An Energy Gap Phenomenon for Willmore Spheres.

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Abstract : *In this work we prove the existence of a threshold strictly larger than* 4π *below which any Willmore sphere in* $\mathbb{R}^m$ *has to be the image by a translation and an homothetie of the standard sphere* S^2*. This result was already proved in [KS1] using a different approach.*

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I Introduction

Let $\vec{\Phi}$ be an immersion of the sphere S^2 into $\mathbb{R}^m$. Denote by $\pi_{\vec{n}_{\vec{\Phi}}}$ the orthonormal projections of vectors in $\mathbb{R}^m$ onto the $m-2$-plane given by $\vec{n}_{\vec{\Phi}}$. With these notations the *second fundamental form*

$$\forall X, Y \in T_p\Sigma \qquad \vec{\mathbb{I}}_p(X,Y) := \pi_{\vec{n}_{\vec{\Phi}}} d^2\vec{\Phi}(X,Y)$$

[1] The *mean curvature vector* of the immersion at p is given by

$$\vec{H} := \frac{1}{2}\, tr_g(\vec{\mathbb{I}}) = \frac{1}{2}\left[\vec{\mathbb{I}}(\varepsilon_1,\varepsilon_1) + \vec{\mathbb{I}}(\varepsilon_2,\varepsilon_2)\right] \quad ,$$

where $(\varepsilon_1, \varepsilon_2)$ is an orthonormal basis of $T_p\Sigma$ for the metric $g_{\vec{\Phi}}$.

In the present paper we are mainly interested with the Lagrangian given by the L^2 norm of the second fundamental form :

$$E(\vec{\Phi}) := \int_\Sigma |\vec{\mathbb{I}}|^2_g \, dvol_g \quad ,$$

An elementary computation gives

$$E(\vec{\Phi}) := \int_\Sigma |\vec{\mathbb{I}}|^2_g \, dvol_g = \int_\Sigma |d\vec{n}_{\vec{\Phi}}|^2_g \, dvol_g \quad .$$

This energy E can be hence seen as being the *Dirichlet Energy* of the Gauss map $\vec{n}_{\vec{\Phi}}$ with respect to the induced metric $g_{\vec{\Phi}}$. The Gauss Bonnet theorem implies that

$$E(\vec{\Phi}) := \int_\Sigma |\vec{\mathbb{I}}|^2_g \, dvol_g = 4 \int_\Sigma |\vec{H}|^2 \, dvol_g - 4\pi\, \chi(\Sigma) \quad , \tag{I.1}$$

where $\chi(\Sigma)$ is the *Euler characteristic* of the surface Σ. The energy

$$W(\vec{\Phi}) := \int_\Sigma |\vec{H}|^2 \, dvol_g \quad ,$$

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[1] In order to define $d^2\vec{\Phi}(X,Y)$ one has to extend locally the vector X or Y by a vector-field but it is not difficult to check that $\pi_{\vec{n}_{\vec{\Phi}}} d^2\vec{\Phi}(X,Y)$ is independent of this extension.

is the so called *Willmore energy.*

It is well known that the standard unit 2-sphere $S^2 \subset \mathbb{R}^3 \subset \mathbb{R}^m$ is, modulo the action of homotheties and translations, the unique minimizer of W among all possible immersions of closed 2-dimensionnal manifolds . Our main result in this paper, theorem I.1 below, reinforces this uniqueness statement by showing roughly that the standard sphere is "isolated" in it's range of energy.

Theorem I.1 *Let $m \geq 3$. There exists $\delta_m > 0$ such that if $\vec{\Phi}$ is a Willmore immersion from S^2 into $\mathbb{R}^m$ satisfying*

$$W(\vec{\Phi}) < 4\pi + \delta_m \quad ,$$

then a translation of $\vec{\Phi}(S^2)$ is homothetic to the standard sphere S^2 in $\mathbb{R}^m$ and we have

$$W(\vec{\Phi}) = 4\pi \quad .$$

□

Remark I.1 *It is conjectured that $\delta_m = 4\pi$ is the optimal constant for which theorem I.1 holds. This conjecture was proved already by R. Bryant for $m = 3$ in [Bry] and by Montiel for $m = 4$ in [Mon].* □

Remark I.2 *Another type of gap phenomenon, for branched Willmore spheres this time, but similar to this one is one of the main step in proving energy quantization results in [BR].* □

II Proof of theorem I.1.

