$\big|F\big|^2(p) = \big\langle F, F \big\rangle(p) = \big\langle \Omega\big(\pi^{-1}(p)\big), \Omega\big(\pi^{-1}(p)\big)\big\rangle_{\bigwedge^2, h} = \big|\Omega\big(\pi^{-1}(p)\big)\big|^2_{\bigwedge^2, h}\,,$$

where the notation of Appendix A is used.

At this stage, we are ready to define the Yang-Mills action

$$YM(A) = \int_M |F(A)|^2\,\eta\,, \tag{B.39}$$

for $A \in \mathcal{A}(P)$. The functional $YM : \mathcal{A}(P) \longrightarrow \mathbb{R}$ that assigns to each connection A its Yang-Mills action $YM(A)$ is called the Yang-Mills functional. We observe that the Yang-Mills functional is gauge invariant, i.e.,

$$YM\big(\sigma(A)\big) = YM(A)\,, \tag{B.40}$$

for all $\sigma \in \mathcal{G}(P)$. Indeed, using (B.37) and (B.38), we have

$$YM\big(\sigma(A)\big) = \int_M \big|F\big(\sigma(A)\big)\big|^2\,\eta = \int_M |F(A)|^2\,\eta = YM(A)\,.$$

Examples of Principal Bundles

Now, we want to illustrate all the geometric objects introduced before by the following two examples which are related to the Abelian and the non-Abelian case of the previous chapters.

Example B.1 (Principal $U(1)$-Bundle). Consider the Abelian matrix Lie group $U(1)$ and $\pi : P \longrightarrow M$ a principal $U(1)$-bundle. We identify the Lie algebra $\mathfrak{u}(1)$ with the algebra $\mathrm{Im}\,\mathbb{C}$ of pure imaginary complex numbers with the trivial Lie bracket. From Cartan's structure equation (B.10), we then easily deduce that the curvature form Ω of any connection form $\omega \in \Omega^1(P, \mathfrak{u}(1))$ coincides with its exterior derivative, i.e.,

$$\Omega = d\omega\,. \tag{B.41}$$

Since $U(1)$ is Abelian, the compatibility conditions (B.22) and (B.24) read as

$$A_j = g_{ij}^{-1}\, A_i\, g_{ij} + g_{ij}^{-1}\, dg_{ij} = A_i + g_{ij}^{-1}\, dg_{ij}\,, \tag{B.42}$$

respectively,

$$F_j = g_{ij}^{-1}\, F_i\, g_{ij} = F_i\,, \tag{B.43}$$

on $U_i \cap U_j \neq \emptyset$. Thus the local field strengths piece together to give a *globally* defined $\mathfrak{u}(1)$-valued two-form F on M. In the $U(1)$-case, we deduce also from (B.37) that

$$F\big(\sigma(A)\big) = F(A)\,. \tag{B.44}$$

This is a pecularity of Abelian gauge fields and generally is not true in the non-Abelian case. Moreover, note that the transition functions can be written as $g_{ij}(p) = e^{-i\,\Lambda(p)}$, for some real-valued function Λ on $U_i \cap U_j \neq \emptyset$. Hence, we have $g_{ij}^{-1}\, dg_{ij} = e^{i\,\Lambda} e^{-i\,\Lambda}(-i\Lambda\, d\Lambda)$ implying that (B.42) becomes

$$A_j = A_i - i\, d\Lambda\,.$$

Remark B.2. In the case of an open subset U of $\mathbb{R}^n$ and the *trivial* principal $U(1)$-bundle $\pi : U \times U(1) \longrightarrow U$, we deduce from (B.41) that

$$F = dA\,, \tag{B.45}$$

where $A = s^*\omega \in \Omega^1(U, \mathfrak{u}(1))$ is now also *globally* defined on U, for some global section of the trivial bundle.

Example B.2 (Trivial Principal $SU(2)$-bundle). Consider the trivial principal $SU(2)$-bundle $\pi : U \times SU(2) \longrightarrow U$ with U an open subset of $\mathbb{R}^n$. The right action of $SU(2)$ on $U \times SU(2)$ is given by

$$\xi g_0 = (p, g)g_0 = (p, gg_0)\,,$$

for all $\xi = (p, g) \in U \times SU(2)$ and $g_0 \in SU(2)$. The scalar product h on the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ introduced in (A.15) is invariant under the adjoint action. To see this, we compute

$$\begin{aligned} h\big(Ad_g(A), Ad_g(B)\big) &= -2\,\mathrm{Tr}\big(Ad_g(A)Ad_g(B)\big) \\ &= -2\,\mathrm{Tr}\big((g\,A\,g^{-1})(g\,B\,g^{-1})\big) \\ &= -2\,\mathrm{Tr}(g\,AB\,g^{-1}) = -2\,\mathrm{Tr}(AB) = h(A, B)\,, \end{aligned}$$

for all $A, B \in \mathfrak{su}(2)$.

There is a natural global section $s : U \longrightarrow U \times SU(2)$ given by $s(p) = (p, e)$. Then, any other global section s^σ has the form

$$s^\sigma(p) = s(p)\sigma(p) = (p, e)\sigma(p) = \big(p, \sigma(p)\big)\,,$$

for some map $\sigma : U \longrightarrow SU(2)$. The section s^σ also gives rise to a bundle automorphism $u : U \times SU(2) \longrightarrow U \times SU(2)$, i.e., a gauge transformation, in the following way:

$$u\big(s(p)g\big) = s^\sigma(p)g = s(p)\sigma(p)g = \big(p, \sigma(p)\big)g = \big(p, \sigma(p)g\big)\,. \tag{B.46}$$

Note that this can be interpreted as an equivalent writing of (B.27) in the case of a trivial bundle (compare also with (B.34)) and hence we identify a gauge transformation with a map $\sigma : U \longrightarrow SU(2)$.

