A Viscosity Method in the Min-max Theory of Minimal Surfaces.

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Abstract : *We present the min-max construction of critical points of the area in arbitrary closed sub-manifold of euclidian spaces by penalization arguments. Precisely, for any immersion of a closed surface Σ, we add to the area functional a term equal to the L^q norm of the second fundamental form of the immersion times a "viscosity" parameter. This relaxation of the area functional satisfies the Palais-Smale condition for $q > 2$. This permits to construct critical points of the relaxed Lagrangian using classical min-max arguments such as the mountain pass lemma. The goal of this work is to describe the passage to the limit when the "viscosity" parameter tends to zero. Under some natural entropy condition, we establish a varifold convergence of these critical points towards a possibly branched smooth closed minimal surface realizing the min-max value. This method opens the door for exploring new min-max values based on homotopy groups of the space of immersions of surfaces into sub-manifolds of arbitrary dimension.*

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I Introduction

The study of minimal surfaces, critical points of the area, has stimulated the development of entire fields in analysis and in geometry. The calculus of variations is one of them. The origin of the field is very much linked to the question of proving the existence of minimal 2-dimensional discs bounding a given curve in the euclidian 3-dimensional space and minimizing the area. This question, known as *Plateau Problem*, has been posed since the XVIIIth century by Joseph-Louis Lagrange, the founder of the Calculus of Variation after Leonhard Euler. This question has been ultimately solved independently by Jesse Douglas and Tibor Radó around 1930. In two words the main strategy of the proofs was to minimize the Dirichlet energy instead of the area, which is lacking coercivity properties, the two lagrangians being identical on conformal maps. After these proofs, successful attempts have been made to solve the Plateau Problem in much more general frameworks. This has been in particular at the origin of the field of *Geometric Measure Theory* during the 50's, where the notions of rectifiable current which were proved to be the *ad-hoc* objects for the minimization process of the area (or the mass in general) in the most general setting.

The search of absolute or even local minimizers is of course the first step in the study of the variations of a given lagrangians but is far from being exhaustive while studying the whole set of critical points. In many problems there is even no minimizer at all, this is for instance the case of closed surfaces in simply connected manifolds with also trivial two dimensional homotopy groups. This problem is already present in the 1-dimensional counter-part of minimal surfaces, the study of closed geodesics. For instance in a sub-manifold of $\mathbb{R}^3$ diffeomorphic to S^2 there is obviously no closed geodesic minimizing the length. In order to construct closed geodesics in such manifold, Birkhoff around 1915 introduced a technic called "min-max" which permits to generate critical points of the length with non trivial index. In two words this technic consists in considering the space of paths of closed curves within a non-trivial homotopy classes

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of paths in the sub-manifold (called "sweep-out") and to minimize, out of all such paths or "sweep-outs", the maximal length of the curves realizing each "sweep-out". In order to do so, one is facing the difficulty posed by a lack of coercivity of the length with respect to this minimization process within this "huge space" of sweep-outs. In order to "project" the problem to a much smaller space of "sweep-outs" in which the length would become more coercive, George Birkhoff replaced each path by a more regular one made of very particular closed curves joining finitely many points with portions of geodesics minimizing the length between these points. This replacement method also called nowadays "curve shortening process" has been generalized in many situations in order to perform min-max arguments.

Back to minimal surfaces, in a series of two works (see [6] and [7]), Tobias Colding and Bill Minicozzi, construct by min-max methods minimal 2 dimensional spheres in riemannian manifolds. The main strategy of the proof combines the original approach of Douglas and Radó, consisting in replacing the area functional by the Dirichlet energy, with a "Birkhoff type" argument of optimal replacements. Locally to any map from a given "sweep-out" one performs a surgery, replacing the map itself by an harmonic extension minimizing the Dirichlet energy. The convergence of such a "harmonic replacement" procedure, corresponding in some sense to Birkhoff "curve shortening procedure" in one dimension, is ensured by a fundamental result regarding the local convexity of the Dirichlet energy into a manifold under small energy assumption and a unique continuation type property. What makes possible the use of the Dirichlet energy instead of the area functional, as in [32], is the fact that the domain S^2 posses only one conformal structure and modulo a re-parametrization any $W^{1,2}$ map can be made almost conformal (due to a fundamental result of Charles Morrey [24]). This is not anymore the case if one wants to extend Colding-Minicozzi's approach to general surfaces. This has been done however successfully by Zhou Xin in [39] and [40] following the original Colding-Minicozzi approach. These papers are based on an involved argument in which to any "sweep-out" of $W^{1,2}-$maps a path of smooth conformal structures together with a path of re-parametrization are assigned in order to be as close as possible to paths of conformal maps.

Because of the finite dimensional nature of the moduli space of conformal structures in 2-D, and the "optimal properties" of the Dirichlet energy, Colding-Minicozzi's min-max method is intrinsically linked to two dimensions as Douglas-Radó's resolution of the Plateau problem was too. The field of Geometric Measure Theory, which was originally designed to remedy to this limitation and to solve the Plateau Problem for arbitrary dimensions in various homology classes, has been initially developed with a minimization perspective and the framework of rectifiable currents as well as the lower semicontinuity of the mass for weakly converging sequences was matching perfectly this goal. In order to solve min-max problems in the general framework of Geometric Measure Theory, the notion of *varifold* has been successfully introduced by William Allard and by Fred Almgren. A complete GMT min-max procedure has been finally set up by Jon Pitts in [26] who introduced the notion of *almost minimizing varifolds* and developed their regularity theory in co-dimension 1. Constructive comparison arguments as well as combinatorial type arguments are also needed in this rather involved and general procedure (The reader is invited also to consult [5] and [19] for thorough presentations of the GMT approaches to min-max procedures).

The aim of the present work is to present a direct min-max approach for constructing minimal surfaces in a given closed submanifold N^n of $\mathbb{R}^m$. The general scheme is simple : one works with a special subspace of C^1 immersions of a given surface Σ, one adds to the area of each of such an immersion $\vec{\Phi}$ a relaxing "curvature type" functional multiplied by a small viscous parameter σ^2

$$A^\sigma(\vec{\Phi}) := \text{Area}(\vec{\Phi}) + \sigma^2 \int_\Sigma \text{curvature terms } dvol_{g_{\vec{\Phi}}} \tag{I.1}$$

where $dvol_{g_{\vec{\Phi}}}$ is the volume form on Σ induced by the immersion $\vec{\Phi}$. The "curvature terms" is chosen in order to ensure that A^σ satisfies the *Palais-Smale property* on the *ad-hoc* corresponding Finsler manifold of C^1-immersions. This offers the suitable framework in which the mountain path lemma can be applied. Once a min-max critical point of A^σ is produced one passes to the limit $\sigma \to 0 \cdots$

More precisely, we introduce the space $\mathcal{E}_{\Sigma,p}$ of $W^{2,2p}-$immersions $\vec{\Phi}$ of a given closed surface Σ for $p>1$ [1] into $N^n\subset\mathbb{R}^m$. It is proved below that this space has a nice structure of *Banach bundle* modeled on the Banach space $W^{2,2p}(\Sigma,\mathbb{R}^n)$. For such immersions we consider the relaxed energy

$$A^\sigma(\vec{\Phi}):=\mathrm{Area}(\vec{\Phi})+\sigma^2\int_\Sigma\left[1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2\right]^p\ dvol_{g_{\vec{\Phi}}}$$

where $g_{\vec{\Phi}}$ and $\vec{\mathbb{I}}_{\vec{\Phi}}$ are respectively the first and second fundamental forms of $\vec{\Phi}(\Sigma)$ in N^n. Unlike previous existing viscous relaxations for min-max problems in the literature, the energy A^σ is intrinsic in the sense that it is invariant under re-parametrization of $\vec{\Phi}$: $A^\sigma(\vec{\Phi})=A^\sigma(\vec{\Phi}\circ\Psi)$ for any smooth diffeomorphism Ψ of Σ. Modulo a choice of parametrization it is proved in [16] and [14] that for a fixed $\sigma\neq 0$ the Lagrangian A^σ satisfies the *Palais-Smale condition.* Hence we can consider applying the mountain path lemma to this Lagrangian. Our main result in the present work is the following convergence theorem.

Theorem I.1. *Let N^n be a closed $n-$dimensional sub-manifold of $\mathbb{R}^m$ with $3\le n\le m-1$ being arbitrary. Let Σ be an arbitrary closed riemanian 2-dimensional manifold. Let $\sigma_k\to 0$ and let $\vec{\Phi}_k$ be a sequence of critical points of*

$$A^{\sigma_k}(\vec{\Phi}):=Area(\vec{\Phi})+\sigma_k^2\int_\Sigma\left[1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2\right]^p\ dvol_{g_{\vec{\Phi}}}$$

in the space of $W^{2,2p}-$immersions of Σ and satisfying the entropy condition

$$\sigma_k^2\int_\Sigma\left[1+|\vec{\mathbb{I}}_{\vec{\Phi}_k}|^2\right]^p\ dvol_{g_{\vec{\Phi}_k}}=o\left(\frac{1}{\log\,\sigma_k^{-1}}\right)\tag{I.2}$$

Then, modulo extraction of a subsequence, there exists a closed riemann surface S with genus$(S)\le$genus(Σ) and a conformal smooth harmonic map $\vec{\Phi}_\infty$ from S into N^n such that

$$\lim_{k\to+\infty}A^{\sigma_k}(\vec{\Phi}_k)=Area(\vec{\Phi}_\infty)$$

Moreover, the oriented varifold associated to $\vec{\Phi}_k$ converges in the sense of Radon measures towards the oriented stationary integer varifold associated to $\vec{\Phi}_\infty$, which is a smooth sub-manifold away from at most finitely many isolated points . □

The main difficulty in proving theorem I.1 in contrast with existing non intrinsic viscous approximations of min-max procedures in the literature is that there is a-priori no $\epsilon-$regularity property <u>independent</u> of the viscosity σ available. Indeed the following result is proved in [22].

Proposition I.1. *There exists $\vec{\Phi}_k\in C^\infty(T^2,S^3)$ and $\sigma_k\to 0$ such that $\vec{\Phi}_k$ is a sequence of immersions, critical points of A^{σ_k}, which is conformal into S^3 from a converging sequence of flat torii $\mathbb{R}^2/\mathbb{Z}+(a_k+i\,b_k)\mathbb{Z}$ towards $\mathbb{R}^2/\mathbb{Z}+(a_\infty+i\,b_\infty)\mathbb{Z}$, for which*

$$\limsup_{k\to+\infty}A^{\sigma_k}(\vec{\Phi}_k)<+\infty$$

such that also $\vec{\Phi}_k$ weakly converges to a limiting map $\vec{\Phi}_\infty$ in $W^{1,2}(\mathbb{R}^2/\mathbb{Z}+(a_\infty+i\,b_\infty)\mathbb{Z},S^3)$ but $\vec{\Phi}_k$ nowhere strongly converges : precisely

$$\forall\ U\ \textit{open set in}\ \mathbb{R}^2/\mathbb{Z}+(a_\infty+i\,b_\infty)\mathbb{Z}\qquad\int_U|\nabla\vec{\Phi}_\infty|^2\ dx^2<\liminf_{k\to+\infty}\int_U|\nabla\vec{\Phi}_k|^2\ dx^2\quad .$$

□

[1] The condition $p>1$ ensures that $\vec{\Phi}$ is C^1. This last fact permits to use the classical definition of an immersion. The case $p=1$ was considered in previous works by the author where the notion of immersion had to be weakened.

In order to overcome this major difficulty in the passage to the limit $\sigma_k \to 0$ we prove a quantization result, lemma III.2, which roughly says that there is a positive number Q_0, depending only on the target $N^n \subset \mathbb{R}^m$, below which for k large enough, under the entropy condition assumption, there is no critical point of A^{σ_k}. This result is used at several stages in the proof. The main strategy goes as follows. We first establish the stationarity of the limiting varifold. The proof is based on an *almost divergence form* of the Euler Lagrange equation associated to A^σ following the approach introduced in [28] for the Willmore Lagrangian in $\mathbb{R}^m$. The existence of such a an *almost divergence form* is due to the symmetry group associated to the same Lagrangian in flat space and the application of Noether theorem (see [1]). As in [22], the exact divergence form in Euclidian space is just an *almost-divergence form* in manifold. Next we choose a conformal parametrization of $\vec{\Phi}_k$ on a possibly degenerating sequence of riemann surfaces (Σ, h_k) (where h_k denotes the constant curvature metric of volume 1 conformally equivalent to $\vec{\Phi}_k^* g_{N^n}$). We use Deligne Mumford compactification in order to make converge (Σ, h_k) towards a nodal riemann surface with punctures (see for instance [12]). We then use the monotonicity formula, deduced from the stationarity, in order to prove that away from a so called *oscillation set* the limiting volume density measure on the thick parts of the limiting nodal surface is absolutely with respect to the Lebesgue measure. We then use the monotonicity formula again in order to prove the quantization result lemma III.2. This quantization result is used in order to show that the limiting volume density measure restricted to the oscillation set is equal to finitely many Dirac masses. The quantization result is again used in order to prove that for the weakly converging sequence $\vec{\Phi}_k$ there is no energy loss neither in the necks in each thick parts of the limiting nodal surface, nor in the collars regions separating possible bubbles, which are possibly formed (see lemma III.6). The previous results are proved to show the rectifiability of the limiting varifold (see lemma III.7). We then prove that there is no measure concentrated on the set of points where the rank of the weak limit $\vec{\Phi}_\infty$ on each thick part and on each bubble is not equal to 2. Finally we use all the previous results to prove a <u>strong</u> $W^{1,2}-$bubble tree convergence of the sequence $\vec{\Phi}_k$ on each thick part (lemma III.11) which gives in particular that the limiting rectifiable stationary varifold is integer. The last lemma, lemma III.14, establishes that the limiting map is a *conformal target harmonic map* on each thick part of the nodal surface and on each bubble. A proof of a *point removability* result realizes the main core of the proof of lemma III.14. Finally we use the regularity result of [30].

Theorem I.1 can be used to prove various existence results of optimal surfaces realizing a min-max energy level. We first define the following notion.

Definition I.1. *A family of subsets $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ of a Banach manifold $\mathcal{M}$ is called* **admissible family** *if for every homeomorphism Ξ of $\mathcal{M}$ isotopic to the identity we have*

$$\forall\, A \in \mathcal{A} \qquad \Xi(A) \in \mathcal{A}$$

□

Example. Consider $\mathcal{M} := W^{2,2p}_{imm}(\Sigma, S^3)$ for some closed surface Σ and take for any $q \in \mathbb{N}$ and $c \in \pi_q(\mathrm{Imm}(\Sigma), S^3)$ then the following family is admissible

$$\mathcal{A} := \left\{ \vec{\Phi} \in C^0(S^q, W^{2,q}_{imm}(S^2, \mathbb{R}^3)) \ ; \qquad \text{s.t. } [\vec{\Phi}] = c \right\}$$

□

Our second main result is the following.

Theorem I.2. *Let $\mathcal{A}$ be an admissible family either in $\mathcal{E}_{\Sigma,p}(N^n)$ such that*

$$\inf_{A \in \mathcal{A}} \max_{\vec{\Phi} \in A} Area(\vec{\Phi}) = \beta^0 > 0 \tag{I.3}$$

then there exists a closed riemann surface S with genus$(S) \leq$genus(Σ) and a smooth conformal harmonic map $\vec{\Phi}_\infty$ from S into N^n such that

$$
Area(\vec{\Phi}_\infty) = \beta^0 \quad .
$$

□

This general existence result has to be put in perspective with the previous min-max existence results discussed above either in GMT (see [26], [34], [5], [19], [20]$\cdots$) or in harmonic map theory (see [6],[7],[39],[40]). First of all theorem I.2 implies all the known existence results proved in these papers for 2 dimensional objects in arbitrary codimensions. It's novelty resides maybe in the fact that it gives a direct approach to the construction of min-max minimal surfaces and does not require any "replacement argument". We do not use the fact that the index in some sense is bounded and any notion like "quasi-minimality". The entropy condition (I.2), which is obtained using Struwe's argument, however is central in the proof. The viscosity approach gives, without any additional work, an upper bound of the genus of the optimal surface. Such lower semicontinuity of the genus has been established in the GMT approach in [9] in co-dimension 1 and was not given by the min-max procedure itself. As in the geodesic case studied recently in [22] and where a passage to the limit in the second derivative is proved, the viscosity approach could possibly give also informations on the limiting index. We hope to study these questions in future works.

The second, and possibly main advantage, of the viscosity method resides in the fact that one can explore min-max within the space of immersions of fixed closed surfaces. The spaces $\text{Imm}(\Sigma, N^n)$ offers a richer topology than the space of *integer rectifiable 2-cycles* $\mathcal{Z}_2(N^n)$ considered by Almgren whose homotopy type is more coarse.

In order to simplify the presentation and in particular the computations of the Euler Lagrange equation to A^σ we are presenting the proof of theorem I.1, in the special case $N^n = S^3$. There is however no argument below which is specific to that case and the proof in the general case follows each step word by word of the S^3 case.

II The viscous relaxation of the area for surfaces.

II.1 The Finsler Manifold of immersions into the spheres with L^q bounded second fundamental form.

For $k \in \mathbb{N}$ and $1 \leq q \leq +\infty$ We recall the definition of $W^{k,q}$ Sobolev function on a closed smooth surface Σ (i.e. Σ is compact without boundary). To that aim we take some reference smooth metric g_0 on Σ

$$
W^{k,q}(\Sigma, \mathbb{R}) := \left\{ f \quad \text{measurable} \quad s.t \quad \nabla^k_{g_0} f \in L^q(\Sigma, g_0) \right\}
$$

where $\nabla^k_{g_0}$ denotes the $k-$th iteration of the Levi-Civita connection associated to Σ. Since the surface is closed the space defined in this way is independent of g_0. Let N^n be a closed $n-$dimensional sub-manifold of $\mathbb{R}^m$ with $3 \leq n \leq m-1$ being arbitrary. The Space of $W^{k,q}$ into N^n is defined as follows

$$
W^{k,q}(\Sigma, N^n) := \left\{ \vec{\Phi} \in W^{k,q}(\Sigma, \mathbb{R}^m) \quad ; \quad \vec{\Phi} \in N^n \quad \text{almost everywhere} \right\}
$$

We have the following well known proposition

Proposition II.1. *Assuming $kq > 2$, the space $W^{k,q}(\Sigma, N^n)$ defines a Banach Manifold modeled on the Banach space $W^{k,q}(\Sigma, \mathbb{R}^m)$.* □

Proof of proposition II.1.

This comes mainly from the fact that, under our assumptions,

$$W^{k,q}(\Sigma,\mathbb{R}^m) \quad \hookrightarrow \quad C^0(\Sigma,\mathbb{R}^m) \quad . \tag{II.1}$$

The Banach manifold structure is then defined as follows. Choose $\delta > 0$ such that each geodesic ball $B_\delta^{N^n}(z)$ for any $z \in N^n$ is strictly convex and the exponential map

$$\exp_z \ : \ V_z \subset T_z N^n \ \longrightarrow \ B_\delta^{N^n}(z)$$

realizes a C^∞ diffeomorphism for some open neighborhood of the origin in $T_z N^n$ into the geodesic ball $B_\delta^{N^n}(z)$. Because of the embedding (II.1) there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned} \forall\ \vec{u}\ ,\ \vec{v} \in W^{k,q}(\Sigma, N^n) \quad & \|\vec{u}-\vec{v}\|_{W^{k,q}} < \varepsilon_0 \\ \Longrightarrow \quad & \|\mathrm{dist}_N(\vec{u}(x),\vec{v}(x))\|_{L^\infty(\Sigma)} < \delta \quad . \end{aligned}$$

We equip now the space $W^{k,q}(\Sigma, N^n)$ with the distance issued from the $W^{k,q}$ norm and for any $\vec{u} \in \mathcal{M} = W^{k,q}(\Sigma, N^n)$ we denote by $B_{\varepsilon_0}^{\mathcal{M}}(\vec{u})$ the open ball in $\mathcal{M}$ of center $\vec{u}$ and radius ε_0.

As a covering of $\mathcal{M}$ we take $(B_{\varepsilon_0}^{\mathcal{M}}(\vec{u}))_{\vec{u}\in\mathcal{M}}$. We denote by

$$E^{\vec{u}} := \Gamma_{W^{k,q}}\left(\vec{u}^{-1}TN\right) := \left\{\vec{w} \in W^{k,q}(\Sigma,\mathbb{R}^m)\ ;\ \vec{w}(x) \in T_{\vec{u}(x)}N^n\ \forall\ x\in\Sigma\right\}$$

this is the Banach space of $W^{k,q}-$sections of the bundle $\vec{u}^{-1}TN$ and for any $\vec{u} \in \mathcal{M}$ and $\vec{v} \in B_{\varepsilon_0}^{\mathcal{M}}(\vec{u})$ we define $\vec{w}^{\,\vec{u}}(\vec{v})$ to be the following element of $E^{\vec{u}}$

$$\forall\ x \in \Sigma \qquad \vec{w}^{\,\vec{u}}(\vec{v})(x) := \exp^{-1}_{\vec{u}(x)}(\vec{v}(x))$$

It is not difficult to see that

$$\vec{w}^{\,\vec{v}} \circ (\vec{w}^{\,\vec{u}})^{-1}\ :\ \vec{w}^{\,u}\left(B_{\varepsilon_0}^{\mathcal{M}}(\vec{u}) \cap B_{\varepsilon_0}^{\mathcal{M}}(\vec{v})\right) \ \longrightarrow \ \vec{w}^{\,v}\left(B_{\varepsilon_0}^{\mathcal{M}}(\vec{u}) \cap B_{\varepsilon_0}^{\mathcal{M}}(\vec{v})\right)$$

defines a C^∞ diffeomorphism. □

For $p > 1$ we define

$$\mathcal{E}_{\Sigma,p} = W^{2,2p}_{imm}(\Sigma^2, N^n) := \left\{\vec{\Phi} \in W^{2,q}(\Sigma^2, N^n)\ ;\ \mathrm{rank}\,(d\Phi_x) = 2 \quad \forall x \in \Sigma^2\right\}$$

The set $W^{2,2p}_{imm}(\Sigma^2, N^n)$ as an open subset of the normal Banach Manifold $W^{2,2p}(\Sigma^2, N^n)$ inherits a Banach Manifold structure.

We equip now the space $W^{2,2p}_{imm}(\Sigma, N^n)$ with a *Finsler manifold structure* on it's tangent bundle (see the definition of Banach Space bundles and Tangent bundle to a Banach manifold in [15]). For the convenience of the reader we recall the notion of Finsler structure.

Definition II.2. *Let $\mathcal{M}$ be a normal Banach manifold and let $\mathcal{V}$ be a Banach Space Bundle over $\mathcal{M}$. A* **Finsler structure** *on $\mathcal{V}$ is a continuous function*

$$\|\cdot\|\ :\ \mathcal{V} \ \longrightarrow \ \mathbb{R}$$

such that for any $x \in \mathcal{M}$

$$\|\cdot\|_x := \|\cdot\||_{\pi^{-1}(\{x\})} \qquad \textit{is a norm on } \mathcal{V}_x \quad .$$

Moreover for any local trivialization τ_i over U_i and for any $x_0 \in U_i$ we define on $\mathcal{V}_x$ the following norm

$$\forall\ \vec{w} \in \pi^{-1}(\{x\}) \qquad \|\vec{w}\|_{x_0} := \|\tau_i^{-1}\left(x_0, \rho(\tau_i(\vec{w}))\right)\|_{x_0}$$

and there exists $C_{x_0} > 1$ such that

$$\forall\ x \in U_i \qquad C_{x_0}^{-1}\ \|\cdot\|_x \le \|\cdot\|_{x_0} \le C_{x_0}\ \|\cdot\|_x \quad .$$

□

Definition II.3. *Let $\mathcal{M}$ be a normal C^p Banach manifold. $T\mathcal{M}$ equipped with a Finsler structure is called a* **Finsler Manifold**. □

Remark II.1. *A Finsler structure on $T\mathcal{M}$ defines in a canonical way a dual Finsler structure on $T^*\mathcal{M}$.* □

The tangent space to $\mathcal{E}_{\Sigma,p}$ at a point $\vec{\Phi}$ is the space $\Gamma_{W^{2,2p}}(\vec{\Phi}^{\,-1}TN^n)$ of $W^{2,2p}-$sections of the bundle $\vec{\Phi}^{\,-1}TN^n$, i.e.

$$T_{\vec{\Phi}}\mathcal{E}_{\Sigma,p} = \left\{\vec{w} \in W^{2,2p}(\Sigma^2, \mathbb{R}^m)\ ;\ \vec{w}(x) \in T_{\vec{\Phi}(x)}N^n \quad \forall\, x \in\ \Sigma^2\right\} \quad .$$

We equip $T_{\vec{\Phi}}\mathcal{E}_{\Sigma,p}$ with the following norm

$$\|\vec{v}\|_{\vec{\Phi}} := \left[\int_\Sigma \left[|\nabla^2\vec{v}|^2_{g_{\vec{\Phi}}} + |\nabla\vec{v}|^2_{g_{\vec{\Phi}}} + |\vec{v}|^2\right]^p\ dvol_{g_{\vec{\Phi}}}\right]^{1/q} + \|\,|\nabla\vec{v}|_{g_{\vec{\Phi}}}\,\|_{L^\infty(\Sigma)}$$

where we keep denoting, for any $j \in \mathbb{N}$, ∇ to be the connection on $(T^*\Sigma)^{\otimes^j} \otimes \vec{\Phi}^{-1}TN$ over Σ defined by $\nabla := \nabla^{g_{\vec{\Phi}}} \otimes \vec{\Phi}^*\nabla^h$ and $\nabla^{g_{\vec{\Phi}}}$ is the Levi Civita connection on $(\Sigma, g_{\vec{\Phi}})$ and ∇^h is the Levi-Civita connection on N^n.

We check for instance that $\nabla^2\vec{v}$ defines a C^0 section of $(T^*\Sigma)^2 \otimes \vec{\Phi}^{-1}TN$.

The fact that we are adding to the $W^{2,2p}$ norm of $\vec{v}$ with respect to $g_{\vec{\Phi}}$ the L^∞ norm of $|\nabla\vec{v}|_{g_{\vec{\Phi}}}$ could look redundant since $W^{2,2p}$ embeds in $W^{1,\infty}$. We are doing it in order to ease the proof of the completeness of the Finsler Space equipped with the Palais distance below.

Observe that, using Sobolev embedding and in particular due to the fact $W^{2,q}(\Sigma, \mathbb{R}^m) \hookrightarrow C^1(\Sigma, \mathbb{R}^m)$ for $q > 2$, the norm $\|\cdot\|_{\vec{\Phi}}$ as a function on the Banach tangent bundle $T\mathcal{E}_{\Sigma,p}$ is obviously continuous.

Proposition II.2. *The norms $\|\cdot\|_{\vec{\Phi}}$ defines a C^2-Finsler structure on the space $\mathcal{E}_{\Sigma,p}$.* □

Proof of proposition II.2. We introduce the following trivialization of the Banach bundle. For any $\vec{\Phi} \in \mathcal{E}_{\Sigma,p}$ we denote $P_{\vec{\Phi}(x)}$ the orthonormal projection in $\mathbb{R}^m$ onto the $n-$dimensional vector subspace of $\mathbb{R}^m$ given by $T_{\vec{\Phi}(x)}N^n$ and for any $\vec{\xi}$ in the ball $B^{\mathcal{E}_{\Sigma,p}}_{\varepsilon_1}(\vec{\Phi})$ for some $\varepsilon_1 > 0$ and any $\vec{v} \in T_{\vec{\xi}}\mathcal{E}_{\Sigma,p} = \Gamma_{W^{2,q}}(\vec{\xi}^{-1}TN)$ we assign the map $\vec{w}(x) := P_{\vec{\Phi}(x)}\vec{v}(x)$. It is straightforward to check that for $\varepsilon_1 > 0$ chosen small enough the map which to $\vec{v}$ assigns $\vec{w}$ is an isomorphism from $T_{\vec{\xi}}\mathcal{E}_{\Sigma,p}$ into $T_{\vec{\Phi}}\mathcal{E}_{\Sigma,p}$ and that there exists $k_{\vec{\Phi}} > 1$ such that $\forall \vec{v} \in TB^{\mathcal{E}_{\Sigma,p}}_{\varepsilon_1}(\vec{\Phi})$

$$k_{\vec{\Phi}}^{-1}\,\|\vec{v}\|_{\vec{\xi}} \le \|\vec{w}\|_{\vec{\Phi}} \le k_{\vec{\Phi}}\,\|\vec{v}\|_{\vec{\xi}}$$

This concludes the proof of proposition II.2. □

II.2 Palais deformation theory applied to the space of $W^{2,2p}-$immersions.

Theorem II.1. **[Palais 1970]** *Let $(\mathcal{M},\|\cdot\|)$ be a Finsler Manifold. Define on $\mathcal{M}\times\mathcal{M}$*

$$d(p,q):=\inf_{\omega\in\Omega_{p,q}}\int_0^1\left\|\frac{d\omega}{dt}\right\|_{\omega(t)}\ dt$$

where

$$\Omega_{p,q}:=\left\{\omega\in C^1([0,1],\mathcal{M})\ ;\ \omega(0)=p\quad \omega(1)=q\right\}\quad .$$

Then d defines a distance on $\mathcal{M}$ and $(\mathcal{M},d)$ defines the same topology as the one of the Banach Manifold. d is called **Palais distance** *of the Finsler manifold $(\mathcal{M},\|\cdot\|)$.* □

Contrary to the first appearance the non degeneracy of d is not straightforward and requires a proof (see [25]). This last result combined with the famous result of Stones on the paracompactness of metric spaces gives the following corollary.

Corollary II.1. *Let $(\mathcal{M},\|\cdot\|)$ be a Finsler Manifold then $\mathcal{M}$ is paracompact.* □

The following result[2] is going to play a central role in adapting Palais deformation theory to our framework of $W^{2,2p}-$immersions.