Let $\vec{\Phi}_k$ be a sequence of Willmore immersions from S^2 into $\mathbb{R}^m$ satisfying

$$\lim_{k\to+\infty} W(\vec{\Phi}_k) = 4\pi \quad .$$

By possibly composing $\vec{\Phi}_k$ with a diffeomorphism of S^2 we can assume that $\vec{\Phi}_k$ is conformal. From the normalization Lemma A.4 of [Ri3] together with lemma III.1 in the same paper (see also [Ri1] section VI.8) we deduce the existence of a sequence of Möbius transformations Ξ_k and a lipschitz diffeomorphism f_k of S^2 such that, modulo extraction of a subsequence $\vec{\xi}_k := \Xi_k \circ \vec{\Phi}_k \circ f_k$ weakly converges to a possibly branched weak lipschitz conformal immersion[2] $\vec{\xi}_\infty$ of S^2 in the following way.

$$\limsup_{k\to+\infty} \mathcal{H}^2(\Xi_k \circ \vec{\Phi}_k \circ f_k(S^2)) < +\infty \qquad , \qquad \Xi_k \circ \vec{\Phi}_k \circ f_k(S^2) \subset B_R(0) \tag{II.2}$$

for some $R > 0$ independent of k, and there exists at most finitely many points $\{a_1 \cdots a_N\}$ in S^2 such that

$$\vec{\xi}_k := \Xi_k \circ \vec{\Phi}_k \circ f_k \rightharpoonup \vec{\xi}_\infty \qquad \text{weakly in } W^{2,2}_{loc} \cap (W^{1,\infty}_{loc})^*(S^2 \setminus \{a_1, \cdots a_N\}) \quad , \tag{II.3}$$

where the convergences are taken w.r.t. g_{S^2}, the standard metric on S^2, moreover

$$\forall K \text{ compact subset of } \Sigma \setminus \{a_1 \cdots a_N\} \quad \limsup_{k\to+\infty} \|\log |d\vec{\xi}_k|_{g_{S^2}}\|_{L^\infty(K)} < +\infty \tag{II.4}$$

Because of (II.3) and (II.4) we have for any $\delta > 0$

$$\int_{S^2\setminus\cup_i B_\delta(a_i)} |\nabla \vec{n}_{\vec{\xi}_\infty}|^2_{g_{S^2}} \ dvol_{g_{S^2}} \leq \liminf_{k\to+\infty} \int_{S^2\setminus\cup_i B_\delta(a_i)} |\nabla \vec{n}_{\vec{\xi}_k}|^2_{g_{S^2}} \ dvol_{g_{S^2}} \quad .$$

[2] See the notion of *weak lipschitz immersions with L^2 bounded second fundamental form* in [Ri1], [Ri3], [BR]...

Simon's monotonicity formula (see [Sim]) implies that

$$4\pi \le \int_{S^2} |\vec{H}_{\vec{\xi}_\infty}|^2\ dvol_{g_\infty} \le \lim_{\delta\to 0} \frac{1}{4}\int_{S^2\setminus\cup_i B_\delta(a_i)} |\nabla\vec{n}_{\vec{\xi}_\infty}|^2_{g_{S^2}}\ dvol_{g_{S^2}} - \frac{1}{2}\int_{S^2} K_{\vec{\xi}_\infty}\ dvol_{g_\infty} \tag{II.5}$$

hence

$$8\pi \le \lim_{\delta\to 0} \int_{S^2\setminus\cup_i B_\delta(a_i)} |\nabla\vec{n}_{\vec{\xi}_\infty}|^2_{g_{S^2}}\ dvol_{g_{S^2}} \tag{II.6}$$

By assumption we have

$$\lim_{k\to+\infty} \int_{S^2} |\nabla\vec{n}_{\vec{\xi}_k}|^2_{g_{S^2}}\ dvol_{g_{S^2}} = 8\pi \tag{II.7}$$

Thus combining (II.5) and (II.7) gives that

$$\lim_{k\to+\infty} \int_{S^2} |\nabla\vec{n}_{\vec{\xi}_k}|^2_{g_{S^2}}\ dvol_{g_{S^2}} = \int_{S^2} |\nabla\vec{n}_{\vec{\xi}_\infty}|^2_{g_{S^2}}\ dvol_{g_{S^2}} = 8\pi \tag{II.8}$$