Because of the triviality of the principal bundle, any connection form $\omega \in \Omega^1\big(U \times SU(2), \mathfrak{su}(2)\big)$ is uniquely determined by its globally defined gauge potential

$$A = s^*\omega \in \Omega^1(U, \mathfrak{su}(2))\,,$$

which can be written as

$$A = A_\alpha\, dx^\alpha\,,$$

where A_α, for $\alpha = 1, \ldots, n$, are $\mathfrak{su}(2)$-valued functions on U. Due to (B.35), the gauge potential A transforms under a gauge transformation σ as

$$\sigma(A) = \sigma^{-1}\, A\, \sigma + \sigma^{-1}\, d\sigma\,. \tag{B.47}$$

Any curvature form $\Omega \in \Omega^2\big(U \times SU(2), \mathfrak{su}(2)\big)$ is also uniquely determined by its globally defined field strength (see (B.10))

$$F = s^*\Omega = dA + A \wedge A\,, \tag{B.48}$$

which can be written as

$$F = \frac{1}{2}\, F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta\,,$$

where, for $\alpha, \beta = 1, \ldots, n$,

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x_\alpha} - \frac{\partial A_\alpha}{\partial x_\beta} + [A_\alpha, A_\beta] \tag{B.49}$$

are $\mathfrak{su}(2)$-valued functions on U. From (B.37), we deduce that F transforms under a gauge transformation σ as

$$F\big(\sigma(A)\big) = \sigma^{-1}\, F(A)\, \sigma\,. \tag{B.50}$$

Moreover, note that the Yang-Mills action in the case of the trivial principal $SU(2)$-bundle reads in terms of the globally defined field strength as

$$\begin{aligned} YM(A) &= \int_U |F(A)|^2\, d^n x \\ &= \int_U \langle F(A), F(A)\rangle\, d^n x \overset{(A.24)}{=} -2 \int_U \mathrm{Tr}\big(F(A) \wedge \star F(A)\big)\,. \end{aligned} \tag{B.51}$$

Appendix C

Covariant Derivatives and Sobolev Spaces

In this third appendix, we define Sobolev spaces for the geometrical objects occuring in Appendix B. For this purpose, we first need the concept of covariant derivative on associated vector bundles. This concept is well-explained in the textbook of S. Kobayashi and K. Nomizu [28], Chapter III (see also G.L. Naber [32], Chapter 4). Sobolev spaces of differential forms are then defined with the help of covariant differentiation. A reference on this topic is G. Schwarz [38], Chapter 1 (see also K. Wehrheim [49], Appendix).

Vector Bundle Associated to a Principal Bundle

Let $\pi : P \longrightarrow M$ be a principal G-bundle and let V be a finite dimensional vector space on which the Lie group G acts on the left through the representation $\rho(g) : V \longrightarrow V$, for all $g \in G$. On $P \times V$, we then define the equivalence relation $(\xi, m) \sim (\xi', m')$ if and only if there exists $g \in G$ such that $(\xi', m') = \big(\xi g, \rho(g^{-1})m\big)$. The quotient space $(P \times V)/\sim$ is denoted by $E = P \times_\rho V$ and there is a well-defined map

$$\begin{aligned} \pi_E : P \times_\rho V &\longrightarrow M\,, \\ [\xi, m] &\longmapsto \pi(p)\,. \end{aligned}$$

If $\big(\pi^{-1}(U), (\pi, \varphi)\big)$ is a local bundle chart for the given principal G-bundle, then the map

$$\begin{aligned} (\pi_E, \varphi_E) : \pi_E^{-1}(U) &\longrightarrow U \times V\,, \\ [\xi, m] &\longmapsto \big(\pi(\xi), \rho(\varphi(\xi))m\big) \end{aligned} \tag{C.1}$$

defines a local bundle chart for the vector bundle $\pi_E : E = P \times_\rho V \longrightarrow M$ associated to $\pi : P \longrightarrow M$ in $\mathcal{P}^G(M)$.

Example C.1. a) The adjoint bundle $Ad(P)$ defined at the beginning of Appendix B is an example for an associated vector bundle.

b) Consider the bundle of linear frames. – A linear frame ξ at a point p of an n-dimensional manifold M is an ordered basis of the tangent space T_pM. The set of all linear frames over all points of M is denoted by $L(M)$. Then, we can construct a principal $GL_n(\mathbb{R})$-bundle $\pi : L(M) \longrightarrow M$ over M (see S. Kobayashi and K. Nomizu, Example 5.2, Chapter I). – To the bundle of linear frames we associate the tensor bundle $\pi : T^r_s(M) \longrightarrow M$ of contravariant type r and covariant type s – or simply of type (r, s). This vector bundle is obtained using the standard action of $GL_n(\mathbb{R})$ on tensors (see S. Kobayashi and K. Nomizu [28], Chapter I, Proposition 2.12).

Tensorial Forms

On principal bundles the tensorial forms play an important role. – Let $P \in \mathcal{P}^G(M)$ and ρ a representation of G on a finite dimensional vector space V. A tensorial s-form of type (ρ, V) is a V-valued s-form α on P such that

$$(R_g)^*\alpha = \rho(g^{-1}) \circ \alpha \,,$$

for all $g \in G$, and

$$\alpha(X_1, \dots, X_s) = 0 \,,$$

whenever at least one of the vector fields $X_1, \dots, X_s \in \mathcal{X}(P)$ is vertical. Tensorial s-forms of type (ρ, V) on P are denoted by $\Omega^s_\rho(P, V)$. The following is well-known: forms $\alpha \in \Omega^s_\rho(P, V)$ can be seen as s-forms on M with values in $E = P \times_\rho V$ given by

$$\bar{\alpha}_{\pi(\xi)}(d\pi_\xi \cdot X_1, \dots, d\pi_\xi \cdot X_s) = \big[\xi, \alpha_\xi(X_1, \dots, X_s)\big] \in (P \times_\rho V)_\xi \,, \tag{C.2}$$

where $X_1, \dots, X_s \in T_\xi P$. In particular, an element s in $\Omega^0_\rho(P, V)$, i.e., a function $s : P \longrightarrow V$ satisfying

$$s(\xi g) = \rho(g^{-1}) s(\xi) \,, \tag{C.3}$$

identifies with a section $\bar{s} : M \longrightarrow E$ of the vector bundle $\pi_E : E = P \times_\rho V \longrightarrow M$ associated to P. – In other words, tensorial s-forms of type (ρ, V) on P can be interpreted as sections of $\bigwedge^s(TM) \otimes P \times_\rho V \longrightarrow M$ which we will denote by $\Omega^s(M, P \times_\rho V)$.