Proposition II.3. *Let $p>1$ and $\mathcal{E}_{\Sigma,p}$ be the space of $W^{2,2p}-$immersions of a closed oriented surface Σ into a closed sub-manifold N^n of $\mathbb{R}^m$*

$$\mathcal{E}_{\Sigma,p}=W^{2,2p}_{imm}(\Sigma^2,N^n):=\left\{\vec{\Phi}\in W^{2,q}(\Sigma^2,N^n)\ ;\ rank(d\Phi_x)=2\quad\forall x\in\Sigma^2\right\}$$

The Finsler Manifold given by the structure

$$\|\vec{v}\|_{\vec{\Phi}}:=\left[\int_\Sigma\left[|\nabla^2\vec{v}|^2_{g_{\vec{\Phi}}}+|\nabla\vec{v}|^2_{g_{\vec{\Phi}}}+|\vec{v}|^2\right]^{q/2}\ dvol_{g_{\vec{\Phi}}}\right]^{1/q}+\|\,|\nabla\vec{v}|_{g_{\vec{\Phi}}}\,\|_{L^\infty(\Sigma)}$$

is <u>complete</u> for the Palais distance. □

Proof of proposition II.3. For any $\vec{\Phi}\in\mathcal{M}$ and $\vec{v}\in T_{\vec{\Phi}}\mathcal{M}$ we introduce the tensor in $(T^*\Sigma)^{\otimes^2}$ given in coordinates by

$$\begin{aligned}\nabla\vec{v}\,\dot{\otimes}\,d\vec{\Phi}+d\vec{\Phi}\,\dot{\otimes}\,\nabla\vec{v}&=\sum_{i,j=1}^2\left[\nabla_{\partial_{x_i}}\vec{v}\cdot\partial_{x_j}\vec{\Phi}+\partial_{x_i}\vec{\Phi}\cdot\nabla_{\partial_{x_j}}\vec{v}\right]\ dx_i\otimes dx_j\\&=\sum_{i,j=1}^2\left[\nabla^h_{\partial_{x_i}\vec{\Phi}}\vec{v}\cdot\partial_{x_j}\vec{\Phi}+\partial_{x_i}\vec{\Phi}\cdot\nabla^h_{\partial_{x_j}\vec{\Phi}}\vec{v}\right]\ dx_i\otimes dx_j\end{aligned}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^m$. Observe that we have

$$\left|\nabla\vec{v}\,\dot{\otimes}\,d\vec{\Phi}+d\vec{\Phi}\,\dot{\otimes}\,\nabla\vec{v}\right|_{g_{\vec{\Phi}}}\le 2\ |\nabla\vec{v}|_{g_{\vec{\Phi}}}$$

Hence, taking a C^1 path $\vec{\Phi}_s$ in $\mathcal{M}$ one has for $\vec{v}:=\partial_s\vec{\Phi}$

$$\begin{aligned}\|\,|d\vec{v}\dot{\otimes}d\vec{\Phi}+d\vec{\Phi}\dot{\otimes}d\vec{v}|^2_{g_{\vec{\Phi}}}\|_{L^\infty(\Sigma)}&=\left\|\sum_{i,j,k,l=1}^2 g^{ij}_{\vec{\Phi}}\ g^{kl}_{\vec{\Phi}}\ \partial_s(g_{\vec{\Phi}})_{ik}\ \partial_s(g_{\vec{\Phi}})_{jl}\right\|_{L^\infty(\Sigma)}\\&=\left\|\ |\partial_s(g_{ij}dx_i\otimes dx_j)|^2_{g_{\vec{\Phi}}}\right\|_{L^\infty(\Sigma)}=\left\|\ |\partial_s g_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right\|_{L^\infty(\Sigma)}\end{aligned}\tag{II.2}$$

[2] As a matter of fact the proof of the completeness with respect to the Palais distance is skipped in various applications of Palais deformation theory in the literature.

Hence

$$\int_0^1 \left\| \ |\partial_s g_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}} \right\|_{L^\infty(\Sigma)} \ ds \le 2 \ \int_0^1 \|\partial_s \vec{\Phi}\|_{\vec{\Phi}_s} \, ds \tag{II.3}$$

We now use the following lemma

Lemma II.1. *Let M_s be a C^1 path into the space of positive n by n symmetric matrix then the following inequality holds*

$$Tr\,(M^{-2}(\partial_s M)^2) \ge \|\partial_s \log M\|^2 = Tr\,((\partial_s \log M)^2)$$

Proof of lemma II.1. We write $M = \exp A$ and we observe that

$$\mathrm{Tr}\,(\exp(-2A)(\partial_s \exp A)^2) = \mathrm{Tr}\,(\partial_s A)^2$$

Then the lemma follows. □

Combining the previous lemma with (II.2) and (II.3) we obtain in a given chart

$$\int_0^1 \|\partial_s \log(g_{ij})\| \ ds \le \int_0^1 \sqrt{\mathrm{Tr}\,((\partial_s \log g_{ij})^2)}\, ds \le 2 \ \int_0^1 \|\partial_s \vec{\Phi}\|_{\vec{\Phi}_s} \, ds \tag{II.4}$$

This implies that in the given chart the log of the matrix $(g_{ij}(s))$ is uniformly bounded for $s \in [0,1]$ and hence $\vec{\Phi}_1$ is an immersion. It remains to show that it has a controlled $W^{2,q}$ norm. We introduce $p = q/2$ and denote

$$\mathrm{Hess}_p(\vec{\Phi}) := \int_\Sigma [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \ dvol_{g_{\vec{\Phi}}}$$

and we compute

$$\begin{aligned} \frac{d}{ds}(\mathrm{Hess}_p(\vec{\Phi})) = p \ &\int_\Sigma \partial_s |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}} \ [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1} \ dvol_{g_{\vec{\Phi}}} \\ &+ \int_\Sigma [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \ \partial_s(dvol_{g_{\vec{\Phi}}}) \end{aligned} \tag{II.5}$$

Classical computations give

$$\partial_s(dvol_{g_{\vec{\Phi}}}) = \left\langle \nabla \partial_s \vec{\Phi}, d\vec{\Phi} \right\rangle_{g_{\vec{\Phi}}} \ dvol_{g_{\vec{\Phi}}}$$

So we have

$$\begin{aligned} \left| \int_\Sigma [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \ \partial_s(dvol_{g_{\vec{\Phi}}}) \right| &\le \| \ |\nabla \partial_s \vec{\Phi}|_{g_{\vec{\Phi}}} \ \|_{L^\infty(\Sigma)} \int_\Sigma [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \ dvol_{g_{\vec{\Phi}}} \\ &\le \|\partial_s \vec{\Phi}\|_{\vec{\Phi}} \ \int_\Sigma [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \ dvol_{g_{\vec{\Phi}}} \end{aligned} \tag{II.6}$$

In local charts we have

$$|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}} = \sum_{i,j,k,l=1}^2 g^{ij}_{\vec{\Phi}} \, g^{kl}_{\vec{\Phi}} \ \left\langle \nabla^h_{\partial_{x_i} \vec{\Phi}} \partial_{x_k} \vec{\Phi}, \nabla^h_{\partial_{x_j} \vec{\Phi}} \partial_{x_l} \vec{\Phi} \right\rangle_h$$

Thus in bounding $\int_\Sigma \partial_s |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}} \ [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1} \ dvol_{g_{\vec{\Phi}}}$ we first have to control terms of the form

$$\left| \int_\Sigma \sum_{i,j,k,l=1}^2 \partial_s g^{ij}_{\vec{\Phi}} \, g^{kl}_{\vec{\Phi}} \ \left\langle \nabla^h_{\partial_{x_i} \vec{\Phi}} \partial_{x_k} \vec{\Phi}, \nabla^h_{\partial_{x_j} \vec{\Phi}} \partial_{x_l} \vec{\Phi} \right\rangle_h \ [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1} \ dvol_{g_{\vec{\Phi}}} \right| \tag{II.7}$$

We write

$$
\begin{aligned}
&\sum_{i,j,k,l=1}^{2} \partial_s g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}} \, \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h \\
&\quad = \sum_{i,j,k,l,t,r=1}^{2} \partial_s g^{ij}_{\vec{\Phi}}\, g_{jt}\, g^{tr} g^{kl}_{\vec{\Phi}} \, \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h \\
&\quad = -\sum_{i,j,k,l,=1}^{2} \left(\sum_{t,r=1}^{2} \partial_s g_{jt}\, g^{tr} \right) \, g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}} \, \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left| \int_\Sigma \sum_{i,j,k,l=1}^{2} \partial_s g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}} \, \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h \, [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1} \, dvol_{g_{\vec{\Phi}}} \right| \\
&\le \; \| \, |\partial_s g_{\vec{\Phi}}|_{g_{\vec{\Phi}}} \|_{L^\infty(\Sigma)} \, \int_\Sigma [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \, dvol_{g_{\vec{\Phi}}} \\
&\le \; \|\partial_s \vec{\Phi}\|_{\vec{\Phi}_s} \, \int_\Sigma [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \, dvol_{g_{\vec{\Phi}}}
\end{aligned}
\tag{II.8}
$$

We have also

$$
\begin{aligned}
&\partial_s \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h \\
&\quad = \left\langle \nabla^h_{\partial_s\vec{\Phi}} \left(\nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi} \right), \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h + \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_s\vec{\Phi}} \left(\nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right) \right\rangle_h
\end{aligned}
$$

By definition we have

$$
\nabla^h_{\partial_s\vec{\Phi}} \left(\nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi} \right) = \nabla^h_{\partial_{x_i}\vec{\Phi}} \left(\nabla^h_{\partial_s\vec{\Phi}} \partial_{x_k}\vec{\Phi} \right) + R^h(\partial_{x_i}\vec{\Phi}, \partial_s\vec{\Phi}) \partial_{x_k}\vec{\Phi}
$$

where we have used the fact that $[\partial_s\vec{\Phi}, \partial_{x_i}\vec{\Phi}] = \vec{\Phi}_*[\partial_s, \partial_{x_i}] = 0$. Using also that $[\partial_s\vec{\Phi}, \partial_{x_k}\vec{\Phi}] = 0$, since ∇^h is torsion free, we have finally

$$
\nabla^h_{\partial_s\vec{\Phi}} \left(\nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi} \right) = \nabla^h_{\partial_{x_i}\vec{\Phi}} \left(\nabla^h_{\partial_{x_k}\vec{\Phi}} \partial_s\vec{\Phi} \right) + R^h(\partial_{x_i}\vec{\Phi}, \partial_s\vec{\Phi}) \partial_{x_k}\vec{\Phi}
\tag{II.9}
$$

where R^h is the Riemann tensor associated to the Levi-Civita connection ∇^h. We have

$$
\nabla^h_{\partial_{x_i}\vec{\Phi}} \left(\nabla^h_{\partial_{x_k}\vec{\Phi}} \partial_s\vec{\Phi} \right) = (\nabla^h)^2_{\partial_{x_i}\vec{\Phi}\partial_{x_k}\vec{\Phi}} \partial_s\vec{\Phi} + \nabla^h_{\nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}} \partial_s\vec{\Phi}
\tag{II.10}
$$

Hence

$$
\begin{aligned}
&\left\langle \nabla^h_{\partial_s\vec{\Phi}} \left(\nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi} \right), \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h = \left\langle (\nabla^h)^2_{\partial_{x_i}\vec{\Phi}\partial_{x_k}\vec{\Phi}} \partial_s\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h \\
&\quad + \left\langle \nabla^h_{\nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}} \partial_s\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h + \left\langle R^h(\partial_{x_i}\vec{\Phi}, \partial_s\vec{\Phi}) \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h
\end{aligned}
\tag{II.11}
$$

Combining all the previous gives then

$$
\begin{aligned}
&\left|\int_\Sigma \sum_{i,j,k,l=1}^2 g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}}\, \partial_s \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h \ dvol_{g_{\vec{\Phi}}}\right| \\
&\quad \le C \int_\Sigma \left|\left\langle \nabla^2\partial_s\vec{\Phi}, \nabla d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}}\right| \ [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1}\ dvol_{g_{\vec{\Phi}}} \\
&\quad\quad + C \int_\Sigma |\nabla\partial_s\vec{\Phi}|_{g_{\vec{\Phi}}}\ |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\ [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1}\ dvol_{g_{\vec{\Phi}}} \\
&\quad\quad + C\,\|R^h\|_{L^\infty(N^n)} \int_\Sigma |\partial_s\vec{\Phi}|_h\ |\nabla d\vec{\Phi}|_{g_{\vec{\Phi}}}\ [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1}\ dvol_{g_{\vec{\Phi}}}
\end{aligned}
\tag{II.12}
$$

Combining all the above we finally obtain that

$$
\left|\partial_s \mathrm{Hess}_p(\vec{\Phi})\right| \le C\,\|\partial_s\vec{\Phi}\|_{\vec{\Phi}}\ \left[\mathrm{Hess}_p(\vec{\Phi}) + \mathrm{Hess}_p(\vec{\Phi})^{1-1/2p}\right] \tag{II.13}
$$

Combining (II.4) and (II.13) we deduce using Gromwall lemma that if we take a C^1 path from $[0,1)$ into $\mathcal{M}$ with finite length for the Palais distance d, the limiting map $\vec{\Phi}_1$ is still a $W^{2,2p}-$immersion of Σ into N^n, which proves the completeness of $(\mathcal{E}_{\Sigma,p}, d)$. □

The following definition is central in Palais deformation theory.

Definition II.4. *Let E be a C^1 function on a Finsler manifold $(\mathcal{M}, \|\cdot\|)$ and $\beta \in E(\mathcal{M})$. On says that E fulfills the* **Palais-Smale condition** *at the level β if for any sequence u_n staisfying*

$$
E(u_n) \longrightarrow \beta \quad \text{and} \quad \|DE_{u_n}\|_{u_n} \longrightarrow 0 \quad ,
$$

then there exists a subsequence $u_{n'}$ and $u_\infty \in \mathcal{M}$ such that

$$
d(u_{n'}, u_\infty) \longrightarrow 0 \quad .
$$

and hence $E(u_\infty) = \beta$ and $DE_{u_\infty} = 0$. □

The following result is the Palais Smale condition for the functional

$$
A^\sigma_p(\vec{\Phi}) := \mathrm{Area}(\vec{\Phi}) + \sigma^2 \int_\Sigma \left[1 + |\mathbb{I}_{\vec{\Phi}}|^2\right]^p \ dvol_{g_{\vec{\Phi}}} \quad .
$$

Theorem II.2. *Let $p > 1$ and $\vec{\Phi}_k$ such that*

$$
\limsup_{k\rightarrow+\infty} A^\sigma_p(\vec{\Phi}_k) < +\infty
$$

and satisfying

$$
\lim_{k\rightarrow+\infty} \sup_{\|\vec{w}\|_{\vec{\Phi}_k}\le 1} \partial A^\sigma_p(\vec{\Phi}_k)\cdot\vec{w} = 0 \quad . \tag{II.14}
$$

Then, modulo extraction of a subsequence, there exists a sequence of $W^{2,2p}-$diffeomorphisms Ψ_k such that $\vec{\Phi}_k$ converges strongly in $\mathcal{E}_{\Sigma,p}$ for the Palais distance to a critical point of A^σ_p. Moreover, if one assume that $\vec{\Phi}_k$ stays inside a fixed ball of the Palais distance one can take $\Psi_k(x) = x$. □

Remark II.2. *The first part of this theorem has been proved in [14] in the flat framework which does not differ much from our case of $W^{2,2p}-$immersions into N^n. The second part is a direct consequence of the proof of proposition II.3 above and is being used below since in the main Palais theorem II.3 the flow issued by the pseudo-gradient maintains the image at a finite Palais distance.* □

Definition II.5. *A family of subsets $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ of a Banach manifold $\mathcal{M}$ is called* **admissible family** *if for every homeomorphism Ξ of $\mathcal{M}$ isotopic to the identity we have*

$$\forall\, A \in \mathcal{A} \qquad \Xi(A) \in \mathcal{A}$$

$\Box$

Example. Consider $\mathcal{M} := W^{2,q}_{imm}(S^2, \mathbb{R}^3)$ and take $c \in \pi_1(\mathrm{Imm}(S^2, \mathbb{R}^3)) = \mathbb{Z}_2 \times \mathbb{Z}$ then the following family is admissible

$$\mathcal{A} := \left\{ \vec{\Phi} \in C^0([0,1], W^{2,q}_{imm}(S^2, \mathbb{R}^3)) \ ; \ \vec{\Phi}(0,\cdot) = \vec{\Phi}(1,\cdot) \quad \text{and } [\vec{\Phi}] = c \right\}$$

$\Box$

We recall the main theorem of Palais deformation theory.

Theorem II.3. [Palais 1970] *Let $(\mathcal{M}, \|\cdot\|)$ be a Banach manifold together with a $C^{1,1}-$Finsler structure. Assume $\mathcal{M}$ is <u>complete</u> for the induced Palais distance d and let $E \in C^1(\mathcal{M})$ satisfying the Palais-Smale condition $(PS)_\beta$ for the level set β. Let $\mathcal{A}$ be an admissible family in $\mathcal{P}(\mathcal{M})$ such that*

$$\inf_{A \in \mathcal{A}} \sup_{u \in A} E(u) = \beta$$

then there exists $u \in \mathcal{M}$ satisfying

$$\left\{ \begin{array}{l} DE_u = 0 \\[2mm] E(u) = \beta \end{array} \right. \tag{II.15}$$

$\Box$

II.3 Struwe's monotonicity trick.

Because of theorem II.2, theorem II.3 can be applied to each of the lagrangian A^σ_p for any admissible family $\mathcal{A}$ of $\mathcal{E}_{\Sigma,p}$ satisfying

$$\inf_{A \in \mathcal{A}} \max_{\vec{\Phi} \in A} \mathrm{Area}(\vec{\Phi}) = \beta^0 > 0 \tag{II.16}$$

However, beside the difficulty of establishing a convergence of any nature to the corresponding sequence of critical points $\vec{\Phi}_\sigma$ given by theorem II.3, although it is clear that

$$\lim_{\sigma \to 0} \inf_{A \in \mathcal{A}} \max_{\vec{\Phi} \in A} A^\sigma_p(\vec{\Phi}_\sigma) = \beta^0$$

nothing excludes *a-priori* that

$$\lim_{\sigma \to 0} \inf_{A \in \mathcal{A}} \max_{\vec{\Phi} \in A} \mathrm{Area}(\vec{\Phi}_\sigma) < \beta^0$$

and it could be that the smoothing part of the lagrangian $\sigma^2 \int_\Sigma \left[1 + |\vec{\mathbb{I}}_{\vec{\Phi}_\sigma}|^2\right]^p \, dvol_{g_{\vec{\Phi}_\sigma}}$ does not go to zero. In order to prevent this unpleasant situation where the smoothed min-max procedure is not approximating properly the limiting min-max procedure, M.Struwe invented a technic consisting in localizing the action of the pseudo-gradient close to the level set $\mathrm{Area}(\vec{\Phi}) = \beta_0$ exclusively. Precisely we have the following result

Theorem II.4. *Let $(\mathcal{M}, \|\cdot\|)$ be a complete Finsler manifold. Let E^σ be a family of C^1 functions for $\sigma \in [0,1]$ on $\mathcal{M}$ such that for every $\vec{\gamma} \in \mathcal{M}$*

$$\sigma \longrightarrow E^\sigma(\vec{\gamma}) \quad \text{ and } \quad \sigma \longrightarrow \partial_\sigma E^\sigma(\vec{\gamma}) \tag{II.17}$$

are increasing and continuous functions with respect to σ. Assume moreover that

$$\|DE^\sigma_{\vec{\gamma}} - DE^\tau_{\vec{\gamma}}\|_{\vec{\gamma}} \le C(\sigma)\ \delta(|\sigma-\tau|)\ f(E^\sigma(\vec{\gamma})) \tag{II.18}$$

where $C(\sigma) \in L^\infty_{loc}((0,1))$, $\delta \in L^\infty_{loc}(\mathbb{R}_+)$ and goes to zero at 0 and $f \in L^\infty_{loc}(\mathbb{R})$. Assume that for every σ the functional E^σ satisfies the Palais Smale condition. Let $\mathcal{A}$ be an admissible family of $\mathcal{M}$ and denote

$$\beta(\sigma) := \inf_{A\in\mathcal{A}} \sup_{\vec{\gamma}\in A} E^\sigma(\vec{\gamma})$$

Then there exists a sequence $\sigma_k \to 0$ and $\vec{\gamma}_j \in \mathcal{M}$ such that

$$E^{\sigma_k}(\vec{\gamma}_k) = \beta(\sigma_k) \quad , \quad DE^{\sigma_k}(\vec{\gamma}_k) = 0$$

Moreover $\vec{\gamma}_k$ satisfies the so called "entropy condition"

$$\partial_{\sigma_k} E^{\sigma_k}(\vec{\gamma}_k) = o\left(\frac{1}{\sigma_k\ \log\left(\frac{1}{\sigma_k}\right)}\right) \quad .$$

□

A proof of this theorem is given for instance in [31]

Theorem II.5. *Let $p>1$ and $\mathcal{A}$ be an admissible family either in $\mathcal{E}_{\Sigma,p}(N^n)$ such that*

$$\inf_{A\in\mathcal{A}} \max_{\vec{\Phi}\in A} Area(\vec{\Phi}) = \beta^0 > 0 \tag{II.19}$$

Then there exists $\sigma_k \to 0$ and a family $\vec{\Phi}_k$ of critical points of $A^{\sigma_k}_p$ satisfying

$$\lim_{k\to+\infty} Area(\vec{\Phi}_k) = \beta^0 \quad \text{ and } \quad \sigma_k^2 \int_\Sigma \left[1 + |\mathbb{I}_{\vec{\Phi}_k}|^2\right]^p\ dvol_{g_{\vec{\Phi}_k}} = o\left(\frac{1}{\log \sigma_k^{-1}}\right) \quad .$$

□

II.4 The first variation of the viscous energies A^σ_p.

Let $\vec{\Phi}$ be a smooth immersion from a closed 2-dimensional manifold Σ into the unit sphere $S^3 \subset \mathbb{R}^4$, let $\vec{w}$ be an infinitesimal immersion satisfying $\vec{w}\cdot\vec{\Phi} \equiv 0$ and denote $\vec{\Phi}_t$: a sequence of immersions into S^3 such that $d\vec{\Phi}/dt(0) = \vec{w}$. The Gauss map of the immersion is given in local coordinates by

$$\vec{n}_t = \star_{\mathbb{R}^4}\left(\vec{\Phi}_t \wedge \frac{\partial_{x_1}\vec{\Phi}_t \wedge \partial_{x_2}\vec{\Phi}_t}{|\partial_{x_1}\vec{\Phi}_t \wedge \partial_{x_2}\vec{\Phi}_t|}\right) \tag{II.20}$$

Assuming $\vec{\Phi}$ is expressed locally in conformal coordinates and denote $e^\lambda = |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$. We have

$$\vec{n}_t = \vec{n} + t\ (a_1\,\vec{e}_1 + a_2\,\vec{e}_2 + b\,\vec{\Phi}) + o(t) \quad , \tag{II.21}$$

where $\vec{e}_i = e^{-\lambda}\,\partial_{x_i}\vec{\Phi}$. Since $\vec{n}_t\cdot\partial_{x_i}\vec{\Phi}_t \equiv 0$ and $\vec{n}_t\cdot\vec{\Phi}_t\equiv 0$ we have

$$\begin{aligned}\frac{d\vec{n}}{dt}(0) &= -\vec{n}\cdot\vec{w}\ \vec{\Phi} - \sum_{i=1}^2 \vec{n}\cdot\partial_{x_i}\vec{w}\ e^{-\lambda}\ \vec{e}_i\\ &= -\vec{n}\cdot\vec{w}\ \vec{\Phi} - \left\langle \vec{n}\cdot d\vec{w}\,,\,d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}} \quad .\end{aligned} \tag{II.22}$$

Since $g_{ij} := \partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{\Phi}$, we have

$$\frac{dg_{ij}}{dt}(0) = \partial_{x_i}\vec{w}\cdot\partial_{x_j}\vec{\Phi} + \partial_{x_j}\vec{w}\cdot\partial_{x_i}\vec{\Phi} \quad . \tag{II.23}$$

Since $\sum_i g_{ki}\, g^{ij} = \delta_{kj}$ and $g_{ki} = e^{2\lambda}\,\delta_{ki}$, we have

$$\frac{dg^{ij}}{dt}(0) = -e^{-4\lambda}\ \left[\partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{w} + \partial_{x_j}\vec{\Phi}\cdot\partial_{x_i}\vec{w}\right] \quad . \tag{II.24}$$

We have also using (II.22) and (II.24)

$$\begin{aligned}\frac{d|d\vec{n}|^2_{g_{\vec{\Phi}}}}{dt} &= \frac{d}{dt}\left(\sum_{i,j=1}^2 g^{ij}\partial_{x_i}\vec{n}\cdot\partial_{x_j}\vec{n}\right)\\ &= -2\ \left\langle d\vec{\Phi}\dot{\otimes} d\vec{w}\,,\,d\vec{n}\dot{\otimes} d\vec{n}\right\rangle_{g_{\vec{\Phi}}} + 2\ \left\langle d\frac{d\vec{n}}{dt}\,;\,d\vec{n}\right\rangle_{g_{\vec{\Phi}}}\\ &= -2\ \left\langle d\vec{\Phi}\dot{\otimes} d\vec{w}\,,\,d\vec{n}\dot{\otimes} d\vec{n}\right\rangle_{g_{\vec{\Phi}}} + 4\,\vec{H}\cdot\vec{w} - 2\ \left\langle d\left\langle\vec{n}\cdot d\vec{w}\,,\,d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}}\ ;\,d\vec{n}\right\rangle_{g_{\vec{\Phi}}}\end{aligned} \tag{II.25}$$

Finally, we have $dvol_{g_{\vec{\Phi}}} = \sqrt{g_{11}g_{22}-g_{12}^2}\ dx_1\wedge dx_2$, hence

$$\frac{d}{dt}(dvol_{g_{\vec{\Phi}}})(0) = \left[\sum_{i=1}^2 \partial_{x_i}\vec{\Phi}\cdot\partial_{x_i}\vec{w}\right]\ dx_1\wedge dx_2 = \left\langle d\vec{\Phi}\,;d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \quad . \tag{II.26}$$

using (II.25) and (II.26) we obtain

$$\left.\frac{d}{dt}\text{Area}(\vec{\Phi}_t)\right|_{t=0} = \int_\Sigma \left\langle d\vec{\Phi}\,;d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \tag{II.27}$$

For any $p>1$ we denote

$$F_p(\vec{\Phi}) := \int_\Sigma \left[1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2\right]^p_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \quad .$$

Using (II.22) and (II.25) we have

$$\begin{aligned}\left.\frac{d}{dt}F_p(\vec{\Phi}_t)\right|_{t=0} &= \int_\Sigma f^p\ \left\langle d\vec{\Phi}\,;d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} - 2\,p\int_\Sigma f^{p-1}\ \left\langle d\vec{\Phi}\dot{\otimes} d\vec{w}\,,\,d\vec{n}\dot{\otimes} d\vec{n}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}}\\ &- 2\,p\int_\Sigma f^{p-1}\ \left\langle d\left\langle\vec{n}\cdot d\vec{w}\,,\,d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}}\ ;\,d\vec{n}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} + 4\,p\int_\Sigma f^{p-1}\ \vec{H}\cdot\vec{w}\ dvol_{g_{\vec{\Phi}}}\end{aligned} \tag{II.28}$$

where $f := \left[1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2\right]$.

II.5 The almost conservation laws satisfied by the critical points of $A_p^\sigma(\vec{\Phi})$.