Since $\vec{n}_{\vec{\xi}_k}$ is weakly converging towards $\vec{n}_{\vec{\xi}_\infty}$, (II.8) implies that the convergence is in fact strong and therefore no bubbling can occur : there is no pint a_i. Observe that since the limiting immersion $\vec{\xi}_\infty$ satisfies $W(\vec{\xi}_\infty) = 4\pi$, by a classical result (see for instance [Ri1]) we have that $\vec{\xi}(S^2) \subset B_R(0)$ is homothetic to the standard $S^2 \subset \mathbb{R}^3 \subset \mathbb{R}^m$.

Now, since $\vec{\xi}_k$ is Willmore and since the L^2 norm of $d\vec{n}_{\vec{\xi}_k}$ nowhere concentrates we can apply the epsilon regularity for Willmore immersions (theorem I.5 in [Ri2]) in order to deduce that the convergence of $\vec{\xi}_k$ towards $\vec{\xi}_\infty$ holds in $C^l(S^2)$ norm for any $l \in \mathbb{N}$. Hence after maybe application of a translation and a dilation $\vec{\xi}_k(S^2)$ can be parametrized as a graph over S^2 : there exists a sequence of maps g_k from S^2 into $\mathbb{R}^{m-3}$ and a sequence of function ε_k such that if $\vec{\phi}_k$ (which is not necessarily conformal) denotes the map from S^2 into R^m given by $\vec{\phi}_k(x) := x(1+\varepsilon_k) + g_k(x)$

i)

$$\vec{\phi}_k(S^2) = \vec{\xi}_k(S^2) \quad .$$

ii)

$$\forall\, l \in \mathbb{N} \qquad \lim_{k\to 0} \|\varepsilon_k\|_{C^l(S^2)} + \|g_k\|_{C^l(S^2)} = 0 \quad .$$

Modulo again a small translation + dilation we can moreover assume that $\vec{\phi}_k(S^2)$ and S^2 intersect each other at the north pole, $North$, in a tangent way which reads

$$\varepsilon_k(North) = 0 \quad , \quad g_k(North) = 0 \quad , \quad d\varepsilon_k(North) = 0 \quad \text{and} \quad dg_k(North) = 0 \quad .$$

We apply now the inversion with respect to the north pole $I_N(x) := (x - North)/|x - North|^2$ and $I_N(S^2)$ is equal to the 2-plane P given by $x_j = 0$ for $j = 3\cdots m$ and the image of $\vec{\phi}_k(S^2)$ has become now a graph over the plane P of the form $(y_1, y_2, f_k(y_1, y_2))$. Assume we have made a translation in order for the north pole to coincide with the origin $(0,0,0)$ prior to apply the inversion and the sphere to which $\vec{\phi}_k(S^2)$ converges is the one given by $x_i = 0$ for $i \ge 4$ and $x_1^2 + x_2^2 + (x_3+1)^2 = 1$. Since $\vec{\phi}_k(S^2)$ is tangent to the 2-plane $x_i = 0$ for $i \ge 3$ and since the curvature of $\vec{\phi}_k(S^2)$ is uniformly bounded, the intersection of $\vec{\phi}_k(S^2)$ with some fixed neighborhood of 0 in $\mathbb{R}^m$ is locally given by a graph of the form $(x_1, x_2, a_k(x_1, x_2))$ and $\|a_k - 1 + \sqrt{1-r^2}\|_{C^l} \to 0$ as $k \to +\infty$ for any $l \in \mathbb{N}$ - where we denote $r^2 = x_1^2 + x_2^2$. Since $\nabla a_k(0,0) = 0$ and since all derivatives of a_k are bounded in a given, small enough neighborhood of 0 the C^∞ function given by $h_k := a_k/r$ is converging in C^l norm towards $h_\infty(x_1,x_2) := (1-\sqrt{1-r^2})/r = O(r)$. We shall now give the explicit norm of f_k at ∞ in terms of a_k. We have

$$I(x_1,x_2,a_k(x_1,x_2)) = \left(\frac{x_1}{r^2+a_k^2},\frac{x_2}{r^2+a_k^2},\frac{a_k}{r^2+a_k^2}(x_1,x_2)\right) \quad .$$