Example C.2. A curvature form Ω on P is a tensorial two-form of type $(Ad, \mathfrak{g})$ on P. To $\Omega \in \Omega^2_{Ad}(P, \mathfrak{g})$, we can associate $\bar{F}$ given by (B.8).

Exterior Covariant Derivative

For tensorial forms we can define an exterior covariant derivative. – Let ω be a connection form on $P \in \mathcal{P}^G(M)$. Recall from Proposition B.4 and (B.23) that ω can be identified with $A = \{A_i\}_{i \in I} \in \mathcal{A}(P)$. The exterior covariant derivative

$$D^A : \Omega^s_\rho(P, V) \longrightarrow \Omega^{s+1}_\rho(P, V) \tag{C.4}$$

is then defined as

$$D^A\alpha(X_1, \dots, X_{s+1}) = d\alpha(X_1^H, \dots, X_{s+1}^H) \,, \tag{C.5}$$

where $X_1^H, \dots, X_{s+1}^H$ are the horizontal components of the vector fields $X_1, \dots, X_{s+1} \in \mathcal{X}(P)$. We already encountered this exterior covariant derivative in (B.7).

Theorem C.1 ([32]). *Let ω be a connection one-form on $P \in \mathcal{P}^G(M)$. Then, the exterior covariant derivative $D^A\alpha$ of $\alpha \in \Omega^s_\rho(P, V)$ reads as*

$$D^A\alpha = d\alpha + \omega\dot{\wedge}\alpha\,, \tag{C.6}$$

where $\dot{\wedge}$ denotes the wedge product between $\mathfrak{g}$-valued and V-valued forms.

Remark C.1. In the particular case of $\alpha \in \Omega^s_{Ad}(P, \mathfrak{g})$ we obtain for the exterior covariant derivative $D^A\alpha = d\alpha + [\omega, \alpha]$ (see (B.13)).

Covariant Derivative on Associated Vector Bundles

Next, we explain how a given connection on a principal bundle induces a covariant derivative for sections on associated vector bundles. – A (local) section $\bar{s} : U \longrightarrow \pi_E^{-1}(U)$ of an associated vector bundle can be written as

$$\bar{s}\big(\pi(\xi)\big) = \big[\xi, s(\xi)\big]\,, \tag{C.7}$$

where the V-valued function s on P satisfies (C.3). Then, for $X \in T_{\pi(\xi)}M$, the covariant derivative $\nabla^A_X\bar{s}$ of $\bar{s}$ in the direction of X is defined by

$$\nabla^A_X\bar{s}\big(\pi(\xi)\big) = [\xi, ds_\xi \cdot \tilde{X}]\,, \tag{C.8}$$

where $\tilde{X} \in T_\xi P$ denotes the horizontal lift of X determined by a given connection A on $P \in \mathcal{P}^G(M)$. Note that the right-hand side of (C.8) is an element of the fiber $(P \times_\rho V)_\xi$. Hence, the covariant derivative evaluated at $\pi(\xi) \in U$ maps the fiber $(P \times_\rho V)_\xi$ into itself with "derivative" properties given in the next proposition.

Proposition C.2 ([28]). *Let $X, Y \in T_pM$ and let s, t be two sections of an associated vector bundle $\pi_E : E \longrightarrow M$ defined in a neighborhood U of p. Then, for their covariant derivative, the following holds:*

(i) The covariant derivative is linear, i.e.,

$$\begin{aligned} \nabla^A_X(s+t) &= \nabla^A_X s + \nabla^A_X t\,, \\ \nabla^A_{X+Y} s &= \nabla^A_X s + \nabla^A_Y s\,, \\ \nabla^A_{\lambda X} s &= \lambda\nabla^A_X s\,, \end{aligned}$$

where λ is a scalar.

(ii) The covariant derivative satisfies the Leibniz rule

$$\nabla^A_X(f\,s) = f\nabla^A_X s + df \cdot X\,s\,,$$

where f is a V-valued function on U.

For more details on covariant differentiation, especially the covariant derivative along a curve and with respect to a vector field, we refer to S. Kobayashi and K. Nomizu [28], Chapter III.1.

Covariant Derivative on the Adjoint Bundle

Once we know how to define covariant derivatives of sections on associated vector bundles, we will investigate them in more details for the two examples given in Example C.1. – We start with the associated adjoint bundle and compute the covariant derivative defined in (C.8) more explicitly. For this purpose, let f be a $\mathfrak{g}$-valued function of type $(Ad, \mathfrak{g})$ on P. For a vertical vector $X^V \in T_\xi P$, we then have

$$df_\xi \cdot X^V = -\big[\omega_\xi(X^V), f(\xi)\big]\,, \tag{C.9}$$

where $\omega \in \Omega^1(P, \mathfrak{g})$ denotes a given connection on P. The last equation is a consequence of Theorem C.1 and the defining equation (C.5) for the exterior covariant derivative. For a vector $X = X^V + X^H \in T_\xi P$ with vertical component X^V and horizontal component X^H, we hence deduce

$$df_\xi \cdot X = df_\xi \cdot X^V + df_\xi \cdot X^H = -\big[\omega_\xi(X), f(\xi)\big] + df_\xi \cdot X^H\,,$$

or, equivalently,

$$df_\xi \cdot X^H = df_\xi \cdot X + \big[\omega_\xi(X), f(\xi)\big]\,.$$

This implies together with the definition (C.8) for the covariant derivative $\nabla^A_X \bar{s}$ of sections $\bar{s} : M \longrightarrow \mathrm{Ad}(P)$ of the adjoint bundle $\mathrm{Ad}(P)$ that

$$\nabla^A_X \bar{s}\big(\pi(\xi)\big) = \Big[\xi, ds_\xi \cdot X + \big[\omega_\xi(X), s(\xi)\big]\Big]\,. \tag{C.10}$$

The last formula for the covariant derivative of sections on the adjoint bundle $\mathrm{Ad}(P)$ can also be obtained in a slightly different way. Namley, let $f \in \Omega^0_{Ad}(P, \mathfrak{g})$ with exterior covariant derivative given by (see Theorem C.1)

$$D^A f = df + [\omega, f]\,. \tag{C.11}$$

Recall that owing to (C.4) we have that $D^A f$ belongs to $\Omega^1_{Ad}(P, \mathfrak{g})$. Thus, we can use (C.2) in order to get a one-form on M with values in $\mathrm{Ad}(P)$ given by

$$\overline{D^A f}_{\pi(\xi)}(d\pi_\xi \cdot X) = [\xi, D^A f_\xi(X)] \overset{(C.11)}{=} \Big[\xi, df_\xi(X) + \big[\omega_\xi(X), f(\xi)\big]\Big]\,.$$

The right-hand side then coincides with (C.10).