The fact that A_p^σ is C^1 in $\mathcal{E}_{\Sigma,p}$ is quite standard for $p>1$ and is proved in [27]. Let $\vec{\Phi}$ be a critical point in $\mathcal{E}_{\Sigma,p}$ of A^σ_p. We then have

$$\begin{array}{l}\vec{\Phi}\wedge d^{*_{g_{\vec{\Phi}}}}\left[\left[1+\sigma^2\, f^p\right]\ d\vec{\Phi}\right]-\,2\,p\,\sigma^2\ \vec{\Phi}\wedge\, d^{*_{g_{\vec{\Phi}}}}\left[\ d^{*_{g_{\vec{\Phi}}}}\left[f^{p-1}\ d\vec{n}\right]\cdot d\vec{\Phi}\ \vec{n}\right]\\[3mm] -\,2\,p\,\sigma^2\ \vec{\Phi}\wedge\, d^{*_{g_{\vec{\Phi}}}}\left[\ f^{p-1}\ (d\vec{n}\,\dot{\otimes}\,d\vec{n})\llcorner_{g_{\vec{\Phi}}}\,d\vec{\Phi}\right]+\ 4\,p\ \sigma^2\ f^{p-1}\ \vec{\Phi}\wedge\vec{H}=0\qquad\text{in }\mathcal{D}'(\Sigma)\end{array}\tag{II.29}$$

where $f:=\left[1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2\right]$ as above and $(d\vec{n}\,\dot{\otimes}\,d\vec{n})\llcorner_{g_{\vec{\Phi}}}\,d\vec{\Phi}$ is the contraction given in local conformal coordinates by

$$(d\vec{n}\,\dot{\otimes}\,d\vec{n})\llcorner_{g_{\vec{\Phi}}}\,d\vec{\Phi}:=e^{-2\lambda}\ \sum_{i,j=1}^2\partial_{x_i}\vec{n}\cdot\partial_{x_j}\vec{n}\ \partial_{x_j}\vec{\Phi}\ dx_i$$

In conformal coordinates the equation becomes

$$\begin{array}{l}\vec{\Phi}\wedge\operatorname{div}\left[\left[1+\sigma^2f^p\right]\nabla\vec{\Phi}-2\,p\,\sigma^2\ e^{-2\lambda}\ f^{p-1}\ \left\langle(\nabla\vec{n}\dot{\otimes}\nabla\vec{n};\nabla\vec{\Phi}\right\rangle\right.\\[3mm] \left.+2\,p\,\sigma^2\ e^{-2\lambda}\ \operatorname{div}\left[f^{p-1}\,\nabla\vec{n}\right]\cdot\nabla\vec{\Phi}\ \vec{n}\right]-4\,p\,\sigma^2\ f^{p-1}\ \vec{\Phi}\wedge\vec{H}=0\end{array}\tag{II.30}$$

We rewrite the first term in the second line.

$$\begin{array}{l}2\,p\,\sigma^2\ e^{-2\lambda}\operatorname{div}\left[f^{p-1}\,\nabla\vec{n}\right]\cdot\nabla\vec{\Phi}\ \vec{n}\\[3mm] \quad=2\,p\,\sigma^2\ e^{-2\lambda}\operatorname{div}\left[f^{p-1}\ \left[\nabla\vec{n}+\ H\,\nabla\vec{\Phi}\right]\right]\cdot\nabla\vec{\Phi}\ \vec{n}-2\,p\,\sigma^2\ \nabla\left[f^{p-1}\ H\ \right]\ \vec{n}\end{array}\tag{II.31}$$

For any $k=1,2$ we compute

$$\sum_{i=1}^2\partial_{x_i}\left[f^{p-1}\ \left[\partial_{x_i}\vec{n}+H\,\partial_{x_i}\vec{\Phi}\right]\right]\cdot\partial_{x_k}\vec{\Phi}\ \vec{n}=-\partial_{x_k}\left[f^{p-1}\ \mathbb{I}^0_{k,k}\right]\ \vec{n}-\partial_{x_{k+1}}\left[f^{p-1}\ \mathbb{I}^0_{k+1,k}\right]\ \vec{n}$$

Denoting $\overline{\nabla}\cdot:=(\partial_{x_1}\cdot,-\partial_{x_2}\cdot)$ and $(\overline{\nabla})^\perp\cdot:=(\partial_{x_2}\cdot,\partial_{x_1}\cdot)$, we have then

$$2\,p\,\sigma^2\ e^{-2\lambda}\operatorname{div}\left[f^{p-1}\ \left[\nabla\vec{n}+\ H\,\nabla\vec{\Phi}\right]\right]\cdot\nabla\vec{\Phi}\ =-2\,p\,\sigma^2\ e^{-2\lambda}\ \left[\overline{\nabla}\left[f^{p-1}\,\mathbb{I}^0_{11}\right]+(\overline{\nabla})^\perp\left[f^{p-1}\,\mathbb{I}^0_{12}\right]\ \right]\tag{II.32}$$

Combining (II.31) and (II.32) gives

$$\begin{array}{l}2\,p\,\sigma^2\ e^{-2\lambda}\operatorname{div}\left[f^{p-1}\,\nabla\vec{n}\right]\cdot\nabla\vec{\Phi}\ \vec{n}=-2\,p\,\sigma^2\ \nabla\left[f^{p-1}\ \vec{H}\ \right]+\,2\,p\,\sigma^2\ f^{p-1}\ H\ \ \nabla\vec{n}\\[3mm] -\,2\,p\,\sigma^2\ e^{-2\lambda}\ \left[\overline{\nabla}\left[f^{p-1}\,\mathbb{I}^0_{11}\right]+(\overline{\nabla})^\perp\left[f^{p-1}\,\mathbb{I}^0_{12}\right]\right]\ \vec{n}\end{array}\tag{II.33}$$

So the equation (II.30) becomes

$$\begin{array}{l}\vec{\Phi}\wedge\operatorname{div}\left[\left[1+\sigma^2f^p\right]\nabla\vec{\Phi}-2\,p\,\sigma^2\ \nabla\left[f^{p-1}\ \vec{H}\ \right]-2\,p\,\sigma^2\ e^{-2\lambda}\,f^{p-1}\ \left\langle\nabla\vec{n}\dot{\otimes}\nabla\vec{n};\nabla\vec{\Phi}\right\rangle\right.\\[3mm] \left.+\,2\,p\,\sigma^2\ f^{p-1}\ H\ \ \nabla\vec{n}\ -\,2\,p\,\sigma^2\ e^{-2\lambda}\ \left[\overline{\nabla}\left[f^{p-1}\,\mathbb{I}^0_{11}\right]+(\overline{\nabla})^\perp\left[f^{p-1}\,\mathbb{I}^0_{12}\right]\right]\ \vec{n}\right]=4\,p\,\sigma^2\ f^{p-1}\ \vec{\Phi}\wedge\vec{H}\end{array}\tag{II.34}$$

The equation (II.29) is also equivalent to the almost conservation law which holds in $\mathcal{D}'(\Sigma)$ and which is due to the translation invariance of the integrand of F_p in $\mathbb{R}^3$ (see [1])

$$
\begin{array}{l}
-\mathrm{div}\left[\left[1+\sigma^2 f^p\right]\nabla\vec{\Phi}-2\,p\,\sigma^2\,\nabla\left[f^{p-1}\,\vec{H}\,\right]-2\,p\,\sigma^2\,e^{-2\lambda}\,f^{p-1}\,\left\langle\nabla\vec{n}\dot{\otimes}\nabla\vec{n};\nabla\vec{\Phi}\right\rangle\right.\\[2mm]
\left.+2\,p\,\sigma^2\,f^{p-1}\,H\;\;\nabla\vec{n}\;-\;2\,p\,\sigma^2\,e^{-2\lambda}\,\left[\nabla\left[f^{p-1}\,\mathbb{I}^0_{11}\right]+(\nabla)^\perp\left[f^{p-1}\,\mathbb{I}^0_{12}\right]\right]\;\vec{n}\right]+4\,p\,\sigma^2\,f^{p-1}\,\vec{H}\\[2mm]
\quad=\left[1+\sigma^2\;(1-p)\;f^p+p\,\sigma^2 f^{p-1}\right]\;|\nabla\vec{\Phi}|^2\,\vec{\Phi}
\end{array}
\tag{II.35}
$$

Finally we end up this section by quoting the following theorem

Theorem II.6. *Let $p\ge 1$ and $\vec{\Phi}$ be an element in the space $\mathcal{E}_{\Sigma,p}$ of $W^{2,2p}-$immersions of a closed surface Σ. Assume $\vec{\Phi}$ is a critical point of $A^\sigma_p(\vec{\Phi})$ then $\vec{\Phi}$ is C^∞ in any conformal parametrization.* $\Box$

Remark II.3. *A proof of theorem II.6 has been given in [14] without balancing constraints and for C^1 immersions into the euclidian space. The method of proof in [14] relies on the work of J.Langer with the decomposition of the immersion into the union of graphs.* $\Box$

III The passage to the limit $\sigma\rightarrow 0$ with controled conformal class.

The goal of the present section is to prove the following theorem

Theorem III.1. *Let $p>1$ and let $\vec{\Phi}_k$ be a sequence of critical points of $A^{\sigma_k}_p$ in the class $\mathcal{E}_{\Sigma,p}$ where $\sigma_k\rightarrow 0$ and satisfying*

$$
0<\limsup_{k\rightarrow+\infty} Area(\vec{\Phi}_k)<+\infty
\tag{III.1}
$$

and

$$
\sigma_k^2\;F_p(\vec{\Phi}_k)=\sigma_k^2\;\int_\Sigma\left[1+|\mathbb{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}\right]^p\,dvol_{g_{\vec{\Phi}_k}}=o\left(\frac{1}{\log(1/\sigma_k)}\right)\quad .
\tag{III.2}
$$

Assume moreover that the conformal class associated to $(\Sigma,g_{\vec{\Phi}_k})$ is precompact in the moduli space, then, modulo extraction of a subsequence, there exists a closed riemann surface S with genus$(S)\le$genus(Σ) and a conformal target harmonic map $\vec{\Phi}_\infty$ from S into N^n such that

$$
\lim_{k\rightarrow+\infty} A^{\sigma_k}(\vec{\Phi}_k)=Area(\vec{\Phi}_\infty)
$$

and the oriented varifold $|T_k|$ equal to the push-forward by $\vec{\Phi}_k$ of Σ converges in the sense of Radon measures towards the oriented stationary integer varifold associated to $\vec{\Phi}_\infty$. The surface S is moreover either equal to the union of Σ with finitely many copies of S^2 or is equal to finitely many copies of S^2. $\Box$

In order to prove theorem III.1 we shall need several lemma.

Lemma III.1. *Under the assumptions of theorem III.1 the sequence of varifolds $|T_k|$ equal to the push forward of Σ by $\vec{\Phi}_k$ converges, modulo extraction of a subsequence, towards a stationary varifold. In particular, introducing the Radon measure in S^3 given by*

$$
<\mu_k,\varphi>:=\int_\Sigma\varphi(\vec{\Phi}_k)\;dvol_{g_{\vec{\Phi}_k}}
\tag{III.3}
$$

μ_k converges modulo extraction of a subsequence to a limiting Radon measure μ_∞ satisfying the following monotonicity formula

$$\forall\ \vec{q}\in supp(\mu_\infty)\quad \forall\, r>0\qquad \frac{d}{dr}\left[\frac{e^{C\,r}\,\mu_\infty(B_r(\vec{q}))}{r^2}\right]\ge 0\quad . \tag{III.4}$$

for some $C>0$ independent of $\vec{q}$ and r. □

Proof of lemma III.1. The monotonicity formula for the limiting measure μ_∞ is a direct consequence of the fact that $|T_k|$ converges towards a stationary varifold (see [2] and [33]). So it would suffices to prove this last fact in order to get (III.4). However the proof of both statements (that can be proven independently of each other) are very similar. In the first case it suffices to prove that for any vectorfield $\vec{X}$ we have

$$\lim_{k\to+\infty}\int_{M_k}\mathrm{div}_{M_k}\vec{X}\ d\mathcal{H}^2=\lim_{k\to+\infty}\int_\Sigma\left[\sum_{i=1}^4\left\langle\partial_{y_i}\vec{X}(\vec{\Phi}_k)\ \nabla\Phi^i_k,\nabla\vec{\Phi}_k\right\rangle-\vec{X}(\vec{\Phi}_k)\cdot\vec{\Phi}_k\ |\nabla\vec{\Phi}_k|^2\right]\ dx^2=0 \tag{III.5}$$

where $M_k:=\vec{\Phi}_k(\Sigma)$ and $\vec{\Phi}_k=(\Phi^1_k,\cdots,\Phi^4_k)$. The computations for proving (III.5) are more or less the same as the one for proving (III.4) and we shall only present the later since we shall revisit them in the forthcoming lemma III.2.

We omit to write explicitly the σ_k and k indices when there is no possible confusion. For any $\vec{q}\in S^3$ and any radius r small enough, *Simon's monotonicity formula* (see [33] chapter 4) applied to $\vec{\Phi}(\Sigma)$ (which is a smooth immersion for any k) that we see as a *varifold* from $\mathbb{R}^4$ gives

$$\begin{aligned}\frac{d}{dr}\left[\frac{1}{r^2}\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}dvol_{g_{\vec{\Phi}}}\right]&=\frac{d}{dr}\left[\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}\frac{|(\vec{n}\wedge\vec{\Phi})\llcorner(\vec{\Phi}-\vec{q})|^2}{|\vec{\Phi}-\vec{q}|^4}\ dvol_{g_{\vec{\Phi}}}\right]\\ &-\frac{1}{2\,r^3}\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot d^{*_g}d\vec{\Phi}\ dvol_{g_{\vec{\Phi}}}\\ &\ge-\frac{1}{2\,r^3}\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot d^{*_g}d\vec{\Phi}\ dvol_{g_{\vec{\Phi}}}\end{aligned} \tag{III.6}$$

Using the form (II.35) of the equation (II.35) we obtain

$$\begin{aligned}&-\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot d^{*_g}d\vec{\Phi}\ dvol_{g_{\vec{\Phi}}}=\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\Delta\vec{\Phi}\ dx^2\\ &=-\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\mathrm{div}\left[\sigma^2\ f^{p-1}\ [f\,\nabla\vec{\Phi}-2\,p\ \left(H\ \nabla\vec{n}-e^{-2\lambda}<\nabla\vec{n}\dot{\otimes}\nabla\vec{n};\nabla\vec{\Phi}>\right)]\right]\\ &+2\ p\ \sigma^2\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\mathrm{div}\left[e^{-2\lambda}\ \left[\nabla\left[f^{p-1}\,\mathbb{I}^0_{11}\right]+(\nabla)^\perp\left[f^{p-1}\,\mathbb{I}^0_{12}\right]\ \right]\vec{n}\right]\ dx^2\\ &+2\ p\ \sigma^2\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\Delta\left[f^{p-1}\ \vec{H}\right]\ dx^2\\ &-\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}\left[1+(1-p)\ \sigma^2\ f^p+p\,\sigma^2\ f^{p-1}\right]\ (\vec{\Phi}-\vec{q})\cdot\vec{\Phi}\,|\nabla\vec{\Phi}|^2\ dx^2\end{aligned} \tag{III.7}$$

Regarding the second to last line observe in one hand that $(\vec{\Phi}-\vec{q})\cdot\vec{\Phi}=1-\cos(\vec{\Phi},\vec{q})\ =O(r^2)$ hence

$$\left|\frac{1}{r^3}\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\vec{\Phi}\,|\nabla\vec{\Phi}|^2\ dx^2\right|\le\frac{C}{r}\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}dvol_{g_{\vec{\Phi}}} \tag{III.8}$$

and in the other hand, again for fixed r and $\vec{q}$, as $k\rightarrow+\infty$

$$
\begin{aligned}
&\left|\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}\left[(1-p)\,\sigma^2\ f^p+p\,\sigma^2\ f^{p-1}\right]\ (\vec{\Phi}-\vec{q})\cdot\vec{\Phi}\,|\nabla\vec{\Phi}|^2\ dx^2\right|\\
&\qquad\le C\ \sigma^2\ F_p(\vec{\Phi})+C\ \sigma^2\ M(T)^{1/p}\ [F_p(\vec{\Phi})]^{1-1/p}\quad\rightarrow 0
\end{aligned}
\tag{III.9}
$$

Integrating by parts each of the two first lines in the r.h.s. of (III.7) gives

$$
\begin{aligned}
&-\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\mathrm{div}\left[\sigma^2\ f^{p-1}\ [f\,\nabla\vec{\Phi}-2\,p\ \left(H\ \nabla\vec{n}-e^{-2\lambda}<\nabla\vec{n}\dot{\otimes}\nabla\vec{n};\nabla\vec{\Phi}>\right)]\right]\\
&+2\ p\ \sigma^2\ \int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\mathrm{div}\left[e^{-2\lambda}\ \left[\overline{\nabla}\left[f^{p-1}\,\mathbb{I}^0_{11}\right]+(\overline{\nabla})^{\perp}\left[f^{p-1}\,\mathbb{I}^0_{12}\right]\ \right]\vec{n}\right]\ dx^2\\
&=\sigma^2\ \int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}f^{p-1}\ \left[f\,|\nabla\vec{\Phi}|^2-2\,p\ H\ \nabla\vec{n}\cdot\nabla\vec{\Phi}+2\,p\,(f-1)\,e^{2\lambda}\right]\ dx^2\\
&-\sigma^2\ \int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}f^p\ \partial_\nu\vec{\Phi}\cdot(\vec{\Phi}-\vec{q})-\ 2\,p\ f^{p-1}\ H\ \partial_\nu\vec{n}\cdot(\vec{\Phi}-\vec{q})dl\\
&+\ 2\ p\ \sigma^2\ \int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}f^{p-1}\ \left\langle\partial_\nu\vec{n}\cdot\nabla\vec{n},\nabla\vec{\Phi}\cdot(\vec{\Phi}-\vec{q})\right\rangle\ dl\\
&+\ 2\ p\ \sigma^2\ \int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}e^{-2\lambda}\ (\vec{\Phi}-\vec{q})\cdot\vec{n}\left[\nu_1\ \partial_{x_1}\left[f^{p-1}\,\mathbb{I}^0_{11}\right]-\nu_2\ \partial_{x_2}\left[f^{p-1}\,\mathbb{I}^0_{11}\right]\right]\ dl\\
&+\ 2\ p\ \sigma^2\ \int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}e^{-2\lambda}\ (\vec{\Phi}-\vec{q})\cdot\vec{n}\left[\nu_1\ \partial_{x_2}\left[f^{p-1}\,\mathbb{I}^0_{12}\right]+\nu_2\ \partial_{x_1}\left[f^{p-1}\,\mathbb{I}^0_{12}\right]\right]\ dl
\end{aligned}
\tag{III.10}
$$

Where ν is the outward unit (in the coordinates) normal to $\vec{\Phi}^{-1}(B^4_r(\vec{q}))$ and is given explicitly by

$$
\nu=(\partial_{x_1}|\vec{\Phi}-\vec{q}|\,,\partial_{x_2}|\vec{\Phi}-\vec{q}|)/|\nabla|\vec{\Phi}-\vec{q}||
$$

This is nothing but the normalized gradient of the function distance to $\vec{q}$. We clearly have

$$
\lim_{k\rightarrow+\infty}\sigma^2\ \int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}f^{p-1}\ \left[f\,|\nabla\vec{\Phi}|^2-2\,p\ H\ \nabla\vec{n}\cdot\nabla\vec{\Phi}+\ 2\,p\,e^{2\lambda}\,(f-1)\right]\ dx^2=0
\tag{III.11}
$$

Multiplying (III.10) by an arbitrary compactly supported function $\chi(r)/r^3$ in $\mathbb{R}^*_+$ and integrating over $\mathbb{R}^*_+$ gives successively

$$
\begin{aligned}
&\sigma^2\ \int_{\mathbb{R}_+}\chi(r)\ \frac{dr}{r^3}\int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}\left[f^p\ \partial_\nu\vec{\Phi}\cdot(\vec{\Phi}-\vec{q})-2\,p\ f^{p-1}\ H\ \partial_\nu\vec{n}\cdot(\vec{\Phi}-\vec{q})\right]\ dl\\
&+2\,p\,\sigma^2\ \int_{\mathbb{R}_+}\chi(r)\ \frac{dr}{r^3}\int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}f^{p-1}\ \left\langle\partial_\nu\vec{n}\cdot\nabla\vec{n},\nabla\vec{\Phi}\cdot(\vec{\Phi}-\vec{q})\right\rangle\ dl\\
&=\sigma^2\ \int_\Sigma\chi(|\vec{\Phi}-\vec{q}|)\ \left[f^p\ \frac{|\nabla|\vec{\Phi}-\vec{q}||^2}{|\vec{\Phi}-\vec{q}|^2}-2\,p\ f^{p-1}\ H\frac{\nabla|\vec{\Phi}-\vec{q}|}{|\vec{\Phi}-\vec{q}|^3}\cdot\left\langle\nabla\vec{n}\cdot(\vec{\Phi}-\vec{q})\right\rangle\right]\ dx^2\\
&+\ 2\,p\,\sigma^2\ \int_\Sigma\chi(|\vec{\Phi}-\vec{q}|)\ f^{p-1}\ \left\langle\frac{\nabla|\vec{\Phi}-\vec{q}|}{|\vec{\Phi}-\vec{q}|^3}\cdot\nabla\vec{n},\nabla\vec{n},\nabla\vec{\Phi}\cdot(\vec{\Phi}-\vec{q})\right\rangle\ dx^2\\
&\qquad\longrightarrow 0\quad\text{ as }k\rightarrow+\infty
\end{aligned}
\tag{III.12}
$$

where we have bound the r.h.s. of (III.12) by a constant depending on χ times $\sigma^2 F_p(\vec{\Phi})$. We also obtain

$$
\begin{aligned}
&- 2\, p\, \sigma^2 \int_{\mathbb{R}_+} \chi(r)\, \frac{dr}{r^3} \int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))} e^{-2\lambda}\, (\vec{\Phi}-\vec{q})\cdot\vec{n}\, \left[\nu_1\, \partial_{x_1}\left[f^{p-1}\, \mathbb{I}^0_{11}\right]\right]\, dl\\
&+ 2\, p\, \sigma^2 \int_{\mathbb{R}_+} \chi(r)\, \frac{dr}{r^3} \int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))} e^{-2\lambda}\, (\vec{\Phi}-\vec{q})\cdot\vec{n}\, \left[\nu_2\, \partial_{x_2}\left[f^{p-1}\, \mathbb{I}^0_{11}\right]\right]\, dl\\
&= -\, p\, \sigma^2 \int_\Sigma \chi(|\vec{\Phi}-\vec{q}|)\, \frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\vec{n}\, \left[e^{-2\lambda}\, \partial_{x_1}|\vec{\Phi}-\vec{q}|^2\, \partial_{x_1}\left[f^{p-1}\, \mathbb{I}^0_{11}\right]\right]\, dx^2\\
&\quad + p\, \sigma^2 \int_\Sigma \chi(|\vec{\Phi}-\vec{q}|)\, \frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\vec{n}\, \left[e^{-2\lambda}\, \partial_{x_2}|\vec{\Phi}-\vec{q}|^2\, \partial_{x_2}\left[f^{p-1}\, \mathbb{I}^0_{11}\right]\right]\, dx^2
\end{aligned}
\tag{III.13}
$$

Integrating by parts the r.h.s of (III.13), we have using again (III.15)

$$
\begin{aligned}
&-\, p\, \sigma^2 \int_\Sigma \chi(|\vec{\Phi}-\vec{q}|)\, \frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\vec{n}\, \left[e^{-2\lambda}\, \partial_{x_1}|\vec{\Phi}-\vec{q}|^2\, \partial_{x_1}\left[f^{p-1}\, \mathbb{I}^0_{11}\right]\right]\\
&\quad + p\, \sigma^2 \int_\Sigma \chi(|\vec{\Phi}-\vec{q}|)\, \frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\vec{n}\, \left[e^{-2\lambda}\, \partial_{x_2}|\vec{\Phi}-\vec{q}|^2\, \partial_{x_2}\left[f^{p-1}\, \mathbb{I}^0_{11}\right]\right]\\
&\quad = \, p\, \sigma^2 \int_\Sigma f^{p-1}\, \mathbb{I}^0_{11}\, \overline{\nabla}\left[\chi(|\vec{\Phi}-\vec{q}|)\, \frac{(\vec{\Phi}-\vec{q})\cdot\vec{n}}{|\vec{\Phi}-\vec{q}|^4}\right] e^{-2\lambda}\, \nabla|\vec{\Phi}-\vec{q}|^2\, dx^2\\
&\quad + p\, \sigma^2 \int_\Sigma f^{p-1}\, \mathbb{I}^0_{11}\cdot\chi(|\vec{\Phi}-\vec{q}|)\, \frac{(\vec{\Phi}-\vec{q})\cdot\vec{n}}{|\vec{\Phi}-\vec{q}|^4}\, \overline{\nabla}\left[e^{-2\lambda}\, \nabla|\vec{\Phi}-\vec{q}|^2\right]\, dx^2
\end{aligned}
\tag{III.14}
$$

We recall that we have respectively

$$
\overline{\nabla}\cdot\left(e^{-2\lambda}\, \nabla\vec{\Phi}\right) = 2\, e^{-2\lambda}\, \vec{\mathbb{I}}^0_{11} \quad \text{and} \quad (\overline{\nabla})^\perp\cdot\left(e^{-2\lambda}\, \nabla\vec{\Phi}\right) = 2\, e^{-2\lambda}\, \vec{\mathbb{I}}^0_{12} \quad . \tag{III.15}
$$

combining these identities with the fact that $\vec{\Phi}$ is conformal we deduce that

$$
\begin{aligned}
\overline{\nabla}\left[e^{-2\lambda}\, \nabla|\vec{\Phi}-\vec{q}|^2\right] &= 2\, \overline{\nabla}\left[e^{-2\lambda}\, \nabla(\vec{\Phi}-\vec{q})\right]\cdot(\vec{\Phi}-\vec{q}) + 2\, e^{-2\lambda}\, \nabla(\vec{\Phi}-\vec{q})\cdot\overline{\nabla}(\vec{\Phi}-\vec{q})\\
&= 4\, e^{-2\lambda}\, \vec{\mathbb{I}}^0_{11}\cdot(\vec{\Phi}-\vec{q})
\end{aligned}
\tag{III.16}
$$

Combining (III.14) and (III.16) we deduce

$$
\begin{aligned}
&\Bigg|-\, p\, \sigma^2 \int_\Sigma \chi(|\vec{\Phi}-\vec{q}|)\, \frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\left[e^{-2\lambda}\, \partial_{x_1}|\vec{\Phi}-\vec{q}|^2\, \partial_{x_1}\left[f^{p-1}\, \vec{\mathbb{I}}^0_{11}\right]\right]\\
&\quad + p\, \sigma^2 \int_\Sigma \chi(|\vec{\Phi}-\vec{q}|)\, \frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\left[e^{-2\lambda}\, \partial_{x_2}|\vec{\Phi}-\vec{q}|^2\, \partial_{x_2}\left[f^{p-1}\, \vec{\mathbb{I}}^0_{11}\right]\right]\Bigg|\\
&\quad \le C\, \sigma^2\, F_p(\vec{\Phi}) + C\, \sigma^2\, M(T)^{1/p}\, [F_p(\vec{\Phi})]^{1-1/p} \quad \to 0
\end{aligned}
\tag{III.17}
$$

So finally deduce that for any χ compactly supported in $\mathbb{R}_+^*$ we have

$$-\int_{\mathbb{R}_+}\chi(r)\ \frac{dr}{r^3}\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\mathrm{div}\left[\sigma^2\ f^{p-1}\ [f\,\nabla\vec{\Phi}-\ 2\,p\ \left(H\ \nabla\vec{n}-e^{-2\lambda}<\nabla\vec{n}\dot{\otimes}\nabla\vec{n};\nabla\vec{\Phi}>\right)]\right]\ dx^2$$

$$+2\ p\ \sigma^2\int_{\mathbb{R}_+}\chi(r)\ \frac{dr}{r^3}\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\mathrm{div}\left[e^{-2\lambda}\ \ \left[\nabla\left[f^{p-1}\,\mathbb{I}^0_{11}\right]+(\overline{\nabla})^\perp\left[f^{p-1}\,\mathbb{I}^0_{12}\right]\ \right]\vec{n}\right]\ dx^2$$

$$\to 0 \tag{III.18}$$

It remains to bound

$$\begin{aligned}
&-\int_{\mathbb{R}_+}\chi(r)\ \frac{dr}{r^3}\sigma^2\ \int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\Delta\left[f^{p-1}\ \vec{H}\right]\ dx^2\\
&\quad=\int_{\mathbb{R}_+}\chi(r)\ \frac{dr}{r^3}\sigma^2\ \int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}\nabla(\vec{\Phi}-\vec{q})\cdot\nabla\left[f^{p-1}\ \vec{H}\right]\ dx^2\\
&\quad-\int_{\mathbb{R}_+}\chi(r)\ \frac{dr}{r^3}\sigma^2\ \int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\partial_\nu\left[f^{p-1}\ \vec{H}\right]\ dl
\end{aligned}\tag{III.19}$$

The last integral in the r.h.s. of (III.19) is equal to

$$\begin{aligned}
&-\int_{\mathbb{R}_+}\chi(r)\ \frac{dr}{r^3}\sigma^2\ \int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\partial_\nu\left[f^{p-1}\ \vec{H}\right]\ dl\\
&\quad=-\sigma^2\ \int_\Sigma\chi(|\vec{\Phi}-\vec{q}|)\ \nabla|\vec{\Phi}-\vec{q}|\cdot\left\langle\nabla\left[f^{p-1}\vec{H}\right],\frac{\vec{\Phi}-\vec{q}}{|\vec{\Phi}-\vec{q}|^3}\right\rangle\ dx^2
\end{aligned}\tag{III.20}$$

We integrate by parts and we observe that in the domain where $\chi(|\vec{\Phi}-\vec{q}|)\neq 0$ we have

$$\Delta|\vec{\Phi}-\vec{q}|=\frac{(\vec{\Phi}-\vec{q})\cdot\Delta\vec{\Phi}}{|\vec{\Phi}-\vec{q}|}+\frac{|\nabla\vec{\Phi}|^2}{|\vec{\Phi}-\vec{q}|}-\frac{|\nabla|\vec{\Phi}-\vec{q}||^2}{|\vec{\Phi}-\vec{q}|}$$

and using the fact that $\Delta\vec{\Phi}=-\vec{\Phi}\,|\nabla\vec{\Phi}|^2+\vec{H}\ |\nabla\vec{\Phi}|^2$ we finally obtain

$$\Delta|\vec{\Phi}-\vec{q}|=-\frac{1-\vec{q}\cdot\vec{\Phi}}{|\vec{\Phi}-\vec{q}|}+\frac{(\vec{\Phi}-\vec{q})\cdot\vec{H}\,|\nabla\vec{\Phi}|^2}{|\vec{\Phi}-\vec{q}|}+\frac{|\nabla\vec{\Phi}|^2}{|\vec{\Phi}-\vec{q}|}-\frac{|\nabla|\vec{\Phi}-\vec{q}||^2}{|\vec{\Phi}-\vec{q}|}\tag{III.21}$$

Hence combining (III.20) and (III.21) we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}_+}\chi(r)\ \frac{dr}{r^3}\sigma^2\ \int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\partial_\nu\left[f^{p-1}\ \vec{H}\right]\ dl\right|\\
&\qquad\le C_\chi\ \sigma^2\ F_p(\vec{\Phi})+C_\chi\ \sigma^2\ M(T)^{1/p}\ [F_p(\vec{\Phi})]^{1-1/p}\quad\to 0
\end{aligned}\tag{III.22}$$

Taking now the first integral in the r.h.s. of (III.19) we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\chi(r)\ \frac{dr}{r^3}\sigma^2\ \int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}\nabla(\vec{\Phi}-\vec{q})\cdot\nabla\left[f^{p-1}\ \vec{H}\right]\ dx^2\\
&\qquad=-\int_{\mathbb{R}_+}\chi(r)\ \frac{dr}{r^3}\sigma^2\ \int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}\Delta\vec{\Phi}\cdot f^{p-1}\ \vec{H}\ dx^2\\
&\qquad+\sigma^2\ \int_\Sigma\chi(|\vec{\Phi}-\vec{q}|)\ \nabla|\vec{\Phi}-\vec{q}|\cdot\left\langle\nabla(\vec{\Phi}-\vec{q}),f^{p-1}\ \vec{H}\right\rangle\ dx^2
\end{aligned}\tag{III.23}$$

So we have also

$$
\begin{aligned}
&\left|\int_{\mathbb{R}_+} \chi(r)\ \frac{dr}{r^3}\sigma^2\ \int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))} \nabla(\vec{\Phi}-\vec{q})\cdot\nabla\left[f^{p-1}\ \vec{H}\right]\ dx^2\right| \\
&\qquad \le C_\chi\ \sigma^2\ F_p(\vec{\Phi})+C_\chi\ \sigma^2\ M(T)^{1/p}\ [F_p(\vec{\Phi})]^{1-1/p} \quad \to 0
\end{aligned}
\tag{III.24}
$$

Combining (III.19) and (III.22) and (III.24) we have

$$
\left|\int_{\mathbb{R}_+} \chi(r)\ \frac{dr}{r^3}\sigma^2\ \int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))} (\vec{\Phi}-\vec{q})\cdot\Delta\left[f^{p-1}\ \vec{H}\right]\ dx^2\right|\to 0
\tag{III.25}
$$

Combining now (III.7), (III.8), (III.9), (III.18) and (III.25) we have that for any fixed non negative $\chi(r)$ compactly supported in $\mathbb{R}^\ast_+$ and any $\vec{q}\in\mathbb{R}^4$

$$
\begin{aligned}
&-\int_0^\infty \chi'(r)\ dr\ \frac{1}{r^2}\int_{\vec{\Phi}_k^{-1}(B^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}_k}} \ge \\
&\quad -C\ \int_0^\infty \chi(r)\ dr\ \frac{1}{r}\int_{\vec{\Phi}_k^{-1}(B^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}_k}}+o_k(1)
\end{aligned}
\tag{III.26}
$$