This then gives

$$f_k(y_1,y_2) = \frac{a_k}{r^2+a_k^2}(x_1,x_2) \qquad \text{where} \qquad y_i := \frac{x_i}{r^2+a_k^2} \quad .$$

The change of variable matrix is given by

$$\nabla_x y := (\partial_{x_i} y_j)_{i=1,2} = \frac{1}{r^2+a_k^2}\ \left(Id - 2\frac{x\otimes x}{r^2} - \frac{x\otimes\nabla h_k^2}{1+h_k^2}\right) \quad . \tag{II.9}$$

We have $det(Id-2x\otimes x/r^2) = -1$ and since the coefficients of this matrix are bounded it is uniformly invertible. Since h_k is converging in C^l norm towards $h_\infty(x_1,x_2) := (1-\sqrt{1-r^2})/r = O(r)$ there exists a $\rho>0$ independent of k such that for any $0<r<\rho$

$$P_k := Id - 2\frac{x\otimes x}{r^2} - \frac{x\otimes\nabla h_k^2}{1+h_k^2}$$

is uniformly invertible that is

$$\limsup_{k\rightarrow+\infty} \|P_k^{-1}\|_{L^\infty(B^2_\rho(0))} < +\infty \quad . \tag{II.10}$$

We deduce from it

$$|(\nabla_x y)^{-1}|(y) \le C\ |y|^{-2} \quad . \tag{II.11}$$

Using the fact that $|\nabla_x P_k| \le C\ r^{-1}$ we deduce, using $\nabla_x P_k^{-1} = P_k^{-1}\,\nabla_x P_k\, P_k^{-1}$ that there exists $C>0$ independent of k such that for $0<r<\rho$

$$|\nabla_x P_k^{-1}| \le C\ r^{-1} \quad . \tag{II.12}$$

We have

$$\begin{aligned}\nabla_y f_k &= (\nabla_x y)^{-1}\,\nabla_x\left(\frac{a_k}{r^2+a_k^2}\right) = (r^2+a_k^2)\ P_k^{-1}\ \nabla_x\left(\frac{a_k}{r^2+a_k^2}\right)\\ &= P_k^{-1}\left[\nabla_x a_k - 2\frac{a_k\ \nabla_x a_k}{r^2+a_k^2} - 2\frac{a_k\ r\ \nabla_x r}{r^2+a_k^2}\right] \quad .\end{aligned}$$

Since $r^{-2}\,|a_k| + r^{-1}\,|\nabla_x a_k| \le C$ on $B_\rho(0)$ independently of k one deduces from the previous identity together with (II.10) that there exists a radius $R>0$ such that

$$\forall y\in\mathbb{R}^2\setminus B_R(0) \quad , \quad \forall k\in\mathbb{N} \qquad |\nabla_y f_k|(y) \le C\ |y|^{-1} \quad . \tag{II.13}$$

Differentiating once more with respect to y gives

$$\begin{aligned}\nabla^2_y f_k = (\nabla_x y)^{-1}&\left[\nabla_x P_k^{-1}\left[\nabla_x a_k - 2\frac{a_k\ \nabla_x a_k}{r^2+a_k^2} - 2\frac{a_k\ r\ \nabla_x r}{r^2+a_k^2}\right]\right]\\ &+(\nabla_x y)^{-1}\left[P_k^{-1}\ \nabla_x\left[\nabla_x a_k - 2\frac{a_k\ \nabla_x a_k}{r^2+a_k^2} - 2\frac{a_k\ r\ \nabla_x r}{r^2+a_k^2}\right]\right]\end{aligned}$$

Using again $r^{-2}\,|a_k| + r^{-1}\,|\nabla_x a_k| + |\nabla^2 a_k| \le C$, (II.11), (II.10) and (II.12) we obtain

$$\forall y\in\mathbb{R}^2\setminus B_R(0) \quad , \quad \forall k\in\mathbb{N} \qquad |\nabla^2_y f_k|(y) \le C\ |y|^{-2} \quad . \tag{II.14}$$