Covariant Derivative for Differential Forms

Considering the second example in Example C.1 we take the tensor bundle $\pi : T^r_s(M) \longrightarrow M$ of type (r, s) which is a vector bundle associated to the principal bundle $\pi : L(M) \longrightarrow M$ of linear frames. Recall that a tensor field K of type (r, s) is a section of $T^r_s(M)$. Hence, given a linear connection on the bundle of linear frames, we can define a type-preserving covariant derivative $\nabla_X K$ of K as explained before. For the particular case of functions f on M it is well known that

$$\nabla_X f = df \cdot X\,, \tag{C.12}$$

where df on the right-hand side means the tangent map of f. This and the action on vector fields $\mathcal{X}(M)$ completely determines the covariant derivative $\nabla_X K$ (see Proposition

C.3 below). Note also that covariant differentiation ∇_X makes sense for a vector X at a point of M, whereas Lie differentiation L_X requires a vector field X (see S. Kobayashi and K. Nomizu [28], Chapter I.3).

The next proposition says how the covariant derivative for differential forms can be obtained from (C.12) and the covariant derivative of vector fields.

Proposition C.3 ([28]). *Let $\alpha \in \Omega^s(M)$. Then, we have*

$$\nabla_X \alpha(X_1, \dots, X_s) = d\big(\alpha(X_1, \dots, X_s)\big) \cdot X - \sum_{i=1}^{s} \alpha\big(X_1, \dots, \nabla_X X_i, \dots, X_s\big)\,, \qquad \text{(C.13)}$$

for vector fields X and $X_1, \dots, X_s \in \mathcal{X}(M)$.

Note that the covariant differential $\nabla\alpha$ of $\alpha \in \Omega^s(M)$ is a tensor field of type $(0, s+1)$ – and not an element of $\Omega^{s+1}(M)$ – which is defined as follows:

$$\nabla\alpha(X_1, \dots, X_s, X) = \nabla_X\big(\alpha(X_1, \dots, X_s)\big)\,. \qquad \text{(C.14)}$$

The next proposition gives a relation between the covariant derivative ∇ for differential forms and the exterior derivative d defined in Appendix A.

Proposition C.4 ([28]). *Let $\alpha \in \Omega^s(M)$ and ∇ a symmetric covariant derivative, i.e.,*

$$\nabla_X Y - \nabla_Y X = [X, Y]\,, \qquad \text{(C.15)}$$

where $[X, Y]$ denotes the Lie-bracket of $X, Y \in \mathcal{X}(M)$. For the exterior derivative $d\alpha \in \Omega^{s+1}(M)$ of α we then have

$$d\alpha = \mathcal{A}(\nabla\alpha)\,, \qquad \text{(C.16)}$$

with $\mathcal{A}$ the alternation operation.

For the proof of the previous proposition we insert (C.15) into the formula (C.13) for the covariant derivative in order to get the characterization (A.26) of the exterior derivative d.

Remark C.2. Doing everything componentwise the covariant differentiation generalizes to vector-valued differential forms as defined in Appendic A.

Covariant Derivative for $\mathrm{Ad}(P)$-valued Differential Forms

In a next step, we combine the covariant derivative for sections on the adjoint bundle $\mathrm{Ad}(P)$ with the covariant derivative for differential forms in order get a covariant derivative for differential forms with values in the bundle $\mathrm{Ad}(P)$. More precisely, motivated by Proposition C.3 and using (C.10), we define for $\mathrm{Ad}(P)$-valued s-forms α on M the covariant derivative

$$\nabla^A_X \alpha(X_1, \dots, X_s) = \nabla^A_X\big(\alpha(X_1, \dots, X_s)\big) - \sum_{i=1}^{s} \alpha\big(X_1, \dots, \nabla_X X_i, \dots, X_s\big)\,. \qquad \text{(C.17)}$$

In the particular case of $\alpha \in \Omega^1(M, \mathrm{Ad}(P))$, the previous formula translates to

$$\nabla^A_X \alpha(X_1) = \nabla^A_X\big(\alpha(X_1)\big) - \alpha(\nabla_X X_1)\,. \tag{C.18}$$

Now, we show that taking the completely anti-symmetric part of the covariant derivative ∇^A we recover the exterior covariant derivative D^A defined in (C.4) and (C.5). This is very similar to Proposition C.4 for usual differential forms. – Applying the alternation operation $\mathcal{A}$ on (C.18), it follows

$$\begin{aligned}\mathcal{A}\big(\nabla^A\alpha\big)(X_1, X) &= \nabla^A_X\big(\alpha(X_1)\big) - \alpha(\nabla_X X_1)\\ &\quad -\nabla^A_{X_1}\big(\alpha(X)\big) - \alpha(\nabla_{X_1} X)\,.\end{aligned}$$

Using (C.10) for the covariant derivative of sections on $\mathrm{Ad}(P)$, this becomes

$$\begin{aligned}\mathcal{A}\big(\nabla^A\alpha\big)(X_1, X) &= d\big(\alpha(X_1)\big)\cdot X + \big[\omega(X), \alpha(X_1)\big] - \alpha(\nabla_X X_1)\\ &\quad -d\big(\alpha(X)\big)\cdot X_1 - \big[\omega(X_1), \alpha(X)\big] - \alpha(\nabla_{X_1} X)\,.\end{aligned}$$