Taking μ_k the Radon measure on $\mathbb{R}^4$ given by (III.3) this can be rewritten as

$$
-\int_0^\infty \chi'(r)\ dr\ \frac{1}{r^2}\mu_k(B^4_r(\vec{q}))\ge -C\ \int_0^\infty \chi(r)\ dr\ \frac{1}{r}\mu_k(B^4_r(\vec{q}))+o_k(1)
$$

We extract a subsequence such that μ_k converges weakly in Radon measure and we finally obtain that for any fixed non negative $\chi(r)$ compactly supported in $\mathbb{R}^\ast_+$ and any $\vec{q}\in\mathbb{R}^4$

$$
-\int_0^\infty \chi'(r)\ dr\ \frac{1}{r^2}\mu_\infty(B^4_r(\vec{q}))\ge -C\ \int_0^\infty \chi(r)\ dr\ \frac{1}{r}\mu_\infty(B^4_r(\vec{q}))
$$

which classically implies (III.4) and lemma III.1 is proved. □

The next result establishes a uniform lower bound of the limiting area for any sequence of immersions satisfying the assumptions of theorem III.1. This result is the "work-horse" in our proof of the main theorem and shall be used crucially at several steps. Precisely we have the following result

Lemma III.2. [Global Energy Quantization] *There exists $Q_0>0$ such that the following holds. Let Σ be a closed surface. Let $p>1$ and let $\vec{\Phi}_k$ be a sequence of critical points of $A^{\sigma_k}_p$ in the class $\mathcal{E}^0_{\Sigma,p}$ where $\sigma_k\to 0$ and satisfying*

$$
\limsup_{k\to+\infty} Area(\vec{\Phi}_k)<+\infty
\tag{III.27}
$$

and

$$
\sigma_k^2\ F_p(\vec{\Phi}_k)=\sigma_k^2\ \int_\Sigma [1+|\mathbb{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}]^p\ dvol_{g_{\vec{\Phi}_k}}=o\left(\frac{1}{\log(1/\sigma_k)}\right)\ Area(\vec{\Phi}_k) \quad .
\tag{III.28}
$$

then,

$$
\liminf_{k\to+\infty} Area(\vec{\Phi}_k)\ge Q_0>0
\tag{III.29}
$$

Proof of lemma III.2. We denote as usual

$$
f(\sigma_k)=\frac{\sigma_k^2\ F_p(\vec{\Phi}_k)}{\mathrm{Area}(\vec{\Phi}_k)}
$$

Let $\eta > 0$ to be fixed later. We omit to write the subscript k. For any $\vec{q} \in \vec{\Phi}(\Sigma)$ we consider the 4-dimensional ball in $\mathbb{R}^4$, $B^4_\sigma(\vec{q})$ centered at $\vec{q}$ with radius σ. We consider the subset E_η of $\vec{\Phi}(\Sigma)$ given by

$$E_\eta := \left\{ \vec{q} \in \vec{\Phi}(\Sigma) \subset S^3 \quad ; \quad \sigma^{-2} \int_{B^4_\sigma(\vec{q}) \cap \vec{\Phi}_k(\Sigma)} dvol_{g_{\vec{\Phi}}} < \eta \right\}$$

From the covering $(B^4_\sigma(\vec{q}))_{\vec{q} \in E_\eta}$ we extract a Besicovitch sub-covering $(B^4_\sigma(\vec{q}_i))_{i \in I}$ such that each point in $\mathbb{R}^4$ is covered by at most N balls where N is a universal number. Using Simon's monotonicity formula we have for each $i \in I$ (see [27])

$$\sigma^{-2} \int_{B^4_\sigma(\vec{q}) \cap \vec{\Phi}_k(\Sigma)} dvol_{g_{\vec{\Phi}}} \geq \frac{2\pi}{3} - \frac{1}{2} \int_{B^4_\sigma(\vec{q}_i)} |\vec{H}^{\mathbb{R}^4}_{\vec{\Phi}}|^2 \, dvol_{g_{\vec{\Phi}}} \quad . \tag{III.30}$$

Considering $\eta = \pi/3$ this imposes

$$\int_{B^4_\sigma(\vec{q}_i)} |\vec{H}^{\mathbb{R}^4}_{\vec{\Phi}}|^2 \, dvol_{g_{\vec{\Phi}}} > \frac{2\,\pi}{3} \quad . \tag{III.31}$$

Hence

$$\int_{\cup_{i \in I} B^4_\sigma(\vec{q}_i)} |\vec{H}^{\mathbb{R}^4}_{\vec{\Phi}}|^2 \, dvol_{g_{\vec{\Phi}}} \geq \frac{1}{N} \sum_{i \in I} \int_{B^4_\sigma(\vec{q}_i)} |\vec{H}^{\mathbb{R}^4}_{\vec{\Phi}}|^2 \, dvol_{g_{\vec{\Phi}}} \geq \frac{2\,\pi}{3\,N} \, \mathrm{card} I \tag{III.32}$$

Combining (III.28) and (III.32) we obtain

$$\sigma^2 \, \frac{2\,\pi}{3\,N} \, \mathrm{card} I \leq f(\sigma) \, \mathrm{Area}(\vec{\Phi}) \tag{III.33}$$

So we have

$$\int_{E_{\pi/3}} dvol_{g_{\vec{\Phi}}} \leq \int_{\cup_{i \in I} B^4_\sigma(\vec{q}_i)} dvol_{g_{\vec{\Phi}}} \leq \frac{\pi}{3} \, \sigma^2 \, \mathrm{card} I \leq f(\sigma) \, \mathrm{Area}(\vec{\Phi}) \tag{III.34}$$

Let $1 > \delta > 0$ to be fixed later . Consider now for $j \in \{1, 2 \cdots \log_2 \sigma^{-1}\}$. We use the notation

$$A(j, \vec{q}) := \int_{B^4_{2^j\sigma}(\vec{q}) \cap \vec{\Phi}(\Sigma)} dvol_{g_{\vec{\Phi}}} \quad \text{and} \quad F(j, \vec{q}) := \sigma^2 \int_{B^4_{2^j\,\sigma}(\vec{q}) \cap \vec{\Phi}(\Sigma)} |\mathbb{I}_{\vec{\Phi}}|^{2p}_{g_{\vec{\Phi}}} \, dvol_{g_{\vec{\Phi}}}$$

$$G^j_\delta := \left\{ \begin{array}{l} \vec{q} \in \vec{\Phi}(\Sigma) \setminus E_{\pi/3} \ ; \ \dfrac{(2^{-2\,j} \ A(j+1, \vec{q}))^{1/p} \ F(j+1, \vec{q})^{1-1/p} + F(j, \vec{q})}{A(j, \vec{q})} \geq \dfrac{f(\sigma)}{\delta} \\ \text{and} \quad A(j+1, \vec{q}) \leq 3\,\pi\, 2^{2j+2} \, \sigma^2 \end{array} \right\}$$

For each $j \in \{1, 2, \cdots, \log_2 \sigma^{-1} - 1\}$ we consider $(B^4_{2^j\sigma}(\vec{q}))_{\vec{q} \in F^j_\delta}$ and we extract from each of these coverings a Vitali sub-covering $(B^4_{2^j\sigma}(\vec{q}_i))_{i \in I_j}$ of G^j_δ such that each point in $\mathbb{R}^4$ is covered by at most $\mathfrak{N}$ balls $B^4_{2^{j+1}\sigma}(\vec{q}_i)$ of double radius where $\mathfrak{N}$ is a universal number. From the union of these Vitali coverings $(B^4_{2^j\sigma}(\vec{q}_i))_{i \in I_j}$ which contains

$$G_\delta := \cup_{j \in \{1,2 \cdots \log_2 \sigma^{-1} - 1\}} G^j_\delta$$

we extract a Besicovitch sub covering that we denote $(B^4_{2^{j_i}\sigma}(\vec{q}_i))_{i \in I}$. For such a covering each point of $\mathbb{R}^4$ is covered by at most N balls $B^4_{2^{j_i}\sigma}(\vec{q}_i)$, moreover we have for any $\alpha > 0$ that

$$\left\| \sum_{i \in I} \mathbf{1}_{B^4_{2^{j_i+1}\sigma}(\vec{q}_i)} \, 2^{\alpha j_i} \right\|_{L^\infty(\mathbb{R}^4)} \leq C \, \mathfrak{N} \sum_{j=0}^{\log_2 \sigma^{-1}} 2^{\alpha\, j} \leq C \, \mathfrak{N} \, \sigma^{-\alpha} \quad , \tag{III.35}$$

where $\mathbf{1}_{B^4_{2^{j_i+1}\sigma}(\vec{q}_i)}$ is the characteristic function of the ball $B^4_{2^{j_i+1}\sigma}(\vec{q}_i)$. For any $j\in\{1,2\cdots\log_2\sigma^{-1}\}$ and $\vec{q}\in G^j_\delta$, the whole support of $\vec{\Phi}(\Sigma)$ cannot be included in $B^4_{2^j}(\vec{q})$ otherwise we would contradict the limiting monotonicity formula (III.4) for σ small enough. Hence, since $\vec{q}\in\vec{\Phi}(\vec{\Sigma})$ for any radius $r\in(2^j\sigma,2^{j+1}\sigma)$ we have $\vec{\Phi}(\Sigma)\cap\partial B_r(\vec{q})\neq\emptyset$ and we can apply lemma A.1. Hence we deduce

$$0<\varepsilon_0(4)<\int_{B^4_{2^{j+1}\sigma}(\vec{q})}|\vec{\mathbb{I}}^{\mathbb{R}^4}_{\vec{\Phi}}|^2\,dvol_{g_{\vec{\Phi}}}\tag{III.36}$$

Since $A(j+1,\vec{q})\le 3\,\pi\,2^{2j+2}\,\sigma^2$ inequality (III.36) implies

$$\begin{aligned}\frac{A(j+1,\vec{q})}{2^{2\,j+2}}&\le\frac{3\,\pi\,\sigma^2}{\varepsilon_0(4)}\int_{B^4_{2^{j+1}\sigma}(\vec{q})}|\vec{\mathbb{I}}^{\mathbb{R}^4}_{\vec{\Phi}}|^2\,dvol_{g_{\vec{\Phi}}}\\&\le\frac{3\,\pi\,\sigma^2}{\varepsilon_0(4)}\,A(j+1,\vec{q})^{1-1/p}\left(\int_{B^4_{2^{j+1}\sigma}(\vec{q})}[1+|\mathbb{I}_{\vec{\Phi}}|^2]^p\,dvol_{g_{\vec{\Phi}}}\right)^{1/p}\end{aligned}\tag{III.37}$$

and we deduce that

$$\frac{A(j+1,\vec{q})}{2^{2\,j}}\le C\,(2^{j+1}\,\sigma)^{2p-2}[F(j+1,\vec{q})+\sigma^2\,A(j+1,\vec{q})]\tag{III.38}$$

So for $\vec{q}\in G^j_\delta$ we have

$$\frac{f(\sigma)}{\delta}\,A(j,\vec{q})\le F(j,\vec{q})+\,C\,(2^{j+1}\,\sigma)^{2-2/p}\,F(j+1,\vec{q})\tag{III.39}$$

summing this identity with respect to $i\in I$ we obtain

$$\begin{aligned}\frac{f(\sigma)}{\delta}\int_{G_\delta}dvol_{g_{\vec{\Phi}}}&\le\frac{f(\sigma)}{\delta}\sum_{i\in I}\int_{B^4_{2^{j_i}\sigma}(\vec{q}_i)}dvol_{g_{\vec{\Phi}}}\\&\le\sum_{i\in I}\sigma^2\int_{B^4_{2^{j_i}\sigma}(\vec{q}_i)}|\mathbb{I}_{\vec{\Phi}}|^{2p}_{g_{\vec{\Phi}}}\,dvol_{g_{\vec{\Phi}}}+\sigma^2\int_\Sigma\sum_{i\in I}\mathbf{1}_{B^4_{2^{j_i+1}\sigma}(\vec{q}_i)}\,2^{\alpha j_i}\,\sigma^\alpha|\mathbb{I}_{\vec{\Phi}}|^{2p}_{g_{\vec{\Phi}}}\,dvol_{g_{\vec{\Phi}}}\end{aligned}\tag{III.40}$$

where $\alpha:=2-2p$. Using (III.35), we then deduce

$$\frac{f(\sigma)}{\delta}\int_{G_\delta}dvol_{g_{\vec{\Phi}}}\le C\,\sigma^2\int_\Sigma|\mathbb{I}_{\vec{\Phi}}|^{2p}_{g_{\vec{\Phi}}}\,dvol_{g_{\vec{\Phi}}}=C\,f(\sigma)\int_\Sigma dvol_{g_{\vec{\Phi}}}\quad.\tag{III.41}$$

We deduce from (III.34) and (III.41)

$$\int_{E_{\pi/3}\cup F_\delta}dvol_{g_{\vec{\Phi}}}\le(C\,\delta+f(\sigma))\int_\Sigma dvol_{g_{\vec{\Phi}}}\tag{III.42}$$

Since $f(\sigma)\to 0$ as $\sigma\to 0$, by taking any $0<\delta<1/N$ we have that for σ small enough $\vec{\Phi}(\Sigma)\setminus(E_{\pi/3}\cup F_\delta)\neq\emptyset$. Let now $\vec{q}\in\vec{\Phi}(\Sigma)\setminus(E_{\pi/3}\cup F_\delta)$ we then have the existence of $j_0=j(\vec{q})$

$$\left\{\begin{array}{l}\displaystyle\int_{B^4_{2^{j_0}\sigma}(\vec{q})}dvol_{g_{\vec{\Phi}}}\ge 2^{2\,j_0}\sigma^2\,\pi/3\qquad\text{and}\\[3mm]\displaystyle\forall j\ge j_0\qquad\frac{f(\sigma)}{\eta}\int_{B^4_{2^j\sigma}(\vec{q})\cap\vec{\Phi}(\Sigma)}dvol_{g_{\vec{\Phi}}}\ge\sigma^2\int_{B^4_{2^j\sigma}(\vec{q})\cap\vec{\Phi}(\Sigma)}|\mathbb{I}_{\vec{\Phi}}|^{2p}_{g_{\vec{\Phi}}}\,dvol_{g_{\vec{\Phi}}}\\[3mm]\displaystyle\quad+\left[\sigma^2\int_{B^4_{2^{j+1}\sigma}(\vec{q})\cap\vec{\Phi}(\Sigma)}|\mathbb{I}_{\vec{\Phi}}|^{2p}_{g_{\vec{\Phi}}}\,dvol_{g_{\vec{\Phi}}}\right]^{1-1/p}\left[2^{-2\,j}\int_{B^4_{2^{j+1}\sigma}(\vec{q})\cap\vec{\Phi}(\Sigma)}dvol_{g_{\vec{\Phi}}}\right]^{1/p}\end{array}\right.\tag{III.43}$$

Let $j\in\{j_0,\cdots,\log_2\sigma^{-1}-1\}$ and let χ be an arbitrary smooth function, bounded by 1, supported in $[2^{j-2}\sigma,2^{j+1}\sigma]$ and such that $|\chi'|\le\ C\ 2^{-j}\sigma^{-1}$. We can estimate each error terms between (III.6) and (III.26) in the computations of the monotonicity formula at fixed k between (III.6) and (III.26) by the mean of the area we obtain

$$
\begin{array}{l}
\displaystyle -\int_0^{+\infty}\chi'(r)\frac{dr}{r^2}\int_{\vec{\Phi}_k^{-1}(B^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}_k}}\ \ge\ \ -C\int_0^{+\infty}\chi(r)\ dr\ \left[\frac{1}{r}+\frac{o(1)}{r^2}\right]\int_{\vec{\Phi}_k^{-1}(B^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}_k}}\\[5mm]
\displaystyle -C\int_0^{+\infty}\chi(r)\frac{dr}{r^3}\int_{\vec{\Phi}_k^{-1}(B^4_r(\vec{q}))}\sigma^2\ |\mathbb{I}_{\vec{\Phi}}|^{2p}_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}_k}}\\[5mm]
\displaystyle -C\,2^{-3j}\,\sigma^{-3}\left[\sigma^2\int_{B^4_{2^{j+1}\,\sigma}(\vec{q})\cap\vec{\Phi}(\Sigma)}|\mathbb{I}_{\vec{\Phi}}|^{2p}_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}}\right]^{1-1/p}\left[2^{-2\,j}\int_{B^4_{2^{j+1}\,\sigma}(\vec{q})\cap\vec{\Phi}(\Sigma)} dvol_{g_{\vec{\Phi}}}\right]^{1/p}
\end{array}
\tag{III.44}
$$

Using (III.43) we deduce that for any $r\in[2^{j_0}\,\sigma,1/2]$

$$
\begin{array}{l}
\displaystyle \frac{d}{dr}\left[\frac{1}{r^2}\int_{\vec{\Phi}_k^{-1}(B^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}_k}}\right]\ge\ \ -\left[\frac{C}{r}+\frac{o(1)}{r^2}\right]\int_{\vec{\Phi}_k^{-1}(B^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}_k}}\\[5mm]
\displaystyle \qquad -C\ \frac{f(\sigma)}{\eta}\ \frac{1}{r^3}\int_{\vec{\Phi}_k^{-1}(B^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}_k}}
\end{array}
\tag{III.45}
$$

Let $Y(r)\ :=\ \frac{1}{r^2}\int_{\vec{\Phi}_k^{-1}(B^4_r(\vec{q}))}\ dvol_{g_{\vec{\Phi}_k}}$, this ordinary differential inequality gives the existence of $C>0$ independent of k such that for $r\in[2^{j_0}\sigma,1/2]$

$$
\frac{d}{dr}\left[e^{C\,r}r^{\frac{C\,f(\sigma)}{\eta}}Y\right]\ge 0
\tag{III.46}
$$

Integrating between $2^{j_0}\,\sigma$ and $1/2$ gives

$$
e^{C/2}\ Y(1/2)\ 2^{-\frac{C\,f(\sigma)}{\eta}}\ge e^{C\,2^{j_0}\,\sigma}(2^{j_0}\,\sigma)^{\frac{C\,f(\sigma)}{\eta}}Y(2^{j_0}\,\sigma)\quad .
$$

Using the fact that $\vec{q}\in\vec{\Phi}(\Sigma)\setminus E_{\pi/3}$ we have then

$$
Y(1/2)\ge e^{-C/2}\ 2^{\frac{C\,f(\sigma)}{\eta}}\ \ e^{C\,2^{j_0}\,\sigma}(2^{j_0}\,\sigma)^{\frac{C\,f(\sigma)}{\eta}}\ \frac{\pi}{3}
\tag{III.47}
$$

Since $f(\sigma)=o(\log_2\sigma^{-1})$ we have $(2^{j_0}\,\sigma)^{\frac{C\,f(\sigma)}{\eta}}=2^{C\,f(\sigma)\,\eta^{-1}\ \log_2(2^{j_0}\,\sigma)}\ge 2^{C\,f(\sigma)\,\eta^{-1}\ \log_2\sigma}\to 1$ as σ goes to zero. So $Q_0:=4^{-1}\,e^{-C/2}\ \pi/3$ satisfies (III.29) and the lemma III.2 is proved. □

We shall now prove the following lemma.

Lemma III.3. [Quasi Equi-integrability] *Under the assumptions of theorem III.1, the sequence of vector-valued functions $\vec{F}_k\ :=\ \vec{\Phi}_k\,|d\vec{\Phi}_k|^2_{h_k}$ over the surface of volume 1 and constant scalar curvature (Σ,h_k) which is conformally equivalent to $(\Sigma,\vec{\Phi}_k^*g_{S^3})$ is equi-integrable with respect to the Lebesgue measure induced by any reference metric g_0 on Σ in the following sense*

$$
\forall\ p\in\Sigma\quad\forall\,\rho>0\quad\limsup_{k\to+\infty}\left|\int_{B_\rho(p)}\vec{\Phi}_k\ dvol_{g_{\vec{\Phi}_k}}\right|\le C\int_{B_{2\rho}(p)}|d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0}
\tag{III.48}
$$

where $B_\rho(p)$ and $B_{2\,\rho}(p)$ are geodesic balls with respect to the reference metric g_0 of center p and radii respectively ρ and 2ρ. □

Proof of lemma III.3. Let $x_0 \in \Sigma$ and let x be a conformal chart for (Σ, h_k) in a fixed open disc U around x_0. We can take the coordinates to be converging strongly in any norm $C^l(\Sigma, g_0)$ for any fixed reference metric g_0 on Σ since (Σ, h_k) is assumed to be pre compact in the moduli space. We can now work on the flat disc D^2 which is mapped conformally and diffeomorphically into (U, h_k). We can assume that $\vec{\Phi}_k$ converges weakly in $W^{1,2}(D^2, S^3)$ to a limiting map $\vec{\Phi}_\infty$ (a-priori non necessarily conformal) . On $\Sigma \times S^3$ we consider the sequence of measures given by

$$
\forall\, F \in C^0(\Sigma \times S^3) \qquad < N_k, F >:= \int_\Sigma F(x, \vec{\Phi}_k)\ dvol_{g_{\vec{\Phi}_k}}
$$

We extract a subsequence that we keep denoting $\vec{\Phi}_k$ such that N_k weakly converges in Radon measures towards a measure N_∞ on $\Sigma \times S^3$ and we have in particular on D^2

$$
\nu_k := |d\vec{\Phi}_k|^2_{h_k}\ dvol_{h_k} = |\nabla \vec{\Phi}_k|^2\ dx^2 \rightharpoonup \nu_\infty \quad \text{ in Radon measures}
$$

We are going to prove the existence of C independent of this subsequence such that

$$
\forall\, B_{2\rho}(x_1) \subset D^2 \qquad \left|< N_\infty, \mathbf{1}_{B_\rho(x_1)} \otimes id_{S^3} >\right| \le C \int_{B_{2\rho}(x_1)} |\nabla \vec{\Phi}_\infty|^2\ dx^2 \tag{III.49}
$$

where $\mathbf{1}_{B_\rho(x_1)} \otimes id_{S^3}$ is the function on $\Sigma \times S^3$ given by

$$
\mathbf{1}_{B_\rho(x_1)} \otimes id_{S^3}(x, \vec{q}) = \left\{ \begin{array}{ll} \vec{q} & \text{for } x \in B_\rho(x_1) \\[2mm] \vec{0} & \text{for } x \in \Sigma \setminus B_\rho(x_1) \end{array} \right. .
$$

Since for any $B_{2\rho}(x_1) \subset D^2$

$$
\int_\rho^{2\rho} dr \liminf_{k\rightarrow+\infty} \int_{\partial B_r(x_1)} |\nabla \vec{\Phi}_k|^2\ dl \le \liminf_{k\rightarrow+\infty} \int_\rho^{2\rho} dr \int_{\partial B_r(x_1)} |\nabla \vec{\Phi}_k|^2\ dl \le \nu_\infty(B_{2\rho})
$$

Using the mean value theorem and Fubini theorem, there exists $r > 0$ and a subsequence that we keep denoting $\vec{\Phi}_k$ such that simultaneously we have

$$
\begin{array}{l}
\displaystyle\limsup_{k\rightarrow+\infty} \int_{\partial B_r(x_1)} |\nabla \vec{\Phi}_k|^2\ dl < +\infty, \\[4mm]
\displaystyle\text{and} \quad \int_{\partial B_r(x_1)} |\nabla \vec{\Phi}_\infty|^2\ dl \le 2\ \frac{\int_{B_{2\rho}(x_1)} |\nabla \vec{\Phi}_\infty|^2\ dx^2}{\rho}
\end{array} \tag{III.50}
$$

Since for any $r \in (\rho, 2\rho)$ we have that $\vec{\Phi}_k \rightharpoonup \vec{\Phi}_\infty$ weakly in $H^{1/2}(\partial B_r(x_1), S^3)$, classical interpolation inequality says

$$
\vec{\Phi}_k \longrightarrow \vec{\Phi}_\infty \text{ strongly in } H^s(\partial B_r(x_1), S^3) \quad \forall\ s < 1
$$

Hence using (III.50) and Cauchy-Schwartz we obtain

$$
\begin{array}{l}
\displaystyle\lim_{k\rightarrow+\infty} \|\vec{\Phi}_k(x) - \vec{\Phi}_k(y)\|^2_{(L^\infty(\partial B_r(x_1)))^2} = \|\vec{\Phi}_\infty(x) - \vec{\Phi}_\infty(y)\|^2_{(L^\infty(\partial B_r(x_1)))^2} \\[4mm]
\displaystyle\le \left[\int_{\partial B_r(x_1)} |\nabla \vec{\Phi}_\infty|\ dl \le\right]^2 \le 8\pi \int_{B_{2\rho}(x_1)} |\nabla \vec{\Phi}_\infty|^2\ dx^2
\end{array} \tag{III.51}
$$

To simplify the presentation we assume

$$s:=\sqrt{8\pi\ \int_{B_{2\rho}(x_1)}|\nabla\vec{\Phi}_\infty|^2\ dx^2}>0$$

the case $s=0$ could be treated in a similar way in the forthcoming arguments except that we would have to introduce a small parameter that can be taken arbitrarily small. Let $\vec{q}:=\vec{\Phi}_\infty(x_2)$ for some fixed arbitrary $x_2\in\partial B_r(x_1)$. For k large enough we have that

$$\vec{\Phi}_k(\partial B_r(x_1))\subset B^4_{2s}(\vec{q})\quad . \tag{III.52}$$

For any $\tau\in[2s,4s]$ we denote by $\omega_k(\tau)$ the connected component of $\vec{\Phi}_k^{-1}(B^4_\tau(\vec{q}))$ in Σ containing $\partial B_r(x_1)$ and denote

$$\Omega_k(\tau):=\omega_k(\tau)\cup B_r(x_1)$$

Observe that $\partial\Omega_k(\tau)\subset\vec{\Phi}_k^{-1}(\partial B^4_\tau(\vec{q}))$ is included in the level sets of $|\vec{\Phi}_k-\vec{q}|$ Let $\chi(t)$ be a smooth cut-off function on $\mathbb{R}_+$ such that $\chi(t)\equiv 1$ for $t\le 2$ and $\chi(t)\equiv 0$ for $t\ge 4$. We denote $\chi_s(x)$ the function given by

$$\left\{\begin{array}{l}\chi_s(x):=\chi\left(\frac{|\vec{\Phi}_k(x)-\vec{q}|}{s}\right)\qquad\text{ on }\Omega_k(4s)\setminus\Omega_k(2s)\\[5mm] \chi_s(x):\equiv 0\qquad\text{ on }\ \Sigma\setminus\Omega_k(4s)\\[5mm] \chi_s(x):\equiv 0\qquad\text{ on }\ \Omega_k(2s)\end{array}\right.$$

Taking (II.34) and integrating by parts exactly as in the proof of lemma III.1 together with (III.15) we get

$$\lim_{k\rightarrow+\infty}\int_\Sigma\chi_s(x)\ \mathrm{div}\left[\vec{\Phi}_k\wedge\nabla\vec{\Phi}_k\right]\llcorner[\vec{\Phi}_k-\vec{q}]\ dx^2=0 \tag{III.53}$$

where the contraction operator $\llcorner$ is defined as $(\vec{a}\wedge\vec{b})\llcorner\vec{c}=<\vec{a},\vec{c}>\ \vec{b}-<\vec{b},\vec{c}>\ \vec{a}$. We then have

$$\lim_{k\rightarrow+\infty}-\int_\Sigma\nabla\chi_s(x)\ \cdot\nabla\vec{\Phi}_k\ \vec{\Phi}_k\cdot\vec{\Phi}_k-\vec{q}\ dx^2+\int_\Sigma\vec{\Phi}_k\ |\nabla\vec{\Phi}_k|^2\ dx^2=0 \tag{III.54}$$

We have $|\nabla\chi_s(x)|\ \le\ \|\dot{\chi}\|_{L^\infty(\mathbb{R}_+)}s^{-1}\ |\nabla\vec{\Phi}_k|(x)\ \mathbf{1}_{\Omega_k(4s)\setminus\Omega_k(2s)}$ where $\mathbf{1}_{\Omega_k(4s)\setminus\Omega_k(2s)}$ is the characteristic function of the set $\Omega_k(4s)\setminus\Omega_k(2s)$. Since on this set $\vec{\Phi}_k(x)\in B^4_{4s}(\vec{q})$, we then have using the monotonicity formula of lemma III.1

$$\begin{array}{l}\displaystyle\limsup_{k\rightarrow+\infty}\left|\int_\Sigma\chi_s(x)\ \vec{\Phi}_k\ |\nabla\vec{\Phi}_k|^2\ dx^2\right|\le\ C\ \mu_\infty(B^4_{4s}(\vec{q}))\le C\ s^2\\[5mm] \displaystyle\qquad\le\ C\ \int_{B_{2\rho}(x_1)}|\nabla\vec{\Phi}_\infty|^2\ dx^2\end{array} \tag{III.55}$$

Since also $\vec{\Phi}_k(\Omega_k(4s)\setminus B_\rho(x_1))\subset B^4_{4s}(\vec{q})$ we have using again (III.4)

$$\begin{array}{l}\displaystyle\limsup_{k\rightarrow+\infty}\left|\int_{\Sigma\setminus B_\rho(x_1)}\chi_s(x)\ \vec{\Phi}_k\ |\nabla\vec{\Phi}_k|^2\ dx^2\right|\le\ \mu_\infty(B^4_{4s}(\vec{q}))\\[5mm] \displaystyle\qquad\le\ C\ s^2\le C\ \int_{B_{2\rho}(x_1)}|\nabla\vec{\Phi}_\infty|^2\ dx^2\end{array} \tag{III.56}$$

Combining (III.55) and (III.56) we obtain (III.48) and lemma III.3 is proved. □

We now introduce two definitions. First we define the *Oscillation set.*

Definition III.6. *Let $\vec{\Phi}_k$ be a sequence of conformal smooth immersions, critical points of*

$$A_p^{\sigma_k}(\vec{\Phi}) := Area(\vec{\Phi}) + \sigma_k^2\, F_p(\vec{\Phi}) = \int_\Sigma \left[1 + \sigma_k^2\ [1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}]^p\right]\ dvol_{g_{\vec{\Phi}}}$$

in the space of weak immersions into S^3 and for $\sigma_k \to 0$. Assume

$$\vec{\Phi}_k \rightharpoonup \vec{\Phi}_\infty \qquad \text{weakly in } W^{1,2}(\Sigma, S^3)$$

where Σ is equipped with a reference metric g_0. Assume the sequence of riemann surfaces $(\Sigma, g_{\vec{\Phi}_k})$ is pre-compact in the moduli space of conformal structures on Σ and assume

$$\nu_k := |d\vec{\Phi}_k|^2_{h_k}\ dvol_{h_k} = |\nabla \vec{\Phi}_k|^2\ dx^2 \rightharpoonup \nu_\infty \qquad \text{in Radon measures}$$

The oscillation set $\mathcal{O} \subset \Sigma$ is the set of points $x \in \Sigma$ such that

$$\mathcal{O} := \left\{ \begin{array}{l} x \in \Sigma\ ; \qquad \nu_\infty(B_\rho(x)) \neq 0 \quad \forall\ \rho > 0 \\[4mm] \text{and} \qquad \liminf_{\rho\to 0} \dfrac{\int_{B_{2\rho}(x)} |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0}}{\nu_\infty(\overline{B_\rho(x)})} = 0 \end{array} \right\} \tag{III.57}$$

□

Now we define the *vanishing set* $\mathcal{V}$.