Since $\vec{\phi}_k(S^2)$ converges as a graph over S^2 in C^l norm to S^2 and since away from the origin, in the ball $B^2(0)$ (which contains $\vec{\phi}_k(S^2)$ for k large enough) the inversion with respect to the origin is a diffeomorphim with uniformly bounded differential, we have for any radius $R>0$

$$\forall\, l\in\mathbb{N}\qquad \lim_{k\rightarrow 0}\|f_k\|_{C^l(B_R(0))}=0\quad . \tag{II.15}$$

From (II.14) and (II.15) we deduce in particular that

$$\forall r>1\qquad \lim_{k\rightarrow 0}\int_{\mathbb{R}^2}|\nabla^2_y f_k|^r\ dy_1\,dy_2=0\quad . \tag{II.16}$$

The mean curvature vector for this graph at the point $(y_1,y_2,f_k(y_1,y_2))$, that we denote $\vec{H}_k(y_1,y_2)$ is given by the sum of (2.13) and (2.14) divided by 2 in [BK]. Hence there exists a smooth function $\vec{G}$ from $(\mathbb{R}^2\otimes\mathbb{R}^{m-2})\times(\mathbb{R}^4\otimes\mathbb{R}^{m-2})$ such that

$$(det(g_k))^{1/4}\,\vec{H}_k(y_1,y_2)=\vec{G}(\nabla f_k,\nabla^2 f_k)$$

where $det(g_k)$ is the determinant of the matrix $g_{k,ij}=\delta_{ij}+\partial_{x_i}f_k\cdot\partial_{x_j}f_k$. Moreover $\vec{G}(p,q)$ satisfies

$$\forall q\in\mathbb{R}^4\otimes\mathbb{R}^{m-2}\qquad \vec{G}(0,q)=\frac{q_{11}+q_{22}}{2}\qquad\text{and}\qquad \partial_{q_{ij}}\vec{G}(0,q)=\frac{\delta_{ij}}{2}\quad .$$

We deduce in particular that, for any $q\in\mathbb{R}^4\otimes\mathbb{R}^{m-2}$ and for any $(i,j)\in\{1,2\}^2$, the linear 1-form on $\mathbb{R}^{m-2}$ given by $(\vec{G}\cdot\partial_{q_{ij}}\vec{G})(0,q)$ identifies to the following vector of $\mathbb{R}^{m-2}$

$$(\vec{G}\cdot\partial_{q_{ij}}\vec{G})(0,q)=\delta_{ij}\ \frac{q_{11}+q_{22}}{2}\quad . \tag{II.17}$$

For any fixed $p\in\mathbb{R}^2\otimes\mathbb{R}^{m-2}$ we have moreover that $\vec{G}(p,q)$ is a linear form in q which implies

$$\forall p\in\mathbb{R}^2\otimes\mathbb{R}^{m-2}\qquad \vec{G}(p,0)=0\quad . \tag{II.18}$$

The Willmore energy of the graph is moreover equal to

$$\int_{\mathbb{R}^2}|\vec{G}(\nabla f_k,\nabla^2 f_k)|^2\ dx_1\,dx_2\quad .$$

Hence any graph realizing a critical point to this Willmore energy satisfies the following Euler Lagrange system

$$\sum_{i,j=1}^2\partial^2_{y_i\,y_j}\left((\vec{G}\cdot\partial_{q_{ij}}\vec{G})(\nabla f_k,\nabla^2 f_k)\right)-\sum_{l=1}^2\partial_{y_l}\left((\vec{G}\cdot\partial_{p_l}\vec{G})(\nabla f_k,\nabla^2 f_k)\right)=0\quad . \tag{II.19}$$

Taking

$$\forall\, i,j=1,2\quad \forall p\in\mathbb{R}^2\otimes\mathbb{R}^{m-2}\quad\text{and}\quad \forall q\in\mathbb{R}^4\otimes\mathbb{R}^{m-2}$$

$$F_{ij}(p,q):=(\vec{G}\cdot\partial_{q_{ij}}\vec{G})(p,q)-(\vec{G}\cdot\partial_{q_{ij}}\vec{G})(0,q)$$

and

$$\forall\, l=1,2\quad \forall p\in\mathbb{R}^2\otimes\mathbb{R}^{m-2}\quad\text{and}\quad \forall q\in\mathbb{R}^4\otimes\mathbb{R}^{m-2}$$

$$L_l(p,q):=(\vec{G}\cdot\partial_{p_l}\vec{G})(p,q)$$

Observe that with our notations, for any p and q, $(\vec{G}\cdot\partial_{p_l}\vec{G})(p,q)$ is a linear form on $\mathbb{R}^{m-2}$ which identifies canonically to a vector in $\mathbb{R}^{m-2}$.