Since (A.27) implies

$$d\alpha(X_1, X) = d\big(\alpha(X)\big)\cdot X_1 - d\big(\alpha(X_1)\big)\cdot X - \alpha\big([X_1, X]\big)\,,$$

we then arrive at

$$\mathcal{A}\big(\nabla^A\alpha\big)(X_1, X) = d\alpha(X_1, X) + \big[\omega(X), \alpha(X_1)\big] - \big[\omega(X_1), \alpha(X)\big]\,,$$

where we assumed that the covariant derivative is symmetric. Using the definition of the wedge product, we can rewrite the last equation as

$$\mathcal{A}(\nabla^A\alpha) = d\alpha + [\omega, \alpha]\,. \tag{C.19}$$

Comparing this with Theorem C.1 for the exterior covariant derivative of tensorial forms and recalling the identification (C.2) of sections of $\bigwedge^1(TM)\otimes\mathrm{Ad}(P) \longrightarrow M$ with $\Omega^1_{Ad}(P, \mathfrak{g})$, we interprete the left-hand side of (C.19) as exterior covariant derivative $D^A\alpha = \mathcal{A}(\nabla^A\alpha)$ of α. In summary, we obtain an exterior covariant derivative

$$D^A : \Omega^s(M, \mathrm{Ad}(P)) \longrightarrow \Omega^{s+1}(M, \mathrm{Ad}(P))\,. \tag{C.20}$$

Sobolev Spaces of Differential Forms

In this second part of the appendix, we define Sobolev spaces of differential forms with the help of the covariant differentiation introduced before. We start again with ordinary differential forms.

Let M be a compact oriented n-dimensional manifold with Riemannian metric g and denote smooth sections of the exterior s-bundle $\pi : \bigwedge^s(TM) \longrightarrow M$ by $\Gamma\big(M, \bigwedge^s(TM)\big)$ which are equipped with a scalar product given by[1]

$$\begin{aligned}\langle\langle\cdot,\cdot\rangle\rangle : \Gamma\big(M, \textstyle\bigwedge^s(TM)\big)\times\Gamma\big(M, \textstyle\bigwedge^s(TM)\big) &\longrightarrow \mathbb{R}\,,\\ (\alpha, \beta) &\longmapsto \int_M \langle\alpha, \beta\rangle\,\eta\,,\end{aligned} \tag{C.21}$$

where $\langle\cdot,\cdot\rangle$ denotes the pointwise scalar product on $\Omega^s(M)$ induced by g (see (A.20)) and η the volume form of M.

[1]Instead of compact manifolds, it is also possible to consider compactly supported sections.

Remark C.3. a) Because of (A.19), we have, for $\alpha, \beta \in \Omega^s(M)$,

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle \, \eta = \int_M \alpha \wedge \star\beta \,. \tag{C.22}$$

b) Since $d(\alpha \wedge \star\beta) = d\alpha \wedge \star\beta - \alpha \wedge \star d_*\beta$ and using Stokes' theorem, we deduce that

$$\langle\langle d\alpha, \beta \rangle\rangle = \int_M d\alpha \wedge \star\beta = \int_M \alpha \wedge \star d_*\beta = \langle\langle \alpha, \delta\beta \rangle\rangle \,, \tag{C.23}$$

for $\alpha \in \Omega^{s-1}(M)$ and $\beta \in \Omega^s(M)$ compactly supported. This shows that d and d_* are formally adjoint operators.

The space $L^2\big(M, \bigwedge^s(TM)\big)$ is then defined as the completion of $\Gamma\big(M, \bigwedge^s(TM)\big)$ with respect to the following L^2-norm:

$$\|\alpha\|_{L^2\left(M, \bigwedge^s(TM)\right)} = \big(\langle\langle \alpha, \alpha \rangle\rangle\big)^{1/2} = \left(\int_M \langle \alpha, \alpha \rangle \, \eta\right)^{1/2} = \left(\int_M |\alpha|^2 \, \eta\right)^{1/2} . \tag{C.24}$$

For general $1 \leq p < \infty$, the L^p-norm on $\Gamma\big(M, \bigwedge^s(TM)\big)$ is given by

$$\|\alpha\|_{L^p\left(M, \bigwedge^s(TM)\right)} = \left(\int_M |\alpha|^p \, \eta\right)^{1/p} , \tag{C.25}$$

and the space $L^p\big(M, \bigwedge^s(TM)\big)$ is then defined as the completion of $\Gamma\big(M, \bigwedge^s(TM)\big)$ with respect to the previous norm. – In the case of $p = \infty$, we define

$$\|\alpha\|_{L^\infty\left(M, \bigwedge^s(TM)\right)} = \sup_{p \in M} |\alpha(p)|_{\bigwedge^s} \,. \tag{C.26}$$

Next, we want to introduce Sobolev spaces for differential forms. – The Levi-Cività connection on the Riemannian manifold M with covariant derivative ∇^M for smooth sections on the tangent bundle $\pi : TM \longrightarrow M$ induces a covariant derivative for differential forms on M via Proposition C.3. More precisely, for $\alpha \in \Omega^s(M)$ we have

$$\nabla_X \alpha(X_1, \ldots, X_s) = d\big(\alpha(X_1, \ldots, X_s)\big) \cdot X - \sum_{i=1}^{s} \alpha(X_1, \ldots, \nabla^M_X X_i, \ldots, X_s) \,,$$

where X and $X_1, \ldots, X_s$ are vector fields on M. For $1 \leq p < \infty$, the $W^{1,p}$-Sobolev norm on $\Gamma\big(M, \bigwedge^s(TM)\big)$ is then given by

$$\|\alpha\|^p_{W^{1,p}\left(M, \bigwedge^s(TM)\right)} = \int_M |\alpha|^p \, \eta + \sum_{i=1}^{n} \int_M |\nabla_{E_i} \alpha|^p \, \eta \,, \tag{C.27}$$

where $\{E_1, \ldots, E_n\}$ is a (local) g-orthonormal basis of TM. The Sobolev space $W^{1,p}\big(M, \bigwedge^s(TM)\big)$ is then defined as the completion of $\Gamma\big(M, \bigwedge^s(TM)\big)$ with respect to that norm. For $k \in \mathbb{N}$, the Sobolev spaces $W^{k,p}\big(M, \bigwedge^s(TM)\big)$ are defined analogously by taking the k-th covariant derivative.