Definition III.7. *Let $\vec{\Phi}_k$ be a sequence of conformal smooth immersions from (Σ, g_k), critical points of*

$$A_p^{\sigma_k}(\vec{\Phi}) := Area(\vec{\Phi}) + \sigma_k^2\, F_p(\vec{\Phi}) = \int_\Sigma \left[1 + \sigma_k^2\ [1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}]^p\right]\ dvol_{g_{\vec{\Phi}}}$$

in the space of weak immersions into S^3 and for $\sigma_k \to 0$. We assume (Σ, g_k) to be pre-compact in the moduli space of conformal structures on Σ. Denote

$$f(\sigma_k) := \frac{\sigma_k^2 \displaystyle\int_\Sigma |\mathbb{I}_{\vec{\Phi}_k}|^{2p}_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}}}{\displaystyle\int_\Sigma dvol_{g_{\vec{\Phi}_k}}} \tag{III.58}$$

We call the "vanishing set" the subset Σ_0 of Σ given by

$$\Sigma_0 := \left\{ x \in \Sigma \quad ; \quad \liminf_{r\to 0}\ \limsup_{k\to+\infty} \frac{f(\sigma_k) \displaystyle\int_{B_r(x)} dvol_{g_{\vec{\Phi}_k}}}{\sigma_k^2 \displaystyle\int_{B_r(x)} |\mathbb{I}_{\vec{\Phi}_k}|^{2p}_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}}} = 0 \right\} \tag{III.59}$$

□

We will need later on the following lemma which justifies the denomination *vanishing set*.

Lemma III.4. [No Limiting Measure on the Vanishing Set] *Let $\vec{\Phi}_k$ be a sequence of conformal smooth immersions from (Σ, g_k) into S^3, critical points of*

$$A_p^{\sigma_k}(\vec{\Phi}) := Area(\vec{\Phi}) + \sigma_k^2\, F_p(\vec{\Phi}) = \int_\Sigma \left[1 + \sigma_k^2\ [1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}]^p\right]\ dvol_{g_{\vec{\Phi}}}$$

in the space of weak immersions into S^3 for $\sigma_k \to 0$. We assume (Σ, g_k) is strongly pre-compact in the Moduli space of Σ. Assume

$$\vec{\Phi}_k \rightharpoonup \vec{\Phi}_\infty \qquad \text{weakly in } W^{1,2}(\Sigma, S^3)$$

and assume the following sequence of Radon measure weakly converges

$$\nu_k := |d\vec{\Phi}_k|^2_{g_k}\ dvol_{g_k}\ \rightharpoonup \nu_\infty \quad ,$$

then we have

$$\nu_\infty(\Sigma_0) = 0 \quad . \tag{III.60}$$

□

Proof of lemma III.4. We have

$$\forall\, x \in \Sigma_0 \quad \forall\, \delta > 0 \quad \forall\, r > 0 \quad \exists\, k_{x,\delta} \in \mathbb{N} \quad \exists\, r > r_x > 0$$

$$\text{s. t.} \qquad \forall\, k \geq k_{x,\delta} \qquad \frac{f(\sigma_k) \displaystyle\int_{B_{r_x}(x)} dvol_{g_{\vec{\Phi}_k}}}{\sigma_k^2 \displaystyle\int_{B_{r_x}(x)} |\mathbb{I}_{\vec{\Phi}_k}|^{2p}_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}}} < \delta \tag{III.61}$$

For any $\delta > 0$ and $j \in \mathbb{N}$ we denote

$$\Sigma_0^j(\delta) := \{x \in \Sigma_0 \quad ; \quad k_{x,\delta} \leq j\}$$

We have clearly $\Sigma_0 = \cup_{j\in\mathbb{N}} \Sigma_0^j(\delta)$. From the covering $(\overline{B_{r_{x,\delta}}(x)})_{x\in\mathcal{O}_0^j(\delta)}$ we extract a Besicovitch sub-covering of $\Sigma_0^j(\delta)$ that we denote $(\overline{B_{r_{x_i,\delta}}(x_i)})_{i\in I}$ in such a way that any point of Σ is covered by at most N balls from this sub-covering. We have for all $k \geq j$

$$\int_{B_{r_{x_i}(x_i)}} dvol_{g_{\vec{\Phi}_k}} \leq \frac{\delta}{f(\sigma_k)}\ \sigma_k^2 \int_{B_{r_{x_i}}(x_i)} |\mathbb{I}_{\vec{\Phi}_k}|^{2p}_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}}$$

summing over $i \in I$ gives

$$\begin{aligned} \nu_k\left(\bigcup_{i\in I} \overline{B_{r_{x_i}(x_i)}}\right) &\leq \sum_{i\in I} \int_{B_{r_{x_i}(x_i)}} dvol_{g_{\vec{\Phi}_k}} \leq \frac{\delta}{f(\sigma_k)}\ \sigma_k^2 \sum_{i\in I} \int_{B_{r_{x_i}}(x_i)} |\mathbb{I}_{\vec{\Phi}_k}|^{2p}_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}} \\ &\leq N\ \frac{\delta}{f(\sigma_k)}\ \sigma_k^2 \int_{\cup_{i\in I} B_{r_{x_i}(x_i)}} |\mathbb{I}_{\vec{\Phi}_k}|^{2p}_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}} \leq N\ \delta \int_\Sigma dvol_{g_{\vec{\Phi}_k}} \end{aligned} \tag{III.62}$$

This implies that

$$\nu_\infty(\Sigma_0^j(\delta)) \leq \limsup_{k\to+\infty} \nu_k\left(\bigcup_{i\in I} \overline{B_{r_{x_i}(x_i)}}\right) \leq N\ \delta\ \nu_\infty(\Sigma) \tag{III.63}$$

This inequality is independent of j and since $\Sigma_0^j(\delta) \subset \Sigma_0^{j+1}(\delta)$ we deduce that

$$\nu_\infty(\Sigma_0) \leq N\ \delta\ \nu_\infty(\Sigma) \tag{III.64}$$

Since this holds for any $\delta > 0$ we have proven

$$\nu_\infty(\Sigma_0) = 0 \quad . \tag{III.65}$$

This completes the proof of lemma III.4. □

The next goal is to prove the following orthogonal decomposition of the limiting measure ν_∞.

Lemma III.5. [Structure of the Limiting Measure] *Under the assumptions of theorem III.1, we have the existence of finitely many points $a_1 \cdots a_n$ in Σ such that the measure ν_∞ decomposes orthogonally as follows*

$$\nu_\infty = m(x)\ \mathcal{L}^2 + \sum_{i=1}^{n}\ \alpha_i\ \delta_{a_i} \tag{III.66}$$

where $\mathcal{L}^2$ is the Lebesgue measure on Σ equipped with the reference metric g_0, m is an L^1 function with respect to the Lebesgue measure and α_i are positive numbers bounded from below by the universal positive number Q_0 given by lemma III.2. □

Proof of lemma III.5. Step 1 We prove that

$$\int_{\mathcal{O}} |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0} = 0 \quad , \tag{III.67}$$

Indeed, for any $\varepsilon > 0$ to any $x \in \mathcal{O}$ we assign r_x such that

$$\int_{B_{r_x}(x)} |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0} \le \int_{B_{2\,r_x}(x)} |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0} \le \varepsilon\ \nu_\infty(\overline{B_{r_x}(x)}) \tag{III.68}$$

Extracting a Besicovitch covering $(\overline{B_{r_i}(x_i)})_{i\in I})$ such that each point of Σ is covered by at most N balls from the covering. We obtain that

$$\int_{\cup_{i\in I} B_{r_i}(x_i)} |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0} \le \varepsilon \sum_{i\in I} \nu_\infty(\overline{B_{r_i}(x_i)}) \le \varepsilon\ N\ \nu_\infty(\Sigma) \quad . \tag{III.69}$$

and since this holds for any $\varepsilon > 0$ we obtain (III.67).

Step 2 : Proof of the **absolute continuity of** ν_∞ with respect to the Lebesgue measure away from the oscillation set $\mathcal{O}$. Precisely we prove in this step

$$\nu_\infty \llcorner (\Sigma \setminus \mathcal{O}) = m\ d\mathcal{L}^2 \tag{III.70}$$

where $m \in L^1(\Sigma)$.

Let $\varepsilon > 0$. Following(III.69), we first include $\mathcal{O}$ in an open subset $\mathcal{O}^\varepsilon$ such that

$$\int_{\mathcal{O}^\varepsilon} |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0} \le \varepsilon \tag{III.71}$$

Let $x \in \Sigma^\varepsilon := \Sigma \setminus \mathcal{O}^\varepsilon$ then there exists $\delta_x > 0$ such that

$$\inf_{\rho>0} \frac{\int_{B_{2\rho}(x)} |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0}}{\nu_\infty(\overline{B_\rho(x)})} \ge \delta_x$$

We denote $F_j := \{x \in \Sigma \setminus \mathcal{O} \quad ; \delta_x > 2^{-j}\}$. We then have

$$\Sigma \setminus \mathcal{O} = \bigcup_{j\in\mathbb{N}} F_j \quad .$$

Let G be a closed subset of $\Sigma^\varepsilon := \Sigma \setminus \mathcal{O}^\varepsilon$ such that $\mathcal{H}^2(G) = 0$. We claim that

$$\nu_\infty(G) = 0 \quad . \tag{III.72}$$

Since $\Sigma^\varepsilon := \Sigma \setminus \mathcal{O}^\varepsilon$ is closed G is compact. Let $\alpha > 0$ to be fixed later on. Since $\mathcal{H}^2(G) = 0$ and since G is compact

$$\exists\, \beta > 0 \qquad \text{s.t.} \qquad \mathcal{H}^2(G_\beta) \le \alpha \quad \text{where } G_\beta := \{x \in \Sigma\ ;\ \text{dist}(x,G) < \beta\}$$

Indeed the closeness of G implies $G := \cap_{n\in\mathbb{N}} G_{1/n}$, $G_{1/n}$ is decreasing for the inclusion and fundamental properties of Hausdorff measures give then $\mathcal{H}^2(G) = \lim_{n\to+\infty} \mathcal{H}^2(\cap_{n\in\mathbb{N}} G_{1/n})$. Let $j \in \mathbb{N}$. From the covering $(\overline{B_{\beta/2}(x)})_{x\in G\cap F_j}$ we extract a Vitalli covering $(\overline{B_{\beta/2}(x_i)})_{i\in I}$ in such a way that the balls $\overline{B_{\beta/6}(x_i)}$ are disjoint. Since all the balls have the same radius $\beta/2$ with centers at distances at least $\beta/3$ each point of Σ is covered by at most N balls $B_\beta(x_i)$ where N is a universal number. since each $x_i \in F_j$

$$\nu_\infty(\overline{B_{\beta/2}(x)}) \le 2^{j+1} \int_{B_\beta(x)} |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0} \tag{III.73}$$

Since all the balls $B_\beta(x_i)$ are included in G_β we have

$$\mathcal{H}^2\left(\bigcup_{i\in I} B_\beta(x_i)\right) \le\ N\ \alpha \tag{III.74}$$

We have moreover

$$\begin{aligned}\nu_\infty(G\cap F_j) &\le \sum_{i\in I} \nu_\infty\left(\bigcup \overline{B_{\beta/2}(x_i)}\right) \le 2^{j+1} \sum_{i\in I} \int_{B_\beta(x_i)} |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0}\\ &\le 2^{j+1}\ N \int_{\cup_{i\in I} B_\beta(x_i)} |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0} \le 2^{j+1}\ N \int_{G_\beta} |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0}\end{aligned} \tag{III.75}$$

Since $|d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0}$ is absolutely continuous with respect to the Lebesgue measure, for any $\eta > 0$ there exists $\alpha > 0$ such that

$$\forall\ E \ \text{ measurable} \qquad \mathcal{H}^2(E) \le \alpha \quad \Longrightarrow \quad \int_E |d\vec{\Phi}_\infty|^2_{g_0}\ dvol_{g_0} \le \eta \quad . \tag{III.76}$$

Hence we finally get combining (III.74), (III.75) and (III.76)

$$\nu_\infty(G\cap F_j) \le 2^{j+1}\ N\ \eta \tag{III.77}$$

For any $j \in \mathbb{N}$ the inequality (III.77) holds for any $\eta > 0$ thus $\nu_\infty(G \cap F_j) = 0$ and we deduce (III.72). Since (III.72) holds true for any closed measurable subset of $\Sigma^\varepsilon := \Sigma \setminus \mathcal{O}^\varepsilon$, then using the fundamental property of Radon measures saying that

$$\forall\ G \text{ measurable} \qquad \nu_\infty(G) = \sup\{\nu_\infty(K)\ ;\ K \subset G\ ; K \text{ compact}\} \quad ,$$

we obtain that ν_∞ for any measurable subset G of $\Sigma \setminus \mathcal{O}^\varepsilon$ satisfying on $\Sigma \setminus \mathcal{O}^\varepsilon$ is absolutely continuous with respect to the Lebesgue measure. By making ε go to zero this implies (III.70).

Step 3 : Detecting the **"bubbles"**. In this step we are just splitting the **oscillation set** $\mathcal{O}$ into it's **vanishing part** $\mathcal{O}_0 := \Sigma_0 \cap \mathcal{O}$ and the **bubble part** $\mathcal{B}$ where we recall that the Σ_0 is the so called *vanishing set* defined in definition III.7 :

$$\mathcal{B} := \mathcal{O} \setminus \left(\mathcal{O} \bigcap \Sigma_0\right) \quad .$$

Recall that we have proved in lemma III.4 $\nu_\infty(\Sigma_0) = 0$ hence

$$\nu_\infty(\mathcal{O}_0) = 0 \quad . \tag{III.78}$$

Step 4: **Finiteness of the bubble set $\mathcal{B}$.** Precisely in this step we are proving that for the constant $Q_0 > 0$ given by lemma III.2 then

$$\forall\, x \in \mathcal{B} \qquad \forall\, r > 0 \quad \nu_\infty(B_r(x)) \geq Q_0 \tag{III.79}$$

Once (III.79) will be established we can then deduce that $\mathcal{B}$ is made of finitely many points. Let then $x \in \mathcal{B}$, then there exists $\delta_x > 0$ and $r_x > 0$ that can be taken as small as one wants such that

$$\forall\, r < r_x \qquad \limsup_{k\rightarrow+\infty} \frac{f(\sigma_k)\displaystyle\int_{B_r(x)} dvol_{g_{\vec{\Phi}_k}}}{\sigma_k^2 \displaystyle\int_{B_r(x)} |\mathbb{I}_{\vec{\Phi}_k}|^{2p}_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}}} \geq \delta_x > 0 \tag{III.80}$$

Let $0 < r_c < r_x$ to be fixed later, let $\vec{\Phi}_{k'}$ a sequence for which

$$\forall\, k' \in \mathbb{N} \quad \frac{f(\sigma_{k'})\displaystyle\int_{B_r(x)} dvol_{g_{\vec{\Phi}_{k'}}}}{\sigma_{k'}^2 \displaystyle\int_{B_r(x)} |\mathbb{I}_{\vec{\Phi}_{k'}}|^{2p}_{g_{\vec{\Phi}_{k'}}}\ dvol_{g_{\vec{\Phi}_{k'}}}} \geq \frac{\delta_x}{2} \tag{III.81}$$

By assumption (III.2) from theorem III.1 we have that $f(\sigma) = o(1/\log\sigma^{-1})$ we are "almost" fulfilling the assumptions of lemma III.2 except that we have a surface with boundary $B_r(x)$ and not a closed surface. So we have to choose a "nice" cut r_c in such a way to be able to apply the arguments of lemma III.2.

Since $x \in \mathcal{O}$, by definition, for any $\eta > 0$ there exists $\rho < r_x$ such that

$$\eta\ \nu_\infty(B_\rho(x)) \geq \int_{B_{2\rho}} |\nabla\vec{\Phi}_\infty|^2\ dx^2 \tag{III.82}$$

Using Fubini and the mean-value theorem, as in (III.51), we can find $r \in [\rho, 2\rho]$ such that

$$\begin{aligned}\lim_{k\rightarrow+\infty} \|\vec{\Phi}_k(x) - \vec{\Phi}_k(y)\|^2_{(L^\infty(\partial B_r(x_1)))^2} &= \|\vec{\Phi}_\infty(x) - \vec{\Phi}_\infty(y)\|^2_{(L^\infty(\partial B_r(x_1)))^2}\\ &\leq \left[\int_{\partial B_r(x_1)} |\nabla\vec{\Phi}_\infty|\ dl \leq\right]^2 \leq 8\pi \int_{B_{2\rho}(x_1)} |\nabla\vec{\Phi}_\infty|^2\ dx^2\end{aligned} \tag{III.83}$$

We take this $r = r_c$ to be our "nice cut". We can assume

$$s := \sqrt{8\pi \int_{B_{2\rho}(x_1)} |\nabla\vec{\Phi}_\infty|^2\ dx^2} > 0$$

the case $s = 0$ could be treated in a similar way but we would have to introduce a new small parameter... Let $\vec{q}_0 := \vec{\Phi}_\infty(x_2)$ for some fixed arbitrary $x_2 \in \partial B_r(x_1)$. For k large enough we have that

$$\vec{\Phi}_k(\partial B_{r_c}(x_1)) \subset B^4_{2s}(\vec{q}_0) \quad . \tag{III.84}$$

Let $R > 4$ to be fixed later. The monotonicity formula (III.4) and (III.82) imply that

$$\mu_\infty(B^4_{R\,s}(\vec{q}_0)) \leq C\ R^2\, s^2 \leq C\, R^2\, \eta\, \nu_\infty(B_\rho(x)) \quad . \tag{III.85}$$

Hence for η chosen in such a way that $C\,R^2\,\eta < 1/2$ we have that for k' large enough (recall that k' is the sequence satisfying (III.81) for our "nice cut" r_c which is fixed now)

$$\int_{B_{r_c}(x)\setminus(\vec{\Phi}_{k'})^{-1}(B^4_{R\,s}(\vec{q}_0))} dvol_{g_{\vec{\Phi}_{k'}}} \geq 4^{-1} \int_{B_{r_c}(x)} dvol_{g_{\vec{\Phi}_{k'}}}$$

Taking the same notations of the proof of lemma III.2 where Σ is replaced by $B_{r_c}(x)$ we can then find $\vec{q}_1 \in \vec{\Phi}_{k'}(B_{r_c}(x)) \setminus (E_{\pi/3} \cup F_\delta \cup B^4_{R\,s}(\vec{q}_0))$. As in the proof of lemma III.2 we shall apply the monotonicity formula centered at this point $\vec{q}_1$ but we will remove from $\vec{\Phi}_{k'}(B_{r_c}(x))$ the balls $B^4_{t\,s}(\vec{q}_0)$ for $t \in [2,4]$. The monotonicity formula with boundary (see for instance [Ri4]) gives for all $r > 0$

$$
\begin{aligned}
&\frac{d}{dr}\left[\frac{1}{r^2}\int_{B_{r_c}(x)\cap\vec{\Phi}^{-1}(B^4_r(\vec{q}_1)\setminus B^4_{t\,s}(\vec{q}_0))} dvol_{g_{\vec{\Phi}}}\right]\\
&\quad=\frac{d}{dr}\left[\int_{B_{r_c}(x)\cap\vec{\Phi}^{-1}(B^4_r(\vec{q}_1)\setminus B^4_{t\,s}(\vec{q}_0))} \frac{|(\vec{n}\wedge\vec{\Phi})\llcorner(\vec{\Phi}-\vec{q}_1)|^2}{|\vec{\Phi}-\vec{q}_1|^4}\, dvol_{g_{\vec{\Phi}}}\right]\\
&\quad-\frac{1}{2\,r^3}\int_{B_{r_c}(x)\cap\vec{\Phi}^{-1}(B^4_r(\vec{q}_1)\setminus B^4_{t\,s}(\vec{q}_0))} (\vec{\Phi}-\vec{q}_1)\cdot d^{*_g}d\vec{\Phi}\; dvol_{g_{\vec{\Phi}}}\\
&\quad-\frac{1}{r^3}\int_{\mathbb{R}^4} <\vec{q}-\vec{q}_1,\vec{\nu}>\; d\mathcal{H}^1\llcorner\left[\vec{\Phi}(B_{r_c}(x))\cap B^4_r(\vec{q}_1)\cap\partial B^4_{t\,s}(\vec{q}_0)\right]\\
&\quad\ge-\frac{1}{2\,r^3}\int_{B_{r_c}(x)\cap\vec{\Phi}^{-1}(B^4_r(\vec{q}_1)\setminus B^4_{t\,s}(\vec{q}_0))} (\vec{\Phi}-\vec{q}_1)\cdot d^{*_g}d\vec{\Phi}\; dvol_{g_{\vec{\Phi}}}\\
&\quad-\frac{1}{r^3}\int_{\mathbb{R}^4} <\vec{q}-\vec{q}_1,\vec{\nu}>\; d\mathcal{H}^1\llcorner\left[\vec{\Phi}(B_{r_c}(x))\cap B^4_r(\vec{q}_1)\cap\partial B^4_{t\,s}(\vec{q}_0)\right]
\end{aligned}
\tag{III.86}
$$

where $\vec{\nu}$ is the outward unit tangent to the surface $\vec{\Phi}_k(B_{r_c}(x)) \setminus B^4_{t\,s}(\vec{q}_0)$ along the boundary

$$
\partial(\vec{\Phi}_k(B_{r_c}(x)) \setminus B^4_{t\,s}(\vec{q}_0)) = \vec{\Phi}_k(B_{r_c}(x)) \cap \partial B^4_{t\,s}(\vec{q}_0)
$$

and perpendicular to this boundary[3]. We consider $\chi(t)$ a smooth non negative function supported in $[1,2]$ satisfying $\int_2^4 \chi(t)\, dt = 1$, $\chi \le 1$ and $|\chi'| \le 1$. We multiply the inequality (III.86) by $\chi(t)$ and we integrate between 2 and 4 this gives

$$
\begin{aligned}
&\frac{d}{dr}\left[\frac{1}{r^2}\int_2^4 \chi(t)\, dt\int_{B_{r_c}(x)\cap\vec{\Phi}^{-1}(B^4_r(\vec{q}_1)\setminus B^4_{t\,s}(\vec{q}_0))} dvol_{g_{\vec{\Phi}}}\right]\\
&\quad\ge-\frac{1}{2\,r^3}\int_2^4 \chi(t)\, dt\int_{B_{r_c}(x)\cap\vec{\Phi}^{-1}(B^4_r(\vec{q}_1)\setminus B^4_{t\,s}(\vec{q}_0))} (\vec{\Phi}-\vec{q}_1)\cdot d^{*_g}d\vec{\Phi}\; dvol_{g_{\vec{\Phi}}}\\
&\quad-\frac{1}{r^3}\int_2^4 \chi(t)\, dt\int_{\mathbb{R}^4} <\vec{q}-\vec{q}_1,\vec{\nu}>\; d\mathcal{H}^1\llcorner\left[\vec{\Phi}(B_{r_c}(x))\cap B^4_r(\vec{q}_1)\cap\partial B^4_{t\,s}(\vec{q}_0)\right]
\end{aligned}
\tag{III.87}
$$

By substituting $d^{*_g}d\vec{\Phi}$ with it's expression deduced from (II.35), exactly as in the proof of the monotonicity formula (III.4) and as in the proof of lemma III.2 the new terms involving σ coming from the boundaries $\partial B^4_{t\,s}(\vec{q}_0)$ in the first integral of the r-h-s of (III.87) tend to zero as k tends to infinity since the distance between the center $\vec{q}_1$ and this boundary is bounded from below by $R\,s > 0$ independently

[3] Observe that $\vec{\Phi}_k(\partial B_{r_c}(x)) \subset B^4_{t\,s}(\vec{q}_0)$ so there is no contribution from $\vec{\Phi}_k(\partial B_{r_c}(x))$ outside $B^4_{t\,s}(\vec{q}_0)$.

of σ. So it remains then to estimate the last term in (III.87). This is done as follows

$$
\begin{aligned}
&\left|\frac{1}{r^3}\ \int_2^4\chi(t)\ dt\int_{\mathbb{R}^4}<\vec{q}-\vec{q}_1,\vec{\nu}>\ d\mathcal{H}^1\llcorner\left[B_{r_c}(x)\cap\vec{\Phi}^{-1}(B^4_r(\vec{q}_1))\cap\vec{\Phi}^{-1}(\partial B^4_{t\,s}(\vec{q}_0))\right]\right|\\
&\quad\le\frac{2\,|\vec{q}_1-\vec{q}_0|}{r^3}\int_2^4 dt\ \mathcal{H}^1(\partial B^4_{t\,s}(\vec{q}_0))\\
&\quad\le\frac{2\,|\vec{q}_1-\vec{q}_0|}{r^3}\int_{\vec{\Phi}^{-1}(B^4_{4\,s}\setminus B^4_{2\,s})}\frac{|d|\vec{\Phi}-\vec{q}_0||_{g_{\vec{\Phi}}}}{s}\ dvol_{g_{\vec{\Phi}}}\le C\ \frac{|\vec{q}_1-\vec{q}_0|}{r^3}\ s
\end{aligned}
\tag{III.88}
$$

where we used successively the coarea formula for the function $|\vec{\Phi}-\vec{q}_0|/s$ and the monotonicity formula (III.4) in the last inequality. Observe that this term appears only for $r>\text{dist}(B^4_{4\,s}(\vec{q}_0),\vec{q}_1)>|\vec{q}_0-\vec{q}_1|/2$. Hence the integral with respect to r between σ and $1/2$ gives

$$
\begin{aligned}
&\left|\int_\sigma^{1/2}\frac{dr}{r^3}\ \int_2^4\chi(t)\ dt\int_{\mathbb{R}^4}<\vec{q}-\vec{q}_1,\vec{\nu}>\ d\mathcal{H}^1\llcorner\left[\vec{\Phi}(B_{r_c}(x))\cap B^4_r(\vec{q}_1)\cap\partial B^4_{t\,s}(\vec{q}_0)\right]\right|\\
&\quad\le\frac{C}{|\vec{q}_1-\vec{q}_0|^2}\ |\vec{q}_1-\vec{q}_0|\ s\le\frac{C}{R}
\end{aligned}
\tag{III.89}
$$

The rest of the argument of the proof of lemma III.2 carries through and we get that

$$
\nu_\infty(B_{r_c}(x))\ge Q_0-C/R\quad .
$$

Since we can take R as large as we want, we obtain (III.79). Hence ν_∞ restricted to $\mathcal{O}$ is equal to a finite sum of Dirac masses and this last step concludes the proof of lemma III.5. □

We shall now prove the following lemma

Lemma III.6. [Absence of Energy in the Necks] *Let $\vec{\Phi}_k$ satisfying the assumptions of theorem III.1. Let $1>\eta_k>0$, $1>\delta_k>0$ and $x_k\in\Sigma$ satisfying*

$$
\lim_{k\rightarrow+\infty}\log\frac{\eta_k}{\delta_k}=+\infty
\tag{III.90}
$$

and such that

$$
\lim_{k\rightarrow 0}\ \sup_{j\in\{1\cdots\log_2(\eta_k/\delta_k)\}}\ \nu_k(B_{2^{j+1}\delta_k}(x_k)\setminus B_{2^j\delta_k}(x_k))=0
\tag{III.91}
$$

Then

$$
\lim_{k\rightarrow 0}\nu_k(B_{\eta_k}(x_k)\setminus B_{\delta_k}(x_k))=0
\tag{III.92}
$$

□

Proof of lemma III.6. We argue by contradiction. If (III.92) does not hold we can then find a subsequence that we denote still $\vec{\Phi}_k$ such that

$$
\lim_{k\rightarrow 0}\nu_k(B_{\eta_k}(x_k)\setminus B_{\delta_k}(x_k))=A>0
\tag{III.93}
$$

Let Q_0 be the universal constant in the lemma III.2. We can assume without loss of generality that

$$
A<Q_0\quad .
\tag{III.94}
$$

Indeed, if this would not be the case we would replace δ_k by a larger number that we keep denoting δ_k and since (III.91) holds we necessarily have (III.90) for this new δ_k. We have for k large enough

$$\frac{\sigma_k^2\displaystyle\int_\Sigma |\mathbb{I}_{\vec{\Phi}_k}|^{2p}_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}}}{\displaystyle\int_{B_{\eta_k}(x_k)\setminus B_{\delta_k}(x_k)} dvol_{g_{\vec{\Phi}_k}}}\le\frac{2\,\nu_\infty(\Sigma)}{A}\,f(\sigma_k) \tag{III.95}$$

Following the approach of step 5 of the proof of lemma III.5, we first select 2 "good cuts" at the two ends of the annulus. So we choose respectively $\delta_{k,c}\in[\delta_k,2\delta_k]$ and $\eta_{k,c}\in[\eta_k/2,\eta_k]$ such that we have respectively

$$s_k^2:=\left[\int_{\partial B_{\delta_{k,c}}(x_k)}|\nabla\vec{\Phi}_k|\ dl\right]^2\le\ 8\pi\ \nu_k(B_{2\,\delta_k}(x_k)\setminus B_{\delta_k}(x_k))\longrightarrow 0$$

and

$$t_k^2:=\left[\int_{\partial B_{\eta_{k,c}}(x_k)}|\nabla\vec{\Phi}_k|\ dl\right]^2\le\ 8\pi\ \nu_k(B_{2\,\eta_k}(x_k)\setminus B_{\eta_k}(x_k))\longrightarrow 0\quad .$$

Let $x_{1,k}\in\partial B_{\delta_{k,c}}(x_k)$ and $x_{2,k}\in\partial B_{\delta_{k,c}}(x_k)$ arbitrary. We have respectively

$$\vec{\Phi}_k(\partial B_{\delta_{k,c}}(x_k))\subset B^4_{s_k}(\vec{\Phi}_k(x_{1,k}))\quad\text{ and }\quad\vec{\Phi}_k(\partial B_{\eta_{k,c}}(x_k))\subset B^4_{t_k}(\vec{\Phi}_k(x_{2,k}))\quad . \tag{III.96}$$

Because of the monotonicity formula (III.4) there exists $s>0$ fixed such that

$$\max_{\vec{q}\in\mathbb{R}^4}\mu_\infty(B^4_s(\vec{q}))<A/4\quad .$$

We then have for k large enough

$$\mu_k\left(\vec{\Phi}_k(B_{\eta_{k,c}}(x_k)\setminus B_{\delta_{k,c}}(x_k))\setminus\left(B^4_s(\vec{\Phi}_k(x_{1,k}))\cup B^4_s(\vec{\Phi}_k(x_{2,k}))\right)\right)\ge A/2$$

As in the step 5 of the proof of lemma III.5, we adopt the notations from the proof of lemma III.2 and replacing Σ by the annulus $B_{\eta_{k,c}}(x_k)\setminus B_{\delta_{k,c}}(x_k)$, we can find $\vec{q}_k$ such that

$$\vec{q}_k\in\vec{\Phi}_k(B_{\eta_{k,c}}(x_k)\setminus B_{\delta_{k,c}}(x_k))\setminus\left(E_{\pi/3}\cup F_\delta\cup B^4_s(\vec{\Phi}_k(x_{1,k}))\cup B^4_s(\vec{\Phi}_k(x_{2,k})\right)\quad .$$

We can carry over one by one the computation of the monotonicity formula centered at $\vec{q}$, controlling the boundary terms induced by the two cuts $\vec{\Phi}_k(\partial B_{\eta_{k,c}}(x_k))$ and $\vec{\Phi}_k(\partial B_{\eta_{k,c}}(x_k))$ which stay at a distance bounded from bellow with respect to $\vec{q}_k$, following the approach of the end of the step 5 of the proof of lemma III.5. It is here even simpler since the lengths of the cuts s_k and t_k shrink to zero in the present case. Hence we obtain

$$A=\lim_{k\to 0}\nu_k(B_{\eta_k}(x_k)\setminus B_{\delta_k}(x_k))\ge Q_0$$

which contradicts (III.94). This concludes the proof of lemma III.6. □

Lemma III.7. [Rectifiability of the Limit] *Let $\vec{\Phi}_k$ satisfying the assumptions of theorem III.1. Then the limiting measure μ_∞ is supported by a rectifiable 2-dimensional subset K of S^3 and we have*

$$\lim_{r\to 0}\frac{\mu_\infty(B^4_r(\vec{q}))}{r^2}>0\qquad\textit{for } \mu_\infty\ \textit{a.e. } \vec{q}\quad . \tag{III.97}$$

□

Proof of lemma III.7. Recall that we denote by $\mathcal{B}=\{a_1\cdots a_n\}$ the blow up set.