The Euler Lagrange system (II.19) becomes then

$$\Delta^2 f_k = -\sum_{i,j=1}^{2} \partial^2_{x_i\,x_j}\left(F_{ij}(\nabla f_k, \nabla^2 f_k)\right) + \sum_{l=1}^{2} \partial_{x_l}\left(L_l(\nabla f_k, \nabla^2 f_k)\right) \quad , \tag{II.20}$$

where the F_{ij} and L_l are smooth functions such that, for any choice of p and q included in the unit balls of respectively $\mathbb{R}^2 \otimes \mathbb{R}^{m-2}$ and $\mathbb{R}^4 \otimes \mathbb{R}^{m-2}$, one has, for any choice of indices,

$$|F_{ij}(p,q)| \le A_{ij}\ |p|\,|q| \qquad \text{and} \qquad |L_l(p,q)| \le B_l\ |q|^2 \quad , \tag{II.21}$$

where (A_{ij}) and (B_l) are families of positive constants independent of p and q in these unit balls.

Let $2 < r < +\infty$. Using the pointwise controls on the F_{ij} and the L_l given by (II.21), classical L^r estimates in elliptic theory (see for instance [GT] chapter 9) gives the existence of a constant C_r independent of k such that

$$\int_{\mathbb{R}^2} |\Delta f_k|^r\ dy_1\,dy_2 \le C_r\ \int_{\mathbb{R}^2} |\nabla f_k|^r\ |\nabla^2 f_k|^r\ dy_1\,dy_2 + C_r\ \left(\int_{\mathbb{R}^2} |\nabla^2 f_k|^{2q}\ dy_1\,dy_2\right)^{r/q} \quad , \tag{II.22}$$

where $q^{-1} - 2^{-1} = r^{-1}$. Classical interpolation inequality (see for instance [GT] chapter 7) gives

$$\left(\int_{\mathbb{R}^2} |\nabla^2 f_k|^{2q}\ dy_1\,dy_2\right)^{1/2q} \le \left(\int_{\mathbb{R}^2} |\nabla^2 f_k|^{2}\ dy_1\,dy_2\right)^{1/4}\ \left(\int_{\mathbb{R}^2} |\nabla^2 f_k|^{r}\ dy_1\,dy_2\right)^{1/2r} \quad . \tag{II.23}$$

Finally classical results on Calderon Zygmund operators (see for instance [GT] chapter 9) give

$$\int_{\mathbb{R}^2} |\nabla^2 f_k|^r\ dy_1\,dy_2 \le C_r\ \int_{\mathbb{R}^2} |\Delta f_k|^r\ dy_1\,dy_2 \quad . \tag{II.24}$$

Combining (II.22), (II.23) and (II.24) gives

$$\int_{\mathbb{R}^2} |\nabla^2 f_k|^r\ dy_1\,dy_2 \le C_r\ \left[\|\nabla f_k\|^r_\infty + \|\nabla^2 f_k\|^r_2\right]\ \int_{\mathbb{R}^2} |\nabla^2 f_k|^r\ dy_1\,dy_2 \quad . \tag{II.25}$$

From (II.13), (II.15) and (II.16) we have

$$\lim_{k\to+\infty} \|\nabla f_k\|^r_\infty + \|\nabla^2 f_k\|^r_2 = 0 \tag{II.26}$$

Thus for k large enough, (II.25) implies that

$$\nabla^2 f_k \equiv 0 \qquad \text{on } \mathbb{R}^2.$$

Sine $\nabla f_k(y)$ tends to zero as $|y|$ tends to infinity (see II.13) we obtain that

$$\nabla f_k \equiv 0 \qquad \text{on } \mathbb{R}^2.$$

Thus for k large enough f_k is a constant which means that $\vec{\phi}_k(S^2)$ is a sphere homothetic to S^2 and we have in particular $W(\vec{Phi}_k) = W(\vec{\xi}_k) = 4\pi$. Theorem I.1 is proved. □

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