Remark C.4. a) In the following, we will simply write $\Omega^s_{W^{k,p}}(M)$ for elements of $W^{k,p}\big(M, \bigwedge^s(TM)\big)$ and $\|\cdot\|_{W^{k,p}(M)}$ for their norms.

b) Again, the generalization to Sobolev spaces of vector-valued differential forms is straightforward when considering each component seperately. For the present thesis the Sobolev spaces $W^{k,p}\big(M, \bigwedge^s(TM) \otimes \mathfrak{g}\big)$ – or simply $\Omega^s_{W^{k,p}}(M, \mathfrak{g})$ – with $\mathfrak{g}$ the Lie algebra of a Lie group G play an important role.

Sobolev Spaces of Ad(P)-valued Differential Forms

Proceeding as in the case of usual differential forms we can now define Sobolev spaces of Ad(P)-valued differential forms. Elements in $\Omega^s(M, \mathrm{Ad}(P))$ considered as smooth sections $\Gamma\big(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\big)$ are equipped with the scalar product

$$\begin{aligned} \langle\langle \cdot, \cdot \rangle\rangle : \Gamma\big(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\big) \times \Gamma\big(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\big) &\longrightarrow \mathbb{R}\,, \\ (\alpha, \beta) &\longmapsto \int_M \langle \alpha, \beta \rangle\, \eta\,, \end{aligned} \tag{C.28}$$

where $\langle \cdot, \cdot \rangle$ denotes the pointwise scalar product on $\Omega^s(M, \mathrm{Ad}(P))$ induced by the Riemannian metric g and the adjoint action invariant scalar product on the Lie algebra $\mathfrak{g}$ (see (B.38)). We then define the space $L^2\big(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\big)$ as the completion of $\Gamma\big(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\big)$ with respect to the following L^2-norm:

$$\|\alpha\|_{L^2\left(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\right)} = \big(\langle\langle \alpha, \alpha \rangle\rangle\big)^{1/2} = \left(\int_M \langle \alpha, \alpha \rangle\, \eta\right)^{1/2} = \left(\int_M |\alpha|^2\, \eta\right)^{1/2}. \tag{C.29}$$

The spaces $L^p\big(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\big)$ for $1 \leq p \leq \infty$ are defined similarly.

The covariant derivative ∇^A (see (C.10)) induced by the connection A on $P \in \mathcal{P}^G(M)$ together with the covariant derivative ∇^M coming from the Levi-Cività connection yield a covariant derivative – again denoted by ∇^A – for $\Omega^s(M, \mathrm{Ad}(P))$ via formula (C.17). For $1 \leq p < \infty$, the $W^{1,p}$-Sobolev norm on $\Gamma\big(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\big)$ is then given by

$$\|\alpha\|^p_{W^{1,p}\left(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\right)} = \int_M |\alpha|^p\, \eta + \sum_{i=1}^n \int_M |\nabla^A_{E_i} \alpha|^p\, \eta\,. \tag{C.30}$$

This leads to the Sobolev spaces $W^{1,p}\big(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\big)$ and also to $W^{k,p}\big(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\big)$ for $k \in \mathbb{N}$.

Remark C.5. a) In the following, we will simply write $\Omega^s_{W^{k,p}}(M, \mathrm{Ad}(P))$ for elements of $W^{k,p}\big(M, \bigwedge^s(TM) \otimes \mathrm{Ad}(P)\big)$ and $\|\cdot\|_{W^{k,p}(M)}$ for their norms.

b) Recalling that the curvature F belongs to $\Omega^2(M, \mathrm{Ad}(P))$ the Yang-Mills action (B.39) can also be written differently as the following L^2-norm:

$$YM(A) = \int_M \langle F(A), F(A) \rangle\, \eta = \langle\langle F(A), F(A) \rangle\rangle = \|F(A)\|^2_{L^2(M)}\,. \tag{C.31}$$

Appendix D

First Attempt to a Strong Density Result in the Non-Abelian Case

The aim of this appendix is to repeat step by step the computations in the proof of the Abelian Theorem 4.6 for the non-Abelian setting. This enables us to see precisely how the strong convergence fails and how the analysis becomes more involved under gauge transformations in the non-Abelian case. – We start exactly as in Section 5.2 and define the cubic decomposition, good and bad cubes, the average on each good cube etc.. Consider again only one fixed good cube and recall that by definition of $\mathcal{F}_{\mathbb{Z}}(B^5)$ we have that $\iota_{\partial C}^* F$ is a curvature form for some principal $SU(2)$-bundle over the boundary of the fixed good cube $C_\varepsilon^{g_{i_0}}(\delta)$. Since (5.12) holds for the good cubes, the second Chern number of this principal $SU(2)$-bundle vanishes implying that it is trivial. Using also the small energy assumption on the boundary of the good cubes, we can hence apply Uhlenbeck's Theorem 3.2 to the smooth approximation of $F \in \mathcal{F}_{\mathbb{Z}}(B^5)$ on the boundary[1] in order to deduce the existence of a smooth Coulomb gauge $\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}$. The fact that the Coulomb gauge $\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}$ is still smooth follows from Proposition 3.4. The Coulomb gauge satisfies

$$d_* \tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta} = 0\,, \tag{D.1}$$

and obeys the bound

$$\|\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\|_{W^{1,2}(\partial C_\varepsilon^{g_{i_0}}(\delta))} \leq C_{Uh}\, \|\iota_{\partial C}^* F\|_{L^2(\partial C_\varepsilon^{g_{i_0}}(\delta))}\,. \tag{D.2}$$

In a next step, we define the harmonic extension $\tilde{A}^{g_{i_0}}_{\varepsilon,\delta}$ of $\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}$ by

$$\begin{cases} \Delta \tilde{A}^{g_{i_0}}_{\varepsilon,\delta} = 0 & \text{on } C_\varepsilon^{g_{i_0}}(\delta) \\ \iota_{\partial C}^* \tilde{A}^{g_{i_0}}_{\varepsilon,\delta} = \tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\,, \quad \iota_{\partial C}^*(d_* \tilde{A}^{g_{i_0}}_{\varepsilon,\delta}) = 0 & \text{on } \partial C_\varepsilon^{g_{i_0}}(\delta)\,. \end{cases} \tag{D.3}$$

and, moreover,

$$A^{g_{i_0}}_{\varepsilon,\delta} = \tilde{A}^{g_{i_0}}_{\varepsilon,\delta} - \bar{A}^{g_{i_0}}_{\varepsilon,\delta}\,,$$

which satisfies

$$\begin{cases} \Delta A^{g_{i_0}}_{\varepsilon,\delta} = 0 & \text{on } C_\varepsilon^{g_{i_0}}(\delta) \\ \iota_{\partial C}^* A^{g_{i_0}}_{\varepsilon,\delta} = \tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta} - \bar{a}^{g_{i_0}}_{\varepsilon,\delta}\,, \quad \iota_{\partial C}^*(d_* A^{g_{i_0}}_{\varepsilon,\delta}) = 0 & \text{on } \partial C_\varepsilon^{g_{i_0}}(\delta)\,. \end{cases} \tag{D.4}$$