Step 1 : Proof of the **rectifiability of the varifold limit of** $|(\vec{\Phi}_k)_\ast[\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)]|$ for any $\varepsilon>0$. Precisely we are going to prove the rectifiability of the limit μ_∞^ε of the following sequence μ_k^ε given by

$$\forall\ \varphi\in C^0(S^3)\qquad <\mu_k^\varepsilon,\varphi>:=\int_{\hat{\Sigma}_\varepsilon}\varphi(\vec{\Phi}_k)\ dvol_{g_{\vec{\Phi}_k}} \tag{III.98}$$

where we use the notation $\hat{\Sigma}_\varepsilon=\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)$. Precisely we are going to prove the existence of a two dimensional subset $K^\varepsilon\subset S^3$ and a function θ^ε in $L^1(K^\varepsilon,d\mathcal{H}^2\llcorner K^\varepsilon)$ such that

$$\mu_\infty^\varepsilon=\theta^\varepsilon\ d\mathcal{H}^2\llcorner K^\varepsilon\quad . \tag{III.99}$$

Since $\vec{\Phi}_k$ weakly converges towards $\vec{\Phi}_\infty$ in $W^{1,2}(\Sigma,S^3)$, it strongly converges in L^2 and hence almost everywhere. Thus Egorov theorem gives, for any $\alpha>0$, the existence of a measurable subset $A^\alpha\subset\Sigma$ such that

$$\vec{\Phi}_k\longrightarrow\vec{\Phi}_\infty\quad\text{uniformly in }\Sigma\setminus A^\alpha\qquad\text{and}\qquad\mathcal{H}^2(A^\alpha)\le\alpha\quad . \tag{III.100}$$

Since $\mathcal{L}^2(\Sigma\setminus A^\alpha)=\sup\{\mathcal{L}^2(K)\ ;\ K\subset\Sigma\setminus A^\alpha\text{ compact }\}$ we can assume that A^α is open. Using moreover the quantitative Lusin type property for Sobolev maps of F.C. Liu (see [18]) we deduce that for any $\alpha>0$ there exists a C^1 map $\vec{\Xi}^\alpha$ from Σ into[4] S^3 and an open measurable subset B^α of Σ such that

$$\left\{\begin{array}{l}\mathcal{H}^2(B^\alpha)\le\alpha\quad ,\\[5mm] \vec{\Phi}_\infty=\vec{\Xi}^\alpha\quad\text{ on }\Sigma\setminus B^\alpha\quad\text{and}\quad d\vec{\Phi}_\infty=d\vec{\Xi}^\alpha\quad\text{ on }\Sigma\setminus B^\alpha\quad ,\\[5mm] \|\vec{\Phi}_\infty-\vec{\Xi}^\alpha\|^2_{W^{1,2}(\Sigma)}\le\alpha\quad .\end{array}\right. \tag{III.101}$$

Let $(\overline{B_{r_i}(x_i)})_{i\in I}$ be a covering of $A^\alpha\cup B^\alpha$ by closed balls such that

$$\pi\sum_{i\in I}r_i^2\le C\ \alpha \tag{III.102}$$

where C only depend on the reference metric g_0. We extract from $(\overline{B_{r_i}(x_i)})_{i\in I}$ a Besicovitch sub-covering such that each point of Σ is covered by at most N balls of this new covering. Since ν_∞ is absolutely continuous with respect to the Lebesgue measure on Σ^ε we have for any $\varepsilon>0$

$$\lim_{\alpha\to 0}\nu_\infty(\hat{\Sigma}_\varepsilon\cap(A^\alpha\cup B^\alpha))=0\quad\text{ and }\quad\lim_{\alpha\to 0}\nu_\infty\left(\bigcup_{i\in I}\overline{B_{r_i}(x_i)}\right)=0 \tag{III.103}$$

in other words we have

$$\int_{\hat{\Sigma}_\varepsilon\cap(A^\alpha\cup B^\alpha)}dvol_{g_{\vec{\Phi}_k}}\le\sum_{i\in I}\int_{\hat{\Sigma}_\varepsilon\cap B_{r_i}(x_i)}dvol_{g_{\vec{\Phi}_k}} \tag{III.104}$$

Since $\nu_\infty(\partial(\hat{\Sigma}_\varepsilon\cap B_{r_i}(x_i)))=0$ for any $i\in I$ we have that

$$\forall\ i\in I\qquad\lim_{k\to+\infty}\nu_k(\hat{\Sigma}_\varepsilon\cap B_{r_i}(x_i))=\nu_\infty(\hat{\Sigma}_\varepsilon\cap B_{r_i}(x_i))=\int_{\hat{\Sigma}_\varepsilon\cap B_{r_i}(x_i)}m(x)\ d\mathcal{L}^2$$

[4]The fact that we can apply Liu's result for maps into $W^{1,2}(\Sigma,S^3)$ comes from the fact that smooth maps in $C^1(\Sigma,S^3)$ are dense in $W^{1,2}(\Sigma,S^3)$ for the $W^{1,2}-$topology.

Hence

$$\begin{aligned}\limsup_{k\to+\infty}\int_{\hat{\Sigma}_\varepsilon\cap(A^\alpha\cup B^\alpha)} dvol_{g_{\vec{\Phi}_k}} &\le \int_{\hat{\Sigma}_\varepsilon\cap(A^\alpha\cup B^\alpha)}\sum_{i\in I}\mathbf{1}_{B_{r_i}(x_i)}\ m(x)\ d\mathcal{L}^2\\ &\le N\ \nu_\infty\left(\hat{\Sigma}_\varepsilon\bigcap\left(\bigcup_{i\in I}B_{r_i}(x_i)\right)\right)\end{aligned}\tag{III.105}$$

Combining this fact with (III.103) we obtain

$$\lim_{\alpha\to0}\limsup_{k\to+\infty}\int_{\hat{\Sigma}_\varepsilon\cap(A^\alpha\cup B^\alpha)} dvol_{g_{\vec{\Phi}_k}}=0\tag{III.106}$$

Denote

$$K^\varepsilon:=\bigcup_{n\in\mathbb{N}}\vec{\Xi}^{1/n^2}(\hat{\Sigma}_\varepsilon\setminus(A^{1/n^2}\cup B^{1/n^2}))$$

Since $\vec{\Xi}^\alpha$ is C^1 on Σ, K^ε is contained in a countable union of the images by C^1 maps of smooth manifolds. It is the a 2-dimensional rectifiable subset of S^3. Denote

$$K^\varepsilon_n:=\bigcup_{l\le n}\vec{\Xi}^{1/l^2}(\hat{\Sigma}_\varepsilon\setminus(A^{1/l^2}\cup B^{1/l^2}))=\vec{\Phi}_\infty\left(\bigcup_{l\le n}\hat{\Sigma}_\varepsilon\setminus(A^{1/n^2}\cup B^{1/n^2})\right)\quad.$$

Observe first that since

$$\mathcal{L}^2(\cup_{l\ge n+1}(A^{1/l^2}\cup B^{1/l^2})\le C\sum_{l\ge n+1}l^{-2}=O(n^{-1})$$

using the area formula we have

$$\mathcal{H}^2\left(K^\varepsilon\setminus K^\varepsilon_n\right)\le\int_{\cup_{l\ge n+1}(A^{1/l^2}\cup B^{1/l^2})}|\nabla\vec{\Phi}_\infty|^2\ dx^2\to0\tag{III.107}$$

Observe also that for all $n\in\mathbb{N}$

$$\lim_{k\to+\infty}\|\mathrm{dist}(\vec{\Phi}_k(x),K^\varepsilon_n)\|_{L^\infty(\hat{\Sigma}_\varepsilon\setminus(A^{1/n^2}\cup B^{1/n^2}))}=0\tag{III.108}$$

Introduce the limiting measure

$$\mu^\varepsilon_{n,\infty}(\varphi):=\lim_{k\to+\infty}\int_{\hat{\Sigma}_\varepsilon\setminus(A^{1/n}\cup B^{1/n})}\varphi(\vec{\Phi}_k)\ dvol_{g_{\vec{\Phi}_k}}$$

Since K^ε_n is a finite union of images by C^1-maps of compact sets $\hat{\Sigma}_\varepsilon\setminus(A^{1/l^2}\cup B^{1/l^2})$ for $l\le n$, it is compact, and denoting

$$F^\delta_n:=\left\{x\in\mathbb{R}^4\quad;\quad\mathrm{dist}\,(x\ ;\ K^\varepsilon_n)\le\delta\right\}$$

we have that

$$K^\varepsilon_n=\bigcap_{\delta>0}F^\delta_n$$

moreover, because of (III.108) we have that

$$\mu^\varepsilon_{n,\infty}(\mathbb{R}^4\setminus F^\delta_n)=0$$

fundamental results in measure theory imply

$$\mu^\varepsilon_{n,\infty}(\mathbb{R}^4\setminus K^\varepsilon_n)=\lim_{\delta\to 0}\mu^\varepsilon_{n,\infty}(\mathbb{R}^4\setminus F^\delta_n)=0$$

Let

$$\mu^\varepsilon_\infty(\varphi):=\lim_{k\to+\infty}\int_{\hat{\Sigma}_\varepsilon}\varphi(\vec{\Phi}_k)\ dvol_{g_{\vec{\Phi}_k}}$$

Because of (III.106) we deduce that $\mu^\varepsilon_\infty=\lim_{n\to+\infty}\mu^\varepsilon_{n,\infty}$. Since now $\mu^\varepsilon_n\le\mu^\varepsilon_\infty\le\mu_\infty$ and since μ_∞ satisfies the monotonicty formula (III.4), using (III.107) we deduce

$$\lim_{n\to+\infty}\sup_{j\in\mathbb{N}}\mu^\varepsilon_{j,\infty}(K^\varepsilon\setminus K^\varepsilon_n)=\lim_{n\to+\infty}\mu^\varepsilon_\infty(K^\varepsilon\setminus K^\varepsilon_n)=0 \tag{III.109}$$

For any $n_0\le n$ we write

$$\mu^\varepsilon_{n,\infty}(\mathbb{R}^4\setminus K^\varepsilon_{n_0})=\mu^\varepsilon_{n,\infty}(\mathbb{R}^4\setminus K^\varepsilon_n)+\mu^\varepsilon_{n,\infty}(K^\varepsilon_n\setminus K^\varepsilon_{n_0})=\mu^\varepsilon_{n,\infty}(K^\varepsilon_n\setminus K^\varepsilon_{n_0})\quad . \tag{III.110}$$

Combining (III.109) and (III.110) we obtain

$$\forall\,\delta>0\quad\exists\,n_0\in\mathbb{N}\quad\text{s.t.}\quad\forall\ n\ge n_0\quad\mu^\varepsilon_{n,\infty}(\mathbb{R}^4\setminus K^\varepsilon_{n_0})\le\delta \tag{III.111}$$

Since $K^\varepsilon_{n_0}$ is closed, using fundamental properties of the weak convergence of Radon measures, we have

$$\mu^\varepsilon_\infty(\mathbb{R}^4\setminus K^\varepsilon_{n_0})\le\limsup_{n\to+\infty}\mu^\varepsilon_{n,\infty}(\mathbb{R}^4\setminus K^\varepsilon_{n_0})\le\delta\quad . \tag{III.112}$$

Combining (III.109) and (III.112) we deduce

$$\mu^\varepsilon_\infty(\mathbb{R}^4\setminus K^\varepsilon)=0\quad . \tag{III.113}$$

Since μ^ε_∞ and μ_∞ is zero on $\mathcal{H}^2$ measure zero sets, Riesz representation theorem gives the existence of θ^ε in $L^1(K^\varepsilon,d\mathcal{H}^2\llcorner K^\varepsilon)$ such that (III.99) holds.

Step 2 Since there is no contribution from the neck regions, the whole limiting measure is obtain by summing the contribution from the domains $\hat{\Sigma}^\varepsilon$ and from the bubbles which are treated exactly as the domains $\hat{\Sigma}^\varepsilon$ after the *ad-hoc* dilation. Hence from Step 1 we deduce the rectifiability of the full measure μ_∞ and lemma III.7 is proved. □

Lemma III.8. [Vanishing of the Limiting Measure on the Degenerating Set] *Let $\mathfrak{L}_{\nabla\vec{\Phi}_\infty}$ be the subset of $\Sigma\setminus\mathcal{B}$ of Lebesgue points for $\nabla\vec{\Phi}_\infty$. We denote by $\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}$ the measurable subset of $\mathfrak{L}_{\nabla\vec{\Phi}_\infty}$ of points where the Lebesgue representative of $\nabla\vec{\Phi}_\infty$ has rank strictly less than 2. Then we have*

$$\nu_\infty(\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty})=0\quad . \tag{III.114}$$

□

Proof of lemma III.8. Let $\mathcal{B}:=\{a_1\cdots a_n\}$ be the blow-up set. For any $\varepsilon>0$ we introduce $\hat{\Sigma}_\varepsilon:=\Sigma\setminus\cup_{l=1}^nB_\varepsilon(a_l)$. Since $\vec{\Phi}_k$ weakly converges towards $\vec{\Phi}_\infty$ in $W^{1,2}(\Sigma,S^3)$, it strongly converges in L^2 and hence almost everywhere. Thus Egorov theorem gives, for any $\alpha>0$, the existence of a measurable subset $A^\alpha\subset\Sigma$ such that

$$\vec{\Phi}_k\longrightarrow\vec{\Phi}_\infty\quad\text{uniformly in }\Sigma\setminus A^\alpha\qquad\text{and}\qquad\mathcal{H}^2(A^\alpha)\le\alpha\quad . \tag{III.115}$$

Using moreover the quantitative Lusin type property for Sobolev maps of F.C. Liu (see [18]) we deduce that for any $\alpha>0$ there exists a C^1 map $\vec{\Xi}^\alpha$ from Σ into[5] S^3 and an open measurable subset B^α of Σ such that

$$
\left\{
\begin{array}{l}
\mathcal{H}^2(B^\alpha)\le\alpha \quad ,\\[5mm]
\vec{\Phi}_\infty=\vec{\Xi}^\alpha\quad \text{ on }\Sigma\setminus B^\alpha\quad\text{ and }\quad d\vec{\Phi}_\infty=d\vec{\Xi}^\alpha\quad\text{ on }\Sigma\setminus B^\alpha\quad ,\\[5mm]
\|\vec{\Phi}_\infty-\vec{\Xi}^\alpha\|^2_{W^{1,2}(\Sigma)}\le\alpha\quad .
\end{array}
\right.
\tag{III.116}
$$

By definition of the Hausdorff measure there exists $(B_{r_i}(x_i))_{i\in I}$, an at most countable covering by open balls of $A^\alpha\setminus B^\alpha$, such that

$$
\pi\sum_{i\in I}r_i^2\le C\ \alpha
\tag{III.117}
$$

And extracting possibly a Besicovitch sub-covering from the covering $(\overline{B_{r_i}(x_i)})_{i\in I}$ we ensure that each point of Σ is covered by at most N balls of the covering where N is universal. Denote $\hat{\Sigma}^\alpha_\varepsilon$ the following compact set

$$
\hat{\Sigma}^\alpha_\varepsilon:=\Sigma\setminus\left((\cup_{l=1}^n B_\varepsilon(a_l))\bigcup(\cup_{i\in I}B_{r_i}(x_i))\right)
$$

We deduce using the area formula

$$
\mathcal{H}^2\left(\vec{\Xi}^\alpha\left(\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}\cap\hat{\Sigma}^\alpha_\varepsilon\right)\right)=\int_{\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}\cap\hat{\Sigma}^\alpha_\varepsilon}|\det(d\vec{\Xi}^\alpha)|dx^2\le C\ \nu_\infty(\Sigma)\ \alpha
\tag{III.118}
$$

Let $\mathcal{U}_\alpha$ be an open neighborhood of $\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}$ such that

$$
\mathcal{H}^2\left(\vec{\Xi}^\alpha(\mathcal{U}_\alpha\cap\hat{\Sigma}^\alpha_\varepsilon)\right)\le 2\ C\ \nu_\infty(\Sigma)\ \alpha
\tag{III.119}
$$

Such a neighborhood exists since $\vec{\Xi}^\alpha$ is C^1 on the whole Σ. We cover the compact set $\vec{\Xi}^\alpha(\overline{\mathcal{U}_\alpha\cap\hat{\Sigma}^\alpha_\varepsilon})$ by open balls $(B^4_{\rho_j}(\vec{q}_j))_{j\in J}$ such that

$$
\sum_{j\in J}\rho_j^2\le 4\ C\ \nu_\infty(\Sigma)\ \alpha
\tag{III.120}
$$

On the compact set $\overline{\mathcal{U}_\alpha}\cap\hat{\Sigma}^\alpha_\varepsilon$ the sequence $\vec{\Phi}_k$ converges uniformly to $\vec{\Phi}_\infty=\vec{\Xi}^\alpha$ on that set. Hence for k large enough we have

$$
\vec{\Phi}_k(\overline{\mathcal{U}_\alpha}\cap\hat{\Sigma}^\alpha_\varepsilon)\subset\cup_{j\in J}B^4_{\rho_j}(\vec{q}_j)\quad .
\tag{III.121}
$$

Hence

$$
\begin{aligned}
\nu_k(\overline{\mathcal{U}_\alpha}\cap\hat{\Sigma}^\alpha_\varepsilon)&=\int_{\overline{\mathcal{U}_\alpha}\cap\hat{\Sigma}^\alpha_\varepsilon}dvol_{g_{\vec{\Phi}_k}}\le\mu_k\left(\cup_{j\in J}B^4_{\rho_j}(\vec{q}_j)\right)\\
&\le\sum_{j\in J}\mu_k\left(B^4_{\rho_j}(\vec{q}_j)\right)
\end{aligned}
\tag{III.122}
$$

We have also

$$
\begin{aligned}
\nu_k\left(\overline{\mathcal{U}_\alpha}\setminus\cup_{l=1}^n B_\varepsilon(a_l)\right)&\le\nu_k(\overline{\mathcal{U}_\alpha}\cap\hat{\Sigma}^\alpha_\varepsilon)+\nu_k\left((\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l))\bigcap(\cup_{i\in I}B_{r_i}(x_i))\right)\\
&\le\nu_k(\overline{\mathcal{U}_\alpha}\cap\hat{\Sigma}^\alpha_\varepsilon)+\sum_{i\in I}\nu_k\left((\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l))\bigcap B_{r_i}(x_i)\right)
\end{aligned}
\tag{III.123}
$$

[5] The fact that we can apply Liu's result for maps into $W^{1,2}(\Sigma,S^3)$ comes from the fact that smooth maps in $C^1(\Sigma,S^3)$ are dense in $W^{1,2}(\Sigma,S^3)$ for the $W^{1,2}-$topology.

Combining (III.122) and (III.123) we obtain

$$\nu_k\left(\overline{\mathcal{U}_\alpha}\setminus\cup_{l=1}^n B_\varepsilon(a_l)\right)\le\sum_{j\in J}\mu_k\left(B^4_{\rho_j}(\vec{q}_j)\right)+\sum_{i\in I}\nu_k\left((\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l))\bigcap B_{r_i}(x_i)\right) \tag{III.124}$$

Using fundamental properties of the convergence of Radon measures we have in one hand

$$\nu_\infty\left(\mathcal{U}_\alpha\setminus\cup_{l=1}^n\overline{B_\varepsilon(a_l)}\right)\le\liminf_{k\to+\infty}\nu_k\left(\overline{\mathcal{U}_\alpha}\setminus\cup_{l=1}^n B_\varepsilon(a_l)\right) \tag{III.125}$$

and in the other hand since $\nu_\infty\left(\partial\left[(\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l))\bigcap B_{r_i}(x_i)\right]\right)=0$, due to the fact that ν_∞ in $\Sigma\setminus\{a_1\cdots a_n\}$ is absolutely continuous with respect to the Lebesgue measure, we have

$$\lim_{k\to+\infty}\nu_k\left((\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l))\bigcap B_{r_i}(x_i)\right)=\nu_\infty\left((\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l))\bigcap B_{r_i}(x_i)\right) \tag{III.126}$$

and finally

$$\limsup_{k\to+\infty}\mu_k\left(B^4_{\rho_j}(\vec{q}_j)\right)\le\mu_\infty(\overline{B^4_{\rho_j}(\vec{q}_j)}) \tag{III.127}$$

Combining (III.124)...(III.127) we obtain

$$\nu_\infty\left(\mathcal{U}_\alpha\setminus\cup_{l=1}^n\overline{B_\varepsilon(a_l)}\right)\le\sum_{j\in J}\mu_\infty(\overline{B^4_{\rho_j}(\vec{q}_j)})+\sum_{i\in I}\nu_\infty\left((\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l))\bigcap B_{r_i}(x_i)\right) \tag{III.128}$$

Using the monotonicity formula (III.4) together with (III.120) we have

$$\sum_{j\in J}\mu_\infty(\overline{B^4_{\rho_j}(\vec{q}_j)})\le\ C\ \sum_{j\in J}\rho_j^2\le C\ \nu_\infty(\Sigma)\ \alpha \tag{III.129}$$

and using (III.117)

$$\begin{aligned}\sum_{i\in I}\nu_\infty\left((\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l))\bigcap B_{r_i}(x_i)\right)&=\int_{\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)}m(x)\ \sum_{i\in I}\mathbf{1}_{B_{r_i}(x_i)}\ d\mathcal{L}^2\\&\le N\ \pi\sum_{i\in I}r_i^2\le C\ \alpha\end{aligned} \tag{III.130}$$

Hence we have proved that

$$\nu_\infty(\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}\setminus\cup_{l=1}^n B_\varepsilon(a_l))\le C\ \alpha\quad. \tag{III.131}$$

This holds for any $\varepsilon>0$ and $\alpha>0$. We then deduce (III.114) and this concludes the proof of lemma III.8. □

Lemma III.9. [Almost Everywhere Approximate Pointwize Convergence and Almost Everywhere Approximate Continuity] *Assume the hypothesis of theorem III.1 are fulfilled and that we have extracted subsequences such that $\vec{\Phi}_k$ converges weakly towards $\vec{\Phi}_\infty$ in $W^{1,2}(\Sigma)$ and ν_k converges towards ν_∞ satisfying (III.66) where $\mathcal{B}:=\{a_1\cdots a_l\}$ the blow-up set. For ν_∞ almost every $x\in\Sigma\setminus\mathcal{B}$ the following holds :*

$$\lim_{r\to0}\limsup_{k\to+\infty}\frac{\displaystyle\int_{B_r(x)}|\vec{\Phi}_k-\vec{\Phi}_\infty|\ dvol_{g_{\Phi_k}}}{\displaystyle\int_{B_r(x)}dvol_{g_{\Phi_k}}}=0 \tag{III.132}$$

Moreover

$$\lim_{r\to 0}\limsup_{k\to+\infty}\frac{\displaystyle\int_{B_r(x)}|\vec{\Phi}_\infty-\vec{\Phi}_\infty(x)|\ dvol_{g_{\Phi_k}}}{\displaystyle\int_{B_r(x)}dvol_{g_{\Phi_k}}}=0 \tag{III.133}$$

□

Proof of lemma III.9. We use the same notations as in the proof of lemma III.8. For any ε we denote $\hat{\Sigma}_\varepsilon:=\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)$. We also denote for any $\alpha>0$ a set A^α such that (III.115) holds and B^α such that (III.116) holds. Let $(B_{r_i}(x_i))_{i\in I}$ be a covering of A^α such that

$$\pi\sum_{i\in I}r_i^2\le C\ \alpha \tag{III.134}$$

where C only depend on the reference metric g_0. We extract from $(B_{r_i}(x_i))_{i\in I}$ a Besicovitch sub-covering such that each point of Σ is covered by at most N balls of this new covering. We identify A^α with this covering. The Lebesgue-Besicovitch Differentiation theorem, for ν_∞ almost every point form $x\in\hat{\Sigma}_\varepsilon\setminus A^\alpha$ gives

$$\lim_{r\to 0}\frac{\nu_\infty((\hat{\Sigma}_\varepsilon\setminus A^\alpha)\cap B_r(x))}{\nu_\infty(B_r(x))}=1\quad . \tag{III.135}$$

Since $\vec{\Phi}_k$ converges uniformly to $\vec{\Phi}_\infty$ on $\hat{\Sigma}_\varepsilon\setminus A^\alpha$ we have

$$\limsup_{k\to+\infty}\frac{\displaystyle\int_{B_r(x)}|\vec{\Phi}_k-\vec{\Phi}_\infty|\ dvol_{g_{\Phi_k}}}{\displaystyle\int_{B_r(x)}dvol_{g_{\Phi_k}}}\le 2\ \limsup_{k\to+\infty}\frac{\displaystyle\int_{A^\alpha\cap B_r(x)}dvol_{g_{\Phi_k}}}{\displaystyle\int_{B_r(x)}dvol_{g_{\Phi_k}}} \tag{III.136}$$

Assume x is a point satisfying (III.135) we have

$$\limsup_{k\to+\infty}\frac{\nu_k(A^\alpha\cap B_r(x))}{\nu_k(B_r(x))}\le\sum_{i\in I}\limsup_{k\to+\infty}\frac{\nu_k(B_{r_i}(x_i)\cap B_r(x))}{\nu_k(B_r(x))}$$

Since $\nu_\infty(\partial(B_{r_i}(x_i)\cap B_r(x)))=0$ and $\nu_\infty(\partial B_r(x))=0$ we have

$$\lim_{k\to+\infty}\frac{\nu_k(B_{r_i}(x_i)\cap B_r(x))}{\nu_k(B_r(x))}=\frac{\nu_\infty(B_{r_i}(x_i)\cap B_r(x))}{\nu_\infty(B_r(x))}$$

Thus

$$\limsup_{k\to+\infty}\frac{\nu_k(A^\alpha\cap B_r(x))}{\nu_k(B_r(x))}\le\frac{\displaystyle\int_{B_r(x)}m(x)\ \sum_{i\in I}\mathbf{1}_{B_{r_i}(x_i)}\ d\mathcal{L}^2}{\nu_\infty(B_r(x))}\le N\ \frac{\nu_\infty(A^\alpha\cap B_r(x))}{\nu_\infty(B_r(x))} \tag{III.137}$$

Combining (III.135), (III.136) and (III.137) we obtain (III.132).