[1]We emphasize that Uhlenbeck's constant $0 \leq \delta_{Uh} \ll 1$ is independent of the chosen cubes, since the L^2-norm on their four-dimensional boundaries is scaling invariant.

Note that because of (5.23) and (D.2) the boundary term is bounded in the $W^{1,2}$-norm. Hence, proceeding as in the proof of Lemma 5.4 below, we obtain the following elliptic estimate from (D.4):

$$\|dA^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))} \leq C\,\|\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta} - \bar{a}^{g_{i_0}}_{\varepsilon,\delta}\|_{W^{1,2}(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,. \tag{D.5}$$

Using a similar Poincaré type estimate as in the proof of Proposition 3.3, we have that

$$\begin{aligned}\|\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta} - \bar{a}^{g_{i_0}}_{\varepsilon,\delta}\|_{W^{1,2}(\partial C^{g_{i_0}}_\varepsilon(\delta))} \quad&\leq\quad C\,\|d\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta} - d\bar{a}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\\ &\qquad +C\,\|d_*\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta} - d_*\bar{a}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,.\end{aligned}$$

But due to (5.22) and (D.1) the second term on the right-hand side vanishes leading to

$$\|dA^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))} \leq C\,\|d\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta} - d\bar{a}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,. \tag{D.6}$$

Now, the triangular inequality is used, in order to get

$$\begin{aligned}&\|d\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta} - d\bar{a}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\\ \leq\;& \|d\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta} + \tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\wedge\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta} - d\bar{a}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\\ &+\|\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\wedge\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\\ \leq\;& \|f(\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}) - \iota^*_{\partial C}\bar{F}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\\ &+\|\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\wedge\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,,\end{aligned} \tag{D.7}$$

where $f(\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}) = d\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta} + \tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\wedge\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}$ denotes the curvature associated to the Coulomb gauge $\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}$. From the continuity of the multiplication $L^4\otimes L^4 \longrightarrow L^2$ and the Sobolev embedding $W^{1,2}\hookrightarrow L^4$ in four dimensions, it follows that

$$\begin{aligned}\|\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\wedge\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \quad&\leq\quad \|\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\|^2_{L^4(\partial C^{g_{i_0}}_\varepsilon(\delta))}\\ &\leq\quad C\,\|\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\|^2_{W^{1,2}(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,.\end{aligned}$$

The bound (D.2) for the Coulomb gauge then gives

$$\|\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\wedge\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \leq C\,C^2_{Uh}\,\|\iota^*_{\partial C}F\|^2_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,,$$

or, equivalently, since the small energy assumption (5.10) holds for the fixed good cube $C^{g_{i_0}}_\varepsilon(\delta)$,

$$\|\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\wedge\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))} \leq C\delta\,\|\iota^*_{\partial C}F\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,.$$

Inserting this and (D.7) into (D.6), we arrive at

$$\begin{aligned}\|dA^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))} \quad&=\quad \|d\tilde{A}^{g_{i_0}}_{\varepsilon,\delta} - d\bar{A}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))}\\ &\leq\quad C\,\|f(\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}) - \iota^*_{\partial C}\bar{F}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\\ &\qquad +C\delta\,\|\iota^*_{\partial C}F\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,.\end{aligned} \tag{D.8}$$

Now, we give an estimate for the wedge product of the harmonic extension of the Coulomb gauge introduced in (D.3). This can be done by using Lemma 5.4 implying that

$$\|\tilde{A}^{g_{i_0}}_{\varepsilon,\delta}\|_{W^{1,5/2}(C^{g_{i_0}}_\varepsilon(\delta))} \le C\,\|\iota^*_{\partial C}F\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,. \tag{D.9}$$

Together with the continuity of the multiplication $L^4 \otimes L^4 \longrightarrow L^2$ and the Sobolev embedding $W^{1,5/2} \hookrightarrow L^5$ in five dimensions, this leads to

$$\begin{aligned}\|\tilde{A}^{g_{i_0}}_{\varepsilon,\delta} \wedge \tilde{A}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))} &\le \|\tilde{A}^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^4(C^{g_{i_0}}_\varepsilon(\delta))} \le C\,\|\tilde{A}^{g_{i_0}}_{\varepsilon,\delta}\|^2_{L^5(C^{g_{i_0}}_\varepsilon(\delta))}\\ &\le C\,\|\tilde{A}^{g_{i_0}}_{\varepsilon,\delta}\|^2_{W^{1,5/2}(C^{g_{i_0}}_\varepsilon(\delta))} \le C\,\|\iota^*_{\partial C}F\|^2_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,.\end{aligned}$$

The fact that we are working on a fixed good cube ensures that (see (5.10))

$$\|\tilde{A}^{g_{i_0}}_{\varepsilon,\delta} \wedge \tilde{A}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))} \le C\delta\,\|\iota^*_{\partial C}F\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,. \tag{D.10}$$

We now rewrite the main estimates (D.8) and (D.10) for the fixed good cube $C^{g_{i_0}}_\varepsilon(\delta)$, using a simple scaling argument, in the following way:

$$\begin{aligned}\|d\tilde{A}^{g_{i_0}}_{\varepsilon,\delta} - d\bar{A}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))} &\le C\varepsilon\,\|f(\tilde{a}^{g_{i_0}}_{Coulomb,\varepsilon,\delta}) - \iota^*_{\partial C}\bar{F}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\\ &\quad + C\varepsilon\delta\,\|\iota^*_{\partial C}F\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,,\end{aligned} \tag{D.11}$$

$$\|\tilde{A}^{g_{i_0}}_{\varepsilon,\delta} \wedge \tilde{A}^{g_{i_0}}_{\varepsilon,\delta}\|_{L^2(C^{g_{i_0}}_\varepsilon(\delta))} \le C\varepsilon\delta\,\|\iota^*_{\partial C}F\|_{L^2(\partial C^{g_{i_0}}_\varepsilon(\delta))}\,, \tag{D.12}$$

where the constants C are independent of $\varepsilon > 0$.

At this stage, we are ready to pass from one fixed good cube to the union $C^g_\varepsilon(\delta) = \bigcup_{i=1}^{N^g_\varepsilon(\delta)} C^{g_i}_\varepsilon(\delta)$ of good cubes and define the *piecewise* smooth $\mathfrak{su}(2)$-valued one-form $\tilde{A}^g_{\varepsilon,\delta}$ on C^g_ε by

$$\tilde{A}^g_{\varepsilon,\delta} = \tilde{A}^{g_i}_{\varepsilon,\delta} \qquad \text{on } C^{g_i}_\varepsilon(\delta)\,. \tag{D.13}$$

The "harmonic" curvature

$$\tilde{F}^g_{\varepsilon,\delta} = dA^g_{\varepsilon,\delta} + A^g_{\varepsilon,\delta} \wedge A^g_{\varepsilon,\delta} \tag{D.14}$$

then belongs to $\mathcal{F}_R(C^g_\varepsilon(\delta))$ and it is only smooth up to gauge transformations.

Since $\tilde{F}^g_{\varepsilon,\delta}$ has been defined after the Coulomb gauge transformations, the strong convergence of $\tilde{F}^g_{\varepsilon,\delta}$ to F fails. In order to see this, we compute

$$\begin{aligned}\|d\tilde{A}^g_{\varepsilon,\delta} + \tilde{A}^g_{\varepsilon,\delta} \wedge \tilde{A}^g_{\varepsilon,\delta} - F\|_{L^2(C^g_\varepsilon(\delta))} &\le \|d\tilde{A}^g_{\varepsilon,\delta} + \tilde{A}^g_{\varepsilon,\delta} \wedge \tilde{A}^g_{\varepsilon,\delta} - \bar{F}^g_{\varepsilon,\delta}\|_{L^2(C^g_\varepsilon(\delta))}\\ &\quad + \|\bar{F}^g_{\varepsilon,\delta} - F\|_{L^2(C^g_\varepsilon(\delta))}\,,\end{aligned} \tag{D.15}$$

where the piecewise constant two-form $\bar{F}^g_{\varepsilon,\delta}$ on $C^g_\varepsilon(\delta)$ is defined in a obvious way. For the first term on the right-hand side of (D.15), we have

$$\begin{aligned}&\|d\tilde{A}^g_{\varepsilon,\delta} + \tilde{A}^g_{\varepsilon,\delta} \wedge \tilde{A}^g_{\varepsilon,\delta} - \bar{F}^g_{\varepsilon,\delta}\|_{L^2(C^g_\varepsilon(\delta))}\\ \le\ &\|d\tilde{A}^g_{\varepsilon,\delta} - d\bar{A}^g_{\varepsilon,\delta}\|_{L^2(C^g_\varepsilon(\delta))} + \|\tilde{A}^g_{\varepsilon,\delta} \wedge \tilde{A}^g_{\varepsilon,\delta}\|_{L^2(C^g_\varepsilon(\delta))}\\ =\ &\sum_{i=1}^{N^g_\varepsilon(\delta)} \|d\tilde{A}^{g_i}_{\varepsilon,\delta} - d\bar{A}^{g_i}_{\varepsilon,\delta}\|_{L^2(C^{g_i}_\varepsilon(\delta))} + \sum_{i=1}^{N^g_\varepsilon(\delta)} \|\tilde{A}^{g_i}_{\varepsilon,\delta} \wedge \tilde{A}^{g_i}_{\varepsilon,\delta}\|_{L^2(C^{g_i}_\varepsilon(\delta))}\end{aligned}$$

Applying the estimates (D.11) and (D.12) to each good cube $C_\varepsilon^{g_i}(\delta)$ this can be bounded by

$$C\varepsilon \sum_{i=1}^{N_\varepsilon^g(\delta)} \Big(\|f(\tilde{a}^{g_i}_{Coulomb,\varepsilon,\delta}) - \iota^*_{\partial C}\bar{F}^{g_i}_{\varepsilon,\delta}\|_{L^2(\partial C_\varepsilon^{g_i}(\delta))} + \delta \, \|\iota^*_{\partial C} F\|_{L^2(\partial C_\varepsilon^{g_i}(\delta))} \Big)$$

Thus, we obtain

$$\begin{aligned} & \|d\tilde{A}^g_{\varepsilon,\delta} + \tilde{A}^g_{\varepsilon,\delta} \wedge \tilde{A}^g_{\varepsilon,\delta} - F\|_{L^2(C_\varepsilon^g(\delta))} \\ \leq \; & C\varepsilon \sum_{i=1}^{N_\varepsilon^g(\delta)} \|f(\tilde{a}^{g_i}_{Coulomb,\varepsilon,\delta}) - \iota^*_{\partial C}\bar{F}^{g_i}_{\varepsilon,\delta}\|_{L^2(\partial C_\varepsilon^{g_i}(\delta))} + C\varepsilon\delta \, \|F\|_{L^2(\partial C_\varepsilon^g(\delta))} \\ & + \|\bar{F}^g_{\varepsilon,\delta} - F\|_{L^2(C_\varepsilon^g(\delta))} \, . \end{aligned} \tag{D.16}$$

Passing to the limit $\varepsilon \to 0$ the second and the third term on the right-hand side converge to zero applying the same arguments as for the convergence on the good cubes in Section 5.2. The first term with the gauge transformed curvatures is however not bounded by the choice (5.8) of the cubic decomposition. – In terms of the metrics introduced in Section 3.3, we are looking for a convergence with respect to the metric γ in order to control the first term on the right-hand side of (D.16). However, this convergence is not a consequence of the convergence assumption (5.8) with respect to the metric d as shown in Proposition 3.5.

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