Let $(B_{r_i}(x_i))_{i\in I}$ be a covering of B^α such that

$$\pi\sum_{i\in I}r_i^2\le C\ \alpha \tag{III.138}$$

where C only depend on the reference metric g_0. We extract from $(B_{r_i}(x_i))_{i\in I}$ a Besicovitch sub-covering such that each point of Σ is covered by at most N balls of this new covering. We identify B^α with this covering. The Lebesgue-Besicovitch Differentiation theorem, for ν_∞ almost every point form $x\in\hat{\Sigma}_\varepsilon\setminus B^\alpha$

$$\lim_{r\to 0}\frac{\nu_\infty((\hat{\Sigma}_\varepsilon\setminus B^\alpha)\cap B_r(x))}{\nu_\infty(B_r(x))}=1\quad . \tag{III.139}$$

We have

$$\limsup_{k\to+\infty}\frac{\displaystyle\int_{B_r(x)}|\vec{\Phi}_\infty-\vec{\Phi}_\infty(x)|\ dvol_{g_{\Phi_k}}}{\displaystyle\int_{B_r(x)}dvol_{g_{\Phi_k}}}\le\limsup_{k\to+\infty}\frac{\displaystyle\int_{B^\alpha\cap B_r(x)}dvol_{g_{\Phi_k}}}{\displaystyle\int_{B_r(x)}dvol_{g_{\Phi_k}}}$$
$$+\limsup_{k\to+\infty}\frac{\displaystyle\int_{B_r(x)}|\vec{\Xi}^\alpha-\vec{\Xi}^\alpha(x)|\ dvol_{g_{\Phi_k}}}{\displaystyle\int_{B_r(x)}dvol_{g_{\Phi_k}}}\tag{III.140}$$

Assume x is a point satisfying (III.135) we have

$$\limsup_{k\to+\infty}\frac{\nu_k(B^\alpha\cap B_r(x))}{\nu_k(B_r(x))}\le\sum_{i\in I}\limsup_{k\to+\infty}\frac{\nu_k(B_{r_i}(x_i)\cap B_r(x))}{\nu_k(B_r(x))}$$

and as in (III.137) we have

$$\limsup_{k\to+\infty}\frac{\nu_k(B^\alpha\cap B_r(x))}{\nu_k(B_r(x))}\le\frac{\displaystyle\int_{B_r(x)}m(x)\ \sum_{i\in I}\mathbf{1}_{B_{r_i}(x_i)}\ d\mathcal{L}^2}{\nu_\infty(B_r(x))}\le N\ \frac{\nu_\infty(B^\alpha\cap B_r(x))}{\nu_\infty(B_r(x))}\tag{III.141}$$

Combining (III.140) and (III.141) we have

$$\limsup_{k\to+\infty}\left|\frac{\displaystyle\int_{B_r(x)}|\vec{\Phi}_\infty-\vec{\Phi}_\infty(x)|\ dvol_{g_{\Phi_k}}}{\displaystyle\int_{B_r(x)}dvol_{g_{\Phi_k}}}\right|\le N\ \frac{\nu_\infty(B^\alpha\cap B_r(x))}{\nu_\infty(B_r(x))}+r\ \|\nabla\vec{\Xi}^\alpha\|_\infty\tag{III.142}$$

Taking now the limit as r goes to zero, since x is a point of $\hat{\Sigma}_\varepsilon\setminus B^\alpha$ satisfying (III.139), we obtain (III.133) and the lemma III.9 is proved. □

Lemma III.10. [Strong L^1 Convergence] *Assume the hypothesis of theorem III.1 are fulfilled and that we have extracted subsequences such that $\vec{\Phi}_k$ converges weakly towards $\vec{\Phi}_\infty$ in $W^{1,2}(\Sigma)$ and ν_k converges towards ν_∞ satisfying (III.66) where $\mathcal{B}:=\{a_1\cdots a_l\}$ the blow-up set. For any $\varepsilon>0$ one has*

$$\int_{\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)}|\vec{\Phi}_k-\vec{\Phi}_\infty|\ dvol_{g_{\Phi_k}}=0\quad.\tag{III.143}$$

□

Proof of lemma III.10. Let $\tilde{\Sigma}^\varepsilon$ be the subset of $\hat{\Sigma}_\varepsilon=\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)$ with full ν_∞ measure such that (III.132) holds. Let $\delta>0$. For any $x\in\tilde{\Sigma}^\varepsilon$ there exists $r_{x,\delta}>0$ and $k_{x,\delta}$ such that

$$\forall\, k\ge k_{x,\delta}\quad\int_{B_{r_x}(x)}|\vec{\Phi}_k-\vec{\Phi}_\infty|\ dvol_{g_{\Phi_k}}\le\delta\int_{B_{r_x}(x)}dvol_{g_{\Phi_k}}$$

From $(B_{r_x}(x))_{x\in\tilde{\Sigma}^\varepsilon}$ we extract an at most countable Besicovitch covering $(B_{r_i}(x_i))_{i\in I}$ of $\tilde{\Sigma}^\varepsilon$. For any $j\in\mathbb{N}$ we denote I^j the subset of I such that

$$\forall\, i\in I^j\qquad\forall\ k\ge j\qquad\int_{B_{r_i}(x_i)}|\vec{\Phi}_k-\vec{\Phi}_\infty|\ dvol_{g_{\Phi_k}}\le\delta\int_{B_{r_i}(x_i)}dvol_{g_{\Phi_k}}\quad.$$

We have $I = \cup_{j\in\mathbb{N}} I^j$ and $I^j \subset I^{j+1}$. Denoting $\mathcal{U}^j := \cup_{i\in I^j} B_{r_i}(x_i)$ and $\mathcal{V}^j := \cup_{i\in I\setminus I^j} B_{r_i}(x_i)$. We have $\mathcal{V}^{j+1} \subset \mathcal{V}^j$ and $\cap_{j\in\mathbb{N}} \mathcal{V}^j = \emptyset$. Thus, since $\tilde{\Sigma}^\varepsilon$ is of ν_∞ full measure in $\hat{\Sigma}_\varepsilon = \Sigma \setminus \cup_{l=1}^n B_\varepsilon(a_l)$ we have

$$\nu_\infty\left(\bigcup_{i\in I^j} B_{r_i}(x_i) \cap \hat{\Sigma}_\varepsilon\right) = \nu_\infty(\tilde{\Sigma}^\varepsilon) = \lim_{j\to+\infty} \nu_\infty\left(\tilde{\Sigma}^\varepsilon \setminus (\mathcal{V}^j \cap \tilde{\Sigma}^\varepsilon)\right) \tag{III.144}$$

We have for any $j \in \mathbb{N}$ and $k \geq j$

$$\begin{aligned}\int_{\tilde{\Sigma}^\varepsilon} |\vec{\Phi}_k - \vec{\Phi}_\infty|\, dvol_{g_{\Phi_k}} &\leq \int_{\mathcal{U}^j} |\vec{\Phi}_k - \vec{\Phi}_\infty|\, dvol_{g_{\Phi_k}} + \int_{\mathcal{V}^{j+1}} |\vec{\Phi}_k - \vec{\Phi}_\infty|\, dvol_{g_{\Phi_k}}\\ &\leq \delta \sum_{i\in I^j} \int_{B_{r_i}(x_i)} dvol_{g_{\Phi_k}} + \sum_{i\in I\setminus I^j} \int_{B_{r_i}(x_i)} dvol_{g_{\Phi_k}}\\ &\leq N\ \delta\ \nu_k(\Sigma) + \sum_{i\in I\setminus I^j} \nu_k(B_{r_i}(x_i))\end{aligned} \tag{III.145}$$

Since $\nu_\infty(\partial B_{r_i}(x_i)) = 0$ for all $i \in I$, we deduce that

$$\begin{aligned}&\limsup_{k\to+\infty} \int_{\tilde{\Sigma}^\varepsilon} |\vec{\Phi}_k - \vec{\Phi}_\infty|\, dvol_{g_{\Phi_k}} \leq N\ \delta\ \nu_\infty(\Sigma) + \sum_{i\in I\setminus I^j} \nu_\infty(B_{r_i}(x_i))\\ &\leq N\ \delta\ \nu_\infty(\Sigma) + \sum_{i\in I\setminus I^j} \int_{\hat{\Sigma}_\varepsilon} \sum_{i\in I\setminus I^j} \mathbf{1}_{B_{r_i}(x_i)}\ m(x)\ d\mathcal{L}^2 \leq N\ \delta\ \nu_\infty(\Sigma) + N\ \nu_\infty(\mathcal{V}^j)\end{aligned} \tag{III.146}$$

Since this holds for any $j \in \mathbb{N}$ and any $\delta > 0$ and since $\mathcal{H}^2(\hat{\Sigma}_\varepsilon \setminus \tilde{\Sigma}^\varepsilon) = 0$ we deduce (III.143) and lemma III.10 is proved. □

Lemma III.11. [Strong Bubble Tree $W^{1,2}$ Convergence] *Under the assumptions of theorem III.1, we have that for any $x_k \in \Sigma$ and r_k converging either to zero or one we can extract a subsequence such that $\vec{\Phi}_k(r_k \cdot + x_k)$ converges strongly in $W^{1,2}$ away from finitely many points.* □

Proof of lemma III.11. Since we have proved that the necks contain no energy at the limit, it suffices to prove the strong convergence of $\vec{\Phi}_k$ towards $\vec{\Phi}_\infty$ in $W^{1,2}_{loc}(\Sigma \setminus \mathcal{B}, S^3)$. The proof will also holds on any bubble so we will have completed the proof of the strong bubble tree $W^{1,2}-$convergence. In order to prove this strong $W^{1,2}_{loc}(\Sigma \setminus \mathcal{B}, S^3)$ convergence it suffices to prove that for any $\varepsilon > 0$

$$\lim_{k\to+\infty} \int_{\hat{\Sigma}_\varepsilon} |d\vec{\Phi}_k|^2_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}} = \int_{\hat{\Sigma}_\varepsilon} |d\vec{\Phi}_\infty|^2_{g_{\vec{\Phi}_\infty}}\ dvol_{g_{\vec{\Phi}_\infty}} \tag{III.147}$$

Since the conformal class of $g_{\vec{\Phi}_k}$ is assumed to be compact in the moduli space, to simplify the presentation we will assume that about every point there exists a fixed chart in which all the $\vec{\Phi}_k$ are conformal (otherwise we would have to take a sequence of conformal charts converging strongly in any C^l topology and this would just make the notation a bit heavier.)

Let $\alpha > 0$ and consider A^α and B^α as in the proof of lemma III.9. We assume that their union coincide with a union of balls from which we have extracted a Besicovitch covering $(B_{r_i}(x_i))_{i\in I}$. we then take an arbitrary point of $\hat{\Sigma}_\varepsilon \setminus (A^\alpha \cup B^\alpha)$ such that (III.132) and (III.133) hold. We also assume that x is a Lebesgue point for $\nabla\vec{\Phi}_\infty$, and that $\vec{\Phi}_\infty(x) = \vec{\Xi}^\alpha(x)$ is a regular value for $\vec{\Xi}^\alpha$ hence $\nabla\vec{\Phi}_\infty = \nabla\vec{\Xi}^\alpha$ has rank 2 at any point in $(\vec{\Xi}^\alpha)^{-1}(\vec{\Xi}^\alpha(x))$.

We also assume that

$$\lim_{r\to 0} \frac{\nu_\infty(B_r(x))}{|B_r(x)|} = m(x) \tag{III.148}$$

Because of the previous results above, all these properties are satisfied for almost every point x for which the Lebesgue representative of $\nabla\vec{\Phi}_\infty$ has rank 2.

Using the area formula we have

$$\mathcal{H}^2(\vec{\Phi}_\infty(B^\alpha))\le\int_{B^\alpha}|\nabla\vec{\Phi}_\infty|^2\ dx^2$$

We can cover the set $\vec{\Phi}_\infty(B^\alpha)$ by balls $(B^4_{\rho_i}(\vec{q}_i))_{i\in I}$ such that

$$\sum_{i\in I}\rho_i^2\le C\ \int_{B^\alpha}|\nabla\vec{\Phi}_\infty|^2\ dx^2$$

We assume these balls realize a Besicovitch covering. We denote $\mathfrak{U}^\alpha=\bigcup_{i\in I}\overline{B^4_{\rho_i}(\vec{q}_i)}$. Because of the monotonicity formula we have

$$\mu_\infty(\mathfrak{U}^\alpha)\le C\ \int_{B^\alpha}|\nabla\vec{\Phi}_\infty|^2\ dx^2 \tag{III.149}$$

Finally we also choose $\vec{\Phi}_\infty(x)\in\vec{\Phi}_\infty(\Sigma)\setminus\mathfrak{U}^\alpha$ such that

$$\lim_{\rho\to 0}\frac{\mu_\infty\left(B^4_\rho(\vec{\Phi}_\infty(x))\bigcap\mathfrak{U}^\alpha\right)}{\mu_\infty\left(B^4_\rho(\vec{\Phi}_\infty(x))\right)}=0\quad . \tag{III.150}$$

By Lebesgue Besicovitch differentiation theorem this is true for μ_∞ almost every point in $\mathbb{R}^4\setminus\mathfrak{U}^\alpha$.

We assume that $(\vec{\Xi}^\alpha)^{-1}(\vec{\Xi}^\alpha(x))\cap\mathfrak{U}^\alpha=\emptyset$. This excludes an $\mathcal{H}^2$ measure of points in K^ε less that $O(\alpha)$, indeed we have

$$\mathcal{H}^2(\vec{\Xi}^\alpha)(\mathfrak{U}^\alpha)\le\int_{\mathfrak{U}^\alpha}|\nabla\vec{\Xi}^\alpha|^2\ dx^2\le 2\ \int_{\mathfrak{U}^\alpha}|\nabla\vec{\Phi}_\infty|^2\ dx^2+2\ \int_\Sigma|\nabla\vec{\Phi}_\alpha-\nabla\vec{\Xi}^\alpha|^2\ dx^2\le C\ \alpha\quad .$$

We finally also assume that there is an approximate tangent plane to the rectifiable set K^ε at the point $\vec{\Phi}_\infty(x)$.

Without loss of generality, modulo the action of rotations we assume that $\vec{\Xi}^\alpha(x)=\vec{\Phi}_\infty(x)=(0,0,1,0)$, that $\partial_{x_1}\vec{\Xi}^\alpha(x)=\partial_{x_1}\vec{\Phi}_\infty(x)=(a,0,0,0)$ and $\partial_{x_2}\vec{\Xi}^\alpha(x)=\partial_{x_2}\vec{\Phi}_\infty(x)=(b,c,0,0)$. We have $a\,c\neq 0$ since $\nabla\vec{\Phi}_\infty$ has rank 2. Moreover the approximate tangent plane at $\vec{\Phi}_\infty(x)$ coincides with $\mathrm{Span}\{(1,0,0,0),(0,1,0,0)\}$. Observe that the existence of this approximate tangent plane and the fact that $\vec{\Xi}^\alpha(x)$ is a regular point for $\vec{\Xi}^\alpha$ forces $\mathrm{Span}\{\partial_{x_1}\vec{\Xi}^\alpha,\partial_{x_2}\vec{\Xi}\}=\{(1,0,0,0),(0,1,0,0)\}$ at any point in $(\vec{\Xi}^\alpha)^{-1}(\vec{\Xi}^\alpha(x))$.

We recall that we adopt the notation $\vec{\Phi}=(\Phi^1,\Phi^2,\Phi^3,\Phi^4)$. We first have for the third coordinate

$$\begin{aligned}\int_{B_r(x)}|\nabla\Phi^3_k|^2\ dx^2&=\int_{B_r(x)}|\Phi^3_k\,\nabla\Phi^3_k|^2\ dx^2+\int_{B_r(x)}(1-|\Phi^3_k|^2)\,|\nabla\Phi^3_k|^2\ dx^2\\&=\int_{B_r(x)}|\Phi^3_k\,\nabla\Phi^3_k|^2\ dx^2+\int_{B_r(x)}(|\Phi^3_\infty(x)|^2-|\Phi^3_k|^2)\,|\nabla\Phi^3_k|^2\ dx^2\end{aligned} \tag{III.151}$$

We have $\Phi^3_k\,\nabla\Phi^3_k=-\Phi^1_k\,\nabla\Phi^1_k-\Phi^2_k\,\nabla\Phi^2_k-\Phi^4_k\,\nabla\Phi^4_k$. Since $\Phi^i_\infty(x)=0$ for $i\neq 3$ we have then

$$\begin{aligned}\int_{B_r(x)}|\nabla\Phi^3_k|^2\ dx^2&\le\ 10\ \int_{B_r(x)}|\vec{\Phi}_k-\vec{\Phi}_\infty(x)|\ dvol_{g_{\vec{\Phi}_k}}\\&\le\ 10\ \int_{B_r(x)}|\vec{\Phi}_\infty-\vec{\Phi}_\infty(x)|\ dvol_{g_{\vec{\Phi}_k}}+\ 10\ \int_{B_r(x)}|\vec{\Phi}_\infty-\vec{\Phi}_k|\ dvol_{g_{\vec{\Phi}_k}}\end{aligned} \tag{III.152}$$

Combining this inequality with (III.132) and (III.133) we obtain

$$\lim_{r\to 0}\limsup_{k\to+\infty}\frac{\displaystyle\int_{B_r(x)}|\nabla\Phi_k^3|^2\ dx^2}{\displaystyle\int_{B_r(x)} dvol_{g_{\vec{\Phi}_k}}}=0 \quad . \tag{III.153}$$

Let $\chi^\alpha_{r,x}$ be a smooth non negative function on $\mathbb{R}^4$ supported in the ball $B^4_{2\,\|\nabla\vec{\Xi}^\alpha\|_\infty\, r}(\vec{\Phi}_\infty(x))$, identically equal to one on $B^4_{\|\nabla\vec{\Xi}^\alpha\|_\infty\, r}(\vec{\Phi}_\infty(x))$ and such that $\|d\chi^\alpha_{r,x}\|_{L^\infty(\mathbb{R}^4)}\le\ r^{-1}\ \|\nabla\vec{\Xi}^\alpha\|^{-1}_\infty$. Multiplying the 4th coordinate of equation (**??**) by $\chi^\alpha_{r,x}(\vec{\Phi}_k)\ \Phi^4_k$ and integrating over Σ gives , arguing exactly as in the proof of lemma III.1,

$$\begin{aligned}\int_{\hat{\Sigma}_\varepsilon}\chi^\alpha_{r,x}(\vec{\Phi}_k)|\nabla\Phi_k^4|^2\ dx^2=&\int_{\hat{\Sigma}_\varepsilon}\chi^\alpha_{r,x}(\vec{\Phi}_k)|\Phi_k^4|^2\ |\nabla\Phi_k^4|^2\ dx^2\\ &-\int_{\hat{\Sigma}_\varepsilon}\Phi^4_k\ \nabla(\chi^\alpha_{r,x}(\vec{\Phi}_k))\cdot\nabla\Phi^4_k+o_k(1)\end{aligned} \tag{III.154}$$

The fact that we are integrating on $\hat{\Sigma}_\varepsilon$ is possible since we can choose ε such that, as in (III.51),

$$\int_{\partial B_\varepsilon(a_l)}|\nabla\vec{\Phi}_\infty|dl\le\left[\int_{B_{2\varepsilon}(a_l)}|\nabla\vec{\Phi}_\infty|^2\ dx^2\right]^2\to 0\quad\text{ and }\quad\vec{\Phi}_k\to\vec{\Phi}_\infty\quad\text{ in }L^\infty(\partial B_\varepsilon(a_l))$$

Hence we can have chosen $\vec{\Phi}(x)$ such that all the balls $B^4_{2\,\|\nabla\vec{\Xi}^\alpha\|_\infty\, r}(\vec{\Phi}_\infty(x))$ do not intersect the cutting circles $\vec{\Phi}_\infty(\partial B_\varepsilon(a_l))$ for r small enough. Using (III.146) we have

$$\begin{aligned}\int_{\hat{\Sigma}_\varepsilon}\chi^\alpha_{r,x}(\vec{\Phi}_\infty)|\nabla\Phi_k^4|^2\ dx^2=&\int_{\hat{\Sigma}_\varepsilon}\chi^\alpha_{r,x}(\vec{\Phi}_\infty)|\Phi_\infty^4|^2\ |\nabla\Phi_k^4|^2\ dx^2\\ &-\int_{\hat{\Sigma}_\varepsilon}\Phi^4_\infty\ \sum_{j=1}^4\partial_{z_j}\chi^\alpha_{r,x}(\vec{\Phi}_\infty)\ \nabla\Phi^j_k\cdot\nabla\Phi^4_k+o_k(1)\end{aligned} \tag{III.155}$$

Observe that $\chi^\alpha_{r,x}(\vec{\Phi}_\infty)|\Phi_\infty^4|^2\le 4\ \chi^\alpha_{r,x}(\vec{\Phi}_\infty)\ \|\nabla\vec{\Xi}^\alpha\|^2_\infty\, r^2$ hence for r small enough we have

$$\chi^\alpha_{r,x}(\vec{\Phi}_\infty)|\Phi_\infty^4|^2\le 2^{-1}\ \chi^\alpha_{r,x}(\vec{\Phi}_\infty) \tag{III.156}$$

Thus

$$\begin{aligned}\int_{\hat{\Sigma}_\varepsilon}\chi^\alpha_{r,x}(\vec{\Phi}_\infty)|\nabla\Phi_k^4|^2\ dx^2\le&-2\int_{\hat{\Sigma}_\varepsilon}\Phi^4_\infty\ \sum_{j=1}^4\partial_{z_j}\chi^\alpha_{r,x}(\vec{\Phi}_\infty)\ \nabla\Phi^j_k\cdot\nabla\Phi^4_k+o_k(1)\\ \le-\,2&\int_{\hat{\Sigma}_\varepsilon\setminus(\hat{\Sigma}_\varepsilon\bigcap B^\alpha)}\Xi^{\alpha,4}\ \sum_{j=1}^4\partial_{z_j}\chi^\alpha_{r,x}(\vec{\Xi}^\alpha)\ \nabla\Phi^j_k\cdot\nabla\Phi^4_k\\ -\,2&\int_{\hat{\Sigma}_\varepsilon\bigcap B^\alpha}\Phi^4_\infty\ \sum_{j=1}^4\partial_{z_j}\chi^\alpha_{r,x}(\vec{\Phi}_\infty)\ \nabla\Phi^j_k\cdot\nabla\Phi^4_k+o_k(1)\end{aligned} \tag{III.157}$$

Since $\vec{\Xi}^\alpha$ is C^1 and since there is an approximate tangent plane to the rectifiable set K^ε and since it coincide with Span$\{(1,0,0,0),(0,1,00)\}$ we have that for every point in the pre-image of $\vec{\Xi}^\alpha(x)=\vec{\Phi}_\infty(x)$ the gradient $\nabla\Xi^{\alpha,4}=0$ hence

$$|\Xi^{\alpha,4}\ \sum_{j=1}^4\partial_{z_j}\chi^\alpha_{r,x}(\vec{\Xi}^\alpha)|\le o(1) \tag{III.158}$$

and

$$
\begin{array}{l}
\displaystyle\limsup_{k\rightarrow 0}\left|\int_{\hat{\Sigma}_\varepsilon\setminus(\hat{\Sigma}_\varepsilon\bigcap B^\alpha)}\Xi^{\alpha,4}\ \sum_{j=1}^4\partial_{z_j}\chi^\alpha_{r,x}(\vec{\Xi}^\alpha)\ \nabla\Phi^j_k\cdot\nabla\Phi^4_k\right|\\
\quad\quad\displaystyle=o(1)\ \mu_\infty(B^4_{2\,\|\nabla\vec{\Xi}^\alpha\|_\infty\,r}(\vec{\Phi}_\infty(x)))=o(r^2)
\end{array}
\tag{III.159}
$$

We have moreover

$$
|\Phi^4_\infty\ \partial_{z_j}\chi^\alpha_{r,x}(\vec{\Phi}_\infty)|\le r\ \|\nabla\vec{\Xi}^\alpha\|_\infty\|d\chi^\alpha_{r,x}\|_\infty\le 1\quad ,
\tag{III.160}
$$

hence, using fundamental properties of the convergence of Radon measure

$$
\begin{array}{l}
\displaystyle\limsup_{k\rightarrow+\infty}\left|-\,2\int_{\hat{\Sigma}_\varepsilon\bigcap B^\alpha}\Phi^4_\infty\ \sum_{j=1}^4\partial_{z_j}\chi^\alpha_{r,x}(\vec{\Phi}_\infty)\ \nabla\Phi^j_k\cdot\nabla\Phi^4_k\right|\\
\quad\displaystyle\le 8\ \limsup_{k\rightarrow+\infty}\mu_k\left(B^4_{2\,\|\nabla\vec{\Xi}^\alpha\|_\infty\,r}(\vec{\Phi}_\infty(x))\cap\vec{\Phi}_\infty(B^\alpha)\right)\\
\quad\displaystyle\le 8\ \limsup_{k\rightarrow+\infty}\sum_{i\in I}\mu_k\left(\overline{B^4_{2\,\|\nabla\vec{\Xi}^\alpha\|_\infty\,r}(\vec{\Phi}_\infty(x))}\cap\overline{B^4_{\rho_i}(\vec{q}_i)}\right)\\
\quad\displaystyle\le 8\ \sum_{i\in I}\mu_\infty\left(\overline{B^4_{2\,\|\nabla\vec{\Xi}^\alpha\|_\infty\,r}(\vec{\Phi}_\infty(x))}\cap\overline{B^4_{\rho_i}(\vec{q}_i)}\right)\\
\quad\displaystyle\le 8\ N\ \mu_\infty\left(\overline{B^4_{2\,\|\nabla\vec{\Xi}^\alpha\|_\infty\,r}(\vec{\Phi}_\infty(x))}\cap\mathfrak{U}^\alpha\right)=o\left(\mu_\infty\left(\overline{B^4_{2\,\|\nabla\vec{\Xi}^\alpha\|_\infty\,r}(\vec{\Phi}_\infty(x))}\right)\right)=o(r^2)
\end{array}
\tag{III.161}
$$

We write

$$
\int_{\hat{\Sigma}_\varepsilon}\chi^\alpha_{r,x}(\vec{\Phi}_\infty)|\nabla\Phi^4_k|^2\ dx^2=\int_{\hat{\Sigma}_\varepsilon\setminus(\hat{\Sigma}_\varepsilon\bigcap B^\alpha)}\cdots+\int_{\hat{\Sigma}_\varepsilon\bigcap B^\alpha}\cdots
\tag{III.162}
$$

We have

$$
\int_{\hat{\Sigma}_\varepsilon\setminus(\hat{\Sigma}_\varepsilon\bigcap B^\alpha)}\chi^\alpha_{r,x}(\vec{\Phi}_\infty)|\nabla\Phi^4_k|^2\ dx^2=\int_{\hat{\Sigma}_\varepsilon\setminus(\hat{\Sigma}_\varepsilon\bigcap B^\alpha)}\chi^\alpha_{r,x}(\vec{\Xi}^\alpha)|\nabla\Phi^4_k|^2\ dx^2
\tag{III.163}
$$

Since $\chi^\alpha_{r,x}$ is identically equal to one on $B^4_{\|\nabla\vec{\Xi}^\alpha\|_\infty\,r}(\vec{\Phi}_\infty(x))$, $\chi^\alpha_{r,x}(\vec{\Xi}^\alpha)$ is identically equal to one on $B_r(x)$ and we have

$$
\int_{\hat{\Sigma}_\varepsilon\setminus(\hat{\Sigma}_\varepsilon\bigcap B^\alpha)}\chi^\alpha_{r,x}(\vec{\Phi}_\infty)|\nabla\Phi^4_k|^2\ dx^2\ge\int_{B_r(x)\setminus(B_r(x)\cap B^\alpha)}|\nabla\Phi^4_k|^2\ dx^2
\tag{III.164}
$$

Combining (III.159), (III.161), (III.164) we obtain

$$
\limsup_{k\rightarrow+\infty}\int_{B_r(x)\setminus(B_r(x)\cap B^\alpha)}|\nabla\Phi^4_k|^2\ dx^2=o(r^2)
\tag{III.165}
$$

We have also, since again $\nu_\infty(\partial(B_r(x)\cap\overline{B_{r_i}(x_i)}))=0$

$$
\begin{array}{l}
\displaystyle\limsup_{k\rightarrow+\infty}\int_{B_r(x)\cap B^\alpha}|\nabla\Phi^4_k|^2\ dx^2\le\limsup_{k\rightarrow+\infty}\sum_{i\in I}\int_{B_r(x)\cap\overline{B_{r_i}(x_i)}}|\nabla\Phi^4_k|^2\ dx^2\\
\quad\displaystyle\le\sum_{i\in I}\nu_\infty(B_r(x)\cap\overline{B_{r_i}(x_i)})\le\ N\ \nu_\infty(B_r(x)\cap B^\alpha)=o(\nu_\infty(B_r(x))
\end{array}
\tag{III.166}
$$

since x has been chosen such that. Observe that $\nu_\infty(B_r(x))\ge 2^{-1}\ \int_{B_r(x)}|\nabla\vec{\Phi}|^2\ dx^2=\frac{\pi}{2}(a^2+b^2+c^2)r^2+o(r^2)$. Since we are assuming $a\ne 0$ and $(b,c)\ne(0,0)$ we have

$$
\limsup_{k\rightarrow+\infty}\int_{B_r(x)}|\nabla\Phi^4_k|^2\ dx^2=o(\nu_\infty(B_r(x)))
\tag{III.167}
$$

We have proved also that

$$\limsup_{k\to+\infty}\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}|\nabla\Phi^4_k|^2\ dx^2=o(\mu_\infty(B^4_\rho(\vec{\Phi}_\infty(x)))) \tag{III.168}$$

Combining (III.153) and (III.167) we have then

$$\lim_{r\to 0}\lim_{k\to+\infty}\frac{\displaystyle\int_{B_r(x)}[|\nabla\vec{\Phi}^1_k|^2+|\nabla\vec{\Phi}^2_k|^2]\ dx^2}{\displaystyle\int_{B_r(x)}|\nabla\vec{\Phi}_k|^2\ dx^2}=1 \tag{III.169}$$

as well as

$$\lim_{\rho\to 0}\limsup_{k\to+\infty}\frac{\displaystyle\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}[|\nabla\vec{\Phi}^1_k|^2+|\nabla\vec{\Phi}^2_k|^2]\ dx^2}{\displaystyle\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}|\nabla\vec{\Phi}_k|^2\ dx^2}=1 \tag{III.170}$$

Since $\vec{\Phi}_k$ is conformal we have then

$$\lim_{r\to 0}\lim_{k\to+\infty}\frac{\displaystyle\int_{B_r(x)}2\ |\partial_{x_1}\vec{\zeta}_k\wedge\partial_{x_2}\vec{\zeta}_k|\ dx^2}{\displaystyle\int_{B_r(x)}|\nabla\vec{\Phi}_k|^2\ dx^2}=1 \tag{III.171}$$

where $\vec{\zeta}_k:=(\Phi^1_k,\Phi^2_k)$ and, combining (III.169) with (III.171)

$$\lim_{r\to 0}\lim_{k\to+\infty}\frac{\displaystyle\int_{B_r(x)}2\ |\partial_{x_1}\vec{\zeta}_k\wedge\partial_{x_2}\vec{\zeta}_k|\ dx^2}{\displaystyle\int_{B_r(x)}|\nabla\vec{\zeta}_k|^2\ dx^2}=1 \tag{III.172}$$

The difficulty at this stage is that we can not remove yet the absolute values inside the upper integral of (III.172). The rest of the argument consists in proving that indeed one can remove the absolute value and the tangent plane $\vec{\Xi}^\alpha_*T_x\Sigma$ is asymptotically covered exactly one time by the limiting varifold associated to the current $(\vec{\Phi}_k)_*[B_r(x)]$. This will imply the strong convergence of $\nabla\vec{\Phi}_k$ towards $\nabla\vec{\Phi}_\infty=\nabla\vec{\Xi}^\alpha$ in $L^2(\Sigma^\alpha_\varepsilon)$, where we recall the notation $\Sigma^\alpha_\varepsilon:=\hat{\Sigma}\setminus((\cup_{l=1}^n B_\varepsilon(a_l))\bigcup(A^\alpha\cup B^\alpha))$. We formulate that differently. Denote by $\tilde{G}_2(S^3)$ to be the Grassmanian of <u>oriented</u> 2 dimensional planes of the tangent bundle to S^3, TS^3. The image by $\vec{\Phi}_k$ of $\Sigma^\alpha_\varepsilon$, induces an <u>oriented integer rectifiable varifold</u> (see [13]) $\mathbf{v}^\alpha_{\varepsilon,k}$ where the choice of orientation of the tangent plane is taken to be the one induced by the push forward by the immersion $\vec{\Phi}_k$ of the one fixed on Σ. The sequence of oriented varifolds $\mathbf{v}_k$ converges to a limiting oriented varifold $\mathbf{v}_\infty$ which is a limiting measure on the oriented 2-Grassmanian $\tilde{G}_2(S^3)$. Denote by $T^+\Sigma$ the tangent bundle to Σ with the positive orientation and $T^-\Sigma$ the same tangent bundle but with the opposite orientation. We see $\vec{\Xi}^\alpha_*(T^+\hat{\Sigma}^\alpha_\varepsilon\cup T^-\hat{\Sigma}^\alpha_\varepsilon)$ as a measurable subset of $\tilde{G}_2(S^3)$. With these notations, the identity (III.169) is in fact equivalent to

$$\mathbf{v}_\infty(\tilde{G}_2(S^3)\setminus\vec{\Xi}^\alpha_*(T^+\hat{\Sigma}^\alpha_\varepsilon\cup T^-\hat{\Sigma}^\alpha_\varepsilon)=0 \tag{III.173}$$

The goal is now to prove

$$\mathbf{v}_\infty(\tilde{G}_2(S^3)\setminus\vec{\Xi}^\alpha_*(T^+\hat{\Sigma}^\alpha_\varepsilon)=0 \tag{III.174}$$

in order to be able to remove the absolute values in the upper integral of (III.172) that will ultimately imply the strong convergence of $\nabla\vec{\Phi}_k$ towards $\nabla\vec{\Phi}_\infty$.

Proof of (III.174). To simplify the presentation, in order not to have to localize in the domain that would make the notations heavier, we shall assume that

$$(\vec{\Xi}^\alpha)^{-1}\left(\vec{\Xi}^\alpha(x)\right)=\{x\}\quad . \tag{III.175}$$

For $i=1\cdots 4$ we denote by $\nabla^{\Sigma_k}y^i$ the vector-field tangent to $\Phi_k(\Sigma)$ given by the projection of the $i-$th canonical vector of $\mathbb{R}^4$ onto $(\vec{\Phi}_k)_*T\Sigma$. We also denote $*_k\nabla^{\Sigma_k}y^i$ the rotation by $\pi/2$ of this vector in the tangent plane to $\Phi_k(\Sigma)$, taking into account the orientation given by the push-forward by $\vec{\Phi}_k$ of the one we fixed on Σ. Denote by $(\vec{\varepsilon}_i)_{i=1\cdots 4}$ the canonical basis of $\mathbb{R}^4$. The identity (III.170) implies that

$$\limsup_{k\rightarrow+\infty}\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}\operatorname{dist}\left(\frac{\partial_{x_1}\vec{\Phi}_k\wedge\partial_{x_2}\vec{\Phi}_k}{|\partial_{x_1}\vec{\Phi}_k\wedge\partial_{x_2}\vec{\Phi}_k|},\pm\,\vec{\varepsilon}_1\wedge\vec{\varepsilon}_2\right)\ |\nabla\vec{\Phi}_k|^2\ dx^2=o(\rho^2) \tag{III.176}$$

recall $\mu_\infty(B^4_\rho(\vec{\Phi}_\infty(x)))\simeq\rho^2$. This also implies

$$\forall\ i=1,2\quad\limsup_{k\rightarrow+\infty}\int_{B^4_\rho(\vec{\Phi}_\infty(x))}|\nabla^{\Sigma_k}y_i-\vec{\varepsilon}_i|\ d\mathcal{H}^2\llcorner\vec{\Phi}_k(\Sigma)=o(\rho^2)\quad . \tag{III.177}$$

For $(\partial_{x_1}\vec{\Phi}_k\wedge\partial_{x_2}\vec{\Phi}_k)\cdot(\vec{\varepsilon}_1\wedge\vec{\varepsilon}_2)\neq 0$ we denote $J_k=\operatorname{sign}\left((\partial_{x_1}\vec{\Phi}_k\wedge\partial_{x_2}\vec{\Phi}_k)\cdot(\vec{\varepsilon}_1\wedge\vec{\varepsilon}_2)\right)$ otherwize we simply take $J_k=0$. Identity (III.176) and (III.177) imply

$$\limsup_{k\rightarrow+\infty}\int_{B^4_\rho(\vec{\Phi}_\infty(x))}[|*_k\nabla^{\Sigma_k}y_1-J_k\,\varepsilon_2|+|*_k\nabla^{\Sigma_k}y_2+J_k\,\varepsilon_1|]\ d\mathcal{H}^2\llcorner\vec{\Phi}_k(\Sigma)=o(\rho^2) \tag{III.178}$$

Let $\vec{T}_k^\rho$ and $\vec{T}_k^{r,x}$ be the following vector-valued one dimensional currents

$$\forall\ \alpha\in\Omega^1(\mathbb{R}^4)\qquad\left\langle\vec{T}^\rho_k,\alpha\right\rangle:=\int_{B^4_\rho(\vec{\Phi}_\infty(x))\cap\vec{\Phi}_k(\Sigma)}\alpha\wedge *_kd\vec{y}=\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}\vec{\Phi}_k^*\alpha\wedge *\,d\vec{\Phi}_k$$

Let φ be a smooth function in $C^\infty_0(B^4_1(0))$ such that $\int_{\mathbb{R}^4}\varphi(y)\ dy^4=1$. Denote $\varphi_{\sigma_k}:=\sigma_k^{-4/p}\varphi(\cdot/\sigma_k^{1/p})$. We recall the definition of the σ_k-*smoothing* $\varphi_{\sigma_k}\star\vec{T}^\rho_k$ of the current $\vec{T}^\rho_k$ (see [10] 4.1.2)

$$\forall\ \alpha\in\Omega^1(\mathbb{R}^4)\qquad\left\langle\varphi_{\sigma_k}\star\vec{T}^\rho_k,\alpha\right\rangle:=\int_{B^4_\rho(\vec{\Phi}_\infty(x))\cap\vec{\Phi}_k(\Sigma)}(\varphi_{\sigma_k}\star\alpha)\wedge *_kd\vec{y}$$

where $\alpha_{\sigma_k}:=\varphi_{\sigma_k}\star\alpha$ denotes the following convolution operation

$$\alpha_{\sigma_k}=\varphi_{\sigma_k}\star\alpha:=\int_{\mathbb{R}^4}\varphi_{\sigma_k}(-z)\ \tau_z^*\alpha\ dz^4$$

where $\tau_z(y)=y+z$. We shall use the following lemma

Lemma III.12. [Convergence of the σ_k-Approximation of $\vec{T}^\rho_k$.] *Under the previous notations we have*

$$\limsup_{k\rightarrow+\infty}\ \sup_{supp(\phi)\subset B^4_\rho(\vec{\Phi}_\infty(x))\ ;\ \|d\phi\|_\infty\le 1}\left\langle\vec{T}^\rho_k-\varphi_{\sigma_k}\star\vec{T}^\rho_k,d\phi\right\rangle=0 \tag{III.179}$$

□

Proof of lemma III.12. Let ϕ be a lipschitz function supported in $B^4_\rho(\vec{\Phi}_\infty(x))$ with $\|d\phi\|_\infty\le 1$. We have

$$\left\langle\vec{T}^\rho_k-\varphi_{\sigma_k}\star\vec{T}^\rho_k,d\phi\right\rangle=\int_{\mathbb{R}^4}dz\ \varphi_{\sigma_k}(-z)\int_{B^4_\rho(\vec{\Phi}_\infty(x))\cap\vec{\Phi}_k(\Sigma)}(d\phi-\tau_z^\ast d\phi)\wedge\ast_k d\vec{y}$$

$$=-\int_{\mathbb{R}^4}dz\ \varphi_{\sigma_k}(-z)\int_{B^4_\rho(\vec{\Phi}_\infty(x))\cap\vec{\Phi}_k(\Sigma)}(\phi(y)-\phi(y+z))\wedge d\ast_k\,d\vec{y}$$

Using the fact that $\|d\phi\|_\infty\le 1$ and that φ_{σ_k} is supported in $B^4_{\sigma_k^{1/p}}(0)$, we have

$$\left|\left\langle\vec{T}^\rho_k-\varphi_{\sigma_k}\star\vec{T}^\rho_k,d\phi\right\rangle\right|\le\sigma_k^{1/p}\ \int_\Sigma[|\vec{H}_k|+1]\ dvol_{g_{\vec{\Phi}_k}}$$

$$\le\left[\sigma_k^2\ \int_\Sigma[|\vec{H}_k|^{2p}+1]\ dvol_{g_{\vec{\Phi}_k}}\right]^{1/2p}\ \text{Area}(\vec{\Phi}_k(\Sigma))^{1-1/2p}=o(1)$$

This concludes the proof of lemma III.12. $\Box$

Lemma III.13. [Asymptotic Vanishing of the Boundary of $\vec{T}^\rho_k$ in $B^4_\rho(\vec{\Phi}_\infty(x))$] *Under the previous notations we have*

$$\limsup_{k\rightarrow+\infty}\sup_{supp(\phi)\subset B^4_\rho(\vec{\Phi}_\infty(x))\ ;\ \|d\phi\|_\infty\le1}\left\langle\vec{T}^\rho_k,d\phi\right\rangle=o(\rho^2)\tag{III.180}$$

and for the two first directions $i=1,2$ we have

$$\limsup_{k\rightarrow+\infty}\sup_{supp(\phi)\subset B^4_\rho(\vec{\Phi}_\infty(x))\ ;\ \|d\phi\|_\infty\le1}\vec{\varepsilon}_i\cdot\left\langle\vec{T}^\rho_k,d\phi\right\rangle=O(\rho^4)\tag{III.181}$$

$\Box$

Proof of lemma III.13. Because of (III.170) it suffices to prove (III.181) for the components along $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$ only. Because of the previous lemma it suffices to prove (III.180) where $\vec{\varepsilon}_i\cdot\vec{T}^\rho_k$ for $i=1,2$ is replaced by $\vec{\varepsilon}_i\cdot\varphi_{\sigma_k}\star\vec{T}^\rho_k$. We assume $\phi(\vec{\Phi}_\infty(x))=0$ in such a way that $\|\phi\|_\infty\le\rho$. We have

$$\left\langle\varphi_{\sigma_k}\star\vec{T}^\rho_k,d\phi\right\rangle=\int_{B^4_\rho(\vec{\Phi}_\infty(x))\cap\vec{\Phi}_k(\Sigma)}d\left(\varphi_{\sigma_k}\star\phi\right)\wedge\ast_k d\vec{y}\tag{III.182}$$

Integrating by parts and using (II.35) we have, omitting to write explicitly the subscript k,

$$\begin{array}{l}
\left\langle\varphi_\sigma\star\vec{T}^\rho,d\phi\right\rangle=\displaystyle\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}\nabla\left(\varphi_\sigma\star\phi(\vec{\Phi})\right)\ \cdot\sigma^2\ f^p\ \nabla\vec{\Phi}\ dx^2\\[2mm]
\displaystyle-2\ p\ \sigma^2\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}e^{-2\lambda}\ \nabla\left(\varphi_\sigma\star\phi(\vec{\Phi})\right)\cdot\left[\nabla\left[f^{p-1}\,\mathbb{I}^0_{11}\right]+(\nabla)^\perp\left[f^{p-1}\,\mathbb{I}^0_{12}\right]\ \right]\vec{n}\ dx^2\\[2mm]
\displaystyle-2\ p\ \sigma^2\ \int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}\nabla\left(\varphi_\sigma\star\phi(\vec{\Phi})\right)\cdot\nabla\left[f^{p-1}\ \vec{H}\right]\ dx^2\\[2mm]
\displaystyle+2\ p\ \sigma^2\ \int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}\nabla\left(\varphi_\sigma\star\phi(\vec{\Phi})\right)\cdot\left[\ f^{p-1}\,H\ \nabla\vec{n}-\ e^{-2\lambda}\ f^{p-1}\ \left\langle\nabla\vec{n}\dot{\otimes}\nabla\vec{n};\nabla\vec{\Phi}\right\rangle\right]\ dx^2\\[2mm]
\displaystyle-\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}\varphi_\sigma\star\phi(\vec{\Phi})\ \left(\left[1+\sigma^2\ (1-p)\,f^p+\,p\,\sigma^2\ f^{p-1}\right]\ \vec{\Phi}\,|\nabla\vec{\Phi}|^2-4\,p\,\sigma^2\ f^{p-1}\ \vec{H}\right)\ dx^2
\end{array}\tag{III.183}$$

Observe that $\|\partial^2_{y_iy_j}(\varphi_\sigma\star\phi)\|_\infty\le\sigma^{-1/p}$ hence integrating by parts $\overline{\nabla}$ and $(\overline{\nabla})^\perp$ in the second and third line of (III.183) as well as integrating by parts ∇ in the fourth line of (III.183) and using (III.15) as in the proof of the monotonicity formula, we obtain that all the terms in the first, second, third and fourth lines of the r.h.s. of (III.183) vanish as k goes to $+\infty$. In the fifth line only the term $\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}\varphi_\sigma\star\phi(\vec{\Phi})\ \vec{\Phi}\,|\nabla\vec{\Phi}|^2dx^2$ is not necessarily converging towards 0. Since we are considering the first and second canonical directions and since Φ^1 and Φ^2 are $O(\rho)$ in $\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))$ and since $\|\phi\|_\infty\le\rho$ we obtain (III.181) and lemma III.13 is proved. $\Box$

Proof of lemma III.11 continued. Denote $\vec{\Phi}_k':=(\Phi^3_k,\Phi^4_k)$. By taking $\phi(y):=h(y_1,y_2)\ \chi_\rho(y_3,y_4)$ where χ_ρ is identically equal to ρ on $B^2_\rho(1,0)$, is non negative, supported in $B^2_{2\rho}(1,0)$, we have for $i=1,2$

$$\limsup_{k\rightarrow+\infty}\ \sup_{\text{supp}(h)\subset B^2_\rho(\vec{\Phi}_\infty(x))\ ;\ \|dh\|_\infty\le\rho^{-1}}\ \vec{\varepsilon}_i\cdot\int_{B^4_{4\rho}(\vec{\Phi}_\infty(x))}\ast_k d\vec{y}\wedge(\chi_\rho\ dh+h\,d\chi_\rho)=O(\rho^4)\tag{III.184}$$

Because of the existence of an approximate tangent plane at $\vec{\Phi}_\infty(x)$, which is equal to Span$\{\vec{\varepsilon}_1,\vec{\varepsilon}_2\}$, the asymptotic mass of the current in $B^4_{4\rho}(\vec{\Phi}_\infty(x))$ contained in the support of $d\chi_\rho$ which is included in $B^2_{4\rho}(0,0)\times(B^2_{2\,\rho}(1,0)\setminus B^2_\rho(1,0))$ is a $o(\rho^2)$. Hence we deduce for $i=1,2$

$$\limsup_{k\rightarrow+\infty}\ \sup_{\text{supp}(h)\subset B^2_\rho(0,0)\ ;\ \|dh\|_\infty\le\rho^{-1}}\int_{B^2_\rho(0,0)\times B^2_\rho(1,0)}\partial_{y_i}h\ d\mathcal{H}^2\llcorner\vec{\Phi}_k(\Sigma)=o(\rho^2)\tag{III.185}$$

This implies, using (III.169),

$$\limsup_{k\rightarrow+\infty}\ \sup_{\text{supp}(h)\subset B^2_\rho(0,0)\ ;\ \|dh\|_\infty\le\rho^{-1}}\int_{B^2_\rho(0,0)}N_k(y)\ \partial_{y_i}h\ d\mathcal{L}^2=o(\rho^2)\tag{III.186}$$

where $N_k(y)$ is the number of pre-images of $y=(y_1,y_2)$ by $\vec{\zeta}_k$. Since $M(B^2_\rho(0,0)\cap\vec{\zeta}_k(\Sigma))\simeq\rho^2$ we then have

$$\limsup_{k\rightarrow+\infty}\ \sup_{\text{supp}(h)\subset B^2_\rho(0,0)\ ;\ \|dh\|_\infty\le\rho^{-1}}\frac{\displaystyle\int_{B^2_\rho(0,0)}N_k(y)\ \partial_{y_i}h\ dy^2}{\displaystyle\int_{B^2_\rho(0,0)}N_k(y)\ dy^2}=o_\rho(1)\tag{III.187}$$

This implies that the number of pre-images is asymptotically constant. We can move the absolute values in the upper integral of (III.172) from inside the integration operation to outside and this proves (III.174). We have then established (III.147) away from a set of Lebesgue measure as small as we wish in $\hat{\Sigma}_\varepsilon$. Since ν_∞ is absolutely continuous with respect to the Lebesgue measure on $\hat{\Sigma}_\varepsilon$ we have then (III.147). $\Box$

We introduce now the following definition

Definition III.8. *Let Σ be a closed riemann surface and N^n be a closed sub-manifold $N^n\subset\mathbb{R}^m$. A map $\vec{\Phi}\in W^{1,2}(\Sigma,N^n)$ is called "target harmonic" if for almost every[6] domain $\Omega\subset\Sigma$ and any smooth function F supported in the complement of an open neighborhood of $\vec{\Phi}(\partial\Omega)$ we have*

$$\int_\Omega\left\langle d(F(\vec{\Phi})),d\vec{\Phi}\right\rangle_{g_0}-F(\vec{\Phi})\ A(\vec{\Phi})(d\vec{\Phi},d\vec{\Phi})_{g_0}\ dvol_{g_0}$$

where g_0 is an arbitrary metric whose conformal structure is the one given by the riemann surface Σ, $<\cdot,\cdot>_{g_0}$ denotes the scalar product in $T^*\Sigma$ issued from g_0, $A(\vec{y})$ denotes the second fundamental form of $N^n\subset\mathbb{R}^m$ at the point $\vec{y}$ $\Box$

[6] The notion of *almost every domain* means for every smooth domain Ω and any smooth function f such that $f^{-1}(0)=\partial\Omega$ and $\nabla f\neq0$ on $\partial\Omega$ then for almost every t close enough to zero and regular value for f one considers the domains contained in Ω or containing Ω and bounded by $f^{-1}(\{t\})$.

Lemma III.14. [Convergence to a Bubble Tree of conformal "target harmonic" maps] *Under the assumptions of theorem III.1, we have that for any $x_k \in \Sigma$ and r_k converging either to zero or one, one can extract a subsequence such that $\vec{\Phi}_k(r_k \cdot + x_k)$ converges strongly in $W^{1,2}$ away from finitely many points towards a conformal target harmonic map into S^3.* $\Box$

Proof of lemma III.14. Because of the strong $W^{1,2}$ convergence away from $\mathcal{B}$ we have the following varifold or Radon measure convergence in the Grassman space $\tilde{G}_2(S^3)$ of oriented two planes from TS^3

$$\lim_{k\to+\infty} \mathbf{v}_{\varepsilon,k} = \mathbf{v}_{\varepsilon,\infty}$$

where we recall the notations : $\mathbf{v}_{\varepsilon,k}$ and $\mathbf{v}_{\varepsilon,\infty}$ are the oriented integer rectifiable varifolds given respectively by the image of $\Sigma \setminus \bigcup_{l=1}^n B_\varepsilon(a_l)$ by $\vec{\Phi}_k$ and the image of $\Sigma \setminus \bigcup_{l=1}^n B_\varepsilon(a_l)$ by $\vec{\Phi}_\infty$.

We are now proving that the induced unoriented varifold from the oriented one $\mathbf{v}_\infty$, given by the image by $\vec{\Phi}_\infty$ of the whole Σ, is a stationary integer rectifiable varifold. More precisely we are going to prove that $\vec{\Phi}_\infty$ realizes a *conformal target harmonic map* into S^3. If $\mathcal{B} = \emptyset$ this has been proved already in lemma III.1. We are now considering the case where $\mathcal{B} = \{a_1 \cdots a_n\} \neq \emptyset$. The remaining task is to "dissociate" $\mathbf{v}_\infty$ from the rest of the limiting varifold, the whole varifold being stationary. Precisely we are going to prove that this dissociation happens at isolated points in S^3 and proceed to a "point removability argument".

As in the proof of (III.51), for any $\rho > 0$ we can chose $r \in [\rho, 2\rho]$ such that, modulo extraction of a subsequence, for all $l = 1 \cdots n$

$$\begin{aligned}\lim_{k\to+\infty} \|\vec{\Phi}_k(x) - \vec{\Phi}_k(y)\|^2_{(L^\infty(\partial B_r(a_l)))^2} &= \|\vec{\Phi}_\infty(x) - \vec{\Phi}_\infty(y)\|^2_{(L^\infty(\partial B_r(a_l)))^2} \\ &\le \left[\int_{\partial B_r(a_l)} |\nabla\vec{\Phi}_\infty|\ dl \le\right]^2 \le 8\pi \int_{B_{2\rho}(a_l)} |\nabla\vec{\Phi}_\infty|^2\ dx^2\end{aligned} \tag{III.188}$$

To simplify the presentation we assume

$$s_\rho := \sup_{l=1\cdots n} \sqrt{8\pi \int_{B_{2\rho}(a_l)} |\nabla\vec{\Phi}_\infty|^2\ dx^2} > 0$$

Let $\vec{q}_{l,\rho} = \vec{\Phi}_\infty(x^l_\rho)$ for some arbitrary choice of $x^l_\rho \in \partial B_r(a_l)$. We have then for k large enough

$$\text{supp}\ \partial\left((\vec{\Phi}_k)_*\left[\Sigma \setminus \bigcup_{l=1}^n B_r(a_l)\right]\right) \subset \bigcup_{l=1}^n B^4_{s_\rho}(\vec{q}_{l,\rho}) \tag{III.189}$$

Let $\vec{X}$ be a smooth vectorfield in TS^3 such that $\text{supp}\,\vec{X} \subset S^3 \setminus \bigcup_{l=1}^n B^4_{s_\rho}(\vec{q}_{l,\rho})$. We can now follow each step of the proof of the stationarity of the limiting varifold in lemma III.1 and obtain that

$$\lim_{k\to+\infty} \int_{\Sigma\setminus\bigcup_{l=1}^n B_r(a_l)} \left[\sum_{i=1}^4 \left\langle \partial_{y_i}\vec{X}(\vec{\Phi}_k)\ \nabla\Phi^i_k ; \nabla\vec{\Phi}_k\right\rangle - \vec{X}(\vec{\Phi}_k)\cdot\vec{\Phi}_k\ |\nabla\vec{\Phi}_k|^2\right]\ dx^2 = 0 \tag{III.190}$$

which implies, using the strong $W^{1,2}-$convergence away from $\mathcal{B} = \{a_1 \cdots a_n\}$,

$$\int_{\Sigma\setminus\bigcup_{l=1}^n B_r(a_l)} \left[\sum_{i=1}^4 \left\langle \partial_{y_i}\vec{X}(\vec{\Phi}_\infty)\ \nabla\Phi^i_\infty ; \nabla\vec{\Phi}_\infty\right\rangle - \vec{X}(\vec{\Phi}_\infty)\cdot\vec{\Phi}_\infty\ |\nabla\vec{\Phi}_\infty|^2\right]\ dx^2 = 0 \tag{III.191}$$

Since $\vec{\Phi}_\infty$ is conformal due to the strong $W^{1,2}$ convergence, (III.191) implies that

$$\left|(\vec{\Phi}_\infty)_*[\Sigma]\llcorner\left(\mathbb{R}^4\setminus\bigcup_{l=1}^n B^4_{s_\rho}(\vec{q}_{l,\rho})\right)\right|$$

realizes an integer rectifiable stationary varifold in $S^3\setminus\bigcup_{l=1}^n B^4_{s_\rho}(\vec{q}_{l,\rho})$. We chose a sequence of radii $\rho_k\to 0$ such that

$$\forall\ l=1\cdots n\qquad\quad \vec{q}_{l,\rho_k}\to\vec{q}_{l,0}\in S^3\quad .$$

Since $s_{\rho_k}\to 0$, $(\vec{\Phi}_\infty)_*[\Sigma]$ is stationary in $S^3\setminus\{\vec{q}_{1,0}\cdots\vec{q}_{n,0}\}$. Let $\chi_\delta(t)=\chi(t/\delta)$ where $\chi\in C^\infty_0([0,2],\mathbb{R}_+)$, χ is identically equal to one on $[0,1]$. For any arbitrary smooth vector field $\vec{X}$ from $\Gamma(TS^3)$ we proceed to the following decomposition :

$$\vec{X}(\vec{q})=\sum_{l=1}^n\chi_\delta(|\vec{q}-\vec{q}_{l,0}|)\ \vec{X}+\vec{X}_\delta(\vec{q})\quad\text{where}\quad \vec{X}_\delta(\vec{q}):=\left[1-\sum_{l=1}^n\chi_\delta(|\vec{q}-\vec{q}_{l,0}|)\ \right]\ \vec{X}$$

Since $\mathrm{Supp}(\vec{X}_\delta)\subset\mathbb{R}^4\setminus\bigcup_{l=1}^n B^4_\delta(\vec{q}_{l,0})$ we have

$$\int_\Sigma\left[\sum_{i=1}^4\left\langle\partial_{y_i}\vec{X}_\delta(\vec{\Phi}_\infty)\ \nabla\Phi^i_\infty;\nabla\vec{\Phi}_\infty\right\rangle-\vec{X}_\delta(\vec{\Phi}_\infty)\cdot\vec{\Phi}_\infty\ |\nabla\vec{\Phi}_\infty|^2\right]\ dx^2=0 \tag{III.192}$$

and we have

$$\begin{array}{l}\displaystyle\left|\int_\Sigma\left[\sum_{i=1}^4\left\langle\partial_{y_i}(\vec{X}-\vec{X}_\delta)(\vec{\Phi}_\infty)\ \nabla\Phi^i_\infty;\nabla\vec{\Phi}_\infty\right\rangle-(\vec{X}-\vec{X}_\delta)(\vec{\Phi}_\infty)\cdot\vec{\Phi}_\infty\ |\nabla\vec{\Phi}_\infty|^2\right]\ dx^2\right|\\[4mm]\displaystyle\qquad\le\|\vec{X}\|_\infty\ \frac{1}{\delta}\sum_{l=1}^n\mu_\infty(B^4_{2\,\delta}(\vec{q}_{l,0}))+\|\nabla\vec{X}\|_\infty\ \sum_{l=1}^n\mu_\infty(B^4_{2\,\delta}(\vec{q}_{l,0}))=O(\delta)\end{array} \tag{III.193}$$

where we are using the monotonicity formula. Combining (III.192) and (III.193) with $\delta\to 0$ we obtain that

$$\int_\Sigma\left[\sum_{i=1}^4\left\langle\partial_{y_i}\vec{X}(\vec{\Phi}_\infty)\ \nabla\Phi^i_\infty;\nabla\vec{\Phi}_\infty\right\rangle-\vec{X}(\vec{\Phi}_\infty)\cdot\vec{\Phi}_\infty\ |\nabla\vec{\Phi}_\infty|^2\right]\ dx^2=0\quad . \tag{III.194}$$

What we have done for the whole Σ can be done for any subdomain Ω assuming that the support of $\vec{X}$ is contained in a complement of an open neighborhood of $\vec{\Phi}_\infty(\partial\Omega)$. We deduce that $\vec{\Phi}_\infty$ is target harmonic from Σ into S^3. The conformality is a consequence of the strong bubble tree $W^{1,2}$ convergence proved in lemma III.11 and lemma III.14 is proved. □

IV The proof of theorem I.1.

We consider the general case where $(\Sigma,g_{\vec{\Phi}_k})$ possibly degenerate in the moduli space. Modulo extraction of a subsequence, following Deligne-Mumford compactification described in section II of [Ri4] we have a "splitting" of the original surface into collars, called also "thin parts" and and a Nodal riemann surface $\tilde{\Sigma}$ called also "thick part". The parts of the collars that contain no bubbles can be treated exactly as the necks in lemma III.6. The "thick parts" as well as the "bubbles" formed either in the thick parts or in the collars can be treated exactly as the surface Σ in the compact case presented in the previous section. The regularity of the limit is a consequence of the main result in [30]. So we deduce theorem I.1.

A Appendix

Lemma A.1. *There exists a universal number $\varepsilon_0(m) > 0$ such that, for any $\vec{\Phi}$ smooth immersion of a smooth surface with boundary Σ into $B_2^m(0) \setminus B_1^m(0)$ and satisfying*

$$Area(\vec{\Phi}(\Sigma)) < 3\,\pi \tag{A.1}$$

and

$$\forall\, r \in (1,2) \quad \vec{\Phi}(\Sigma) \cap \partial B_r^m(0) \neq \emptyset \quad \text{ and } \quad \vec{\Phi}(\partial\Sigma) \subset \partial\,(B_2^m(0) \setminus B_1^m(0)) \quad , \tag{A.2}$$

then

$$\int_\Sigma |d\vec{n}|^2_{g_{\vec{\Phi}}}\ dvol_{\vec{\Phi}} \geq \varepsilon_0(m) \quad . \tag{A.3}$$

□

Proof of lemma A.1. We argue by contradiction. We consider a sequence Σ_k and $\vec{\Phi}_k$ such that

$$\text{Area}(\vec{\Phi}_k(\Sigma_k)) < 3\,\pi \tag{A.4}$$

such that

$$\forall\, r \in (1,2) \quad \vec{\Phi}_k(\Sigma_k) \cap \partial B_r^m(0) \neq \emptyset \quad \text{ and } \quad \vec{\Phi}_k(\partial\Sigma_k) \subset \partial\,(B_2^m(0) \setminus B_1^m(0)) \quad , \tag{A.5}$$

and

$$\lim_{k\to+\infty} \int_{\Sigma_k} |d\vec{n}|^2_{g_{\vec{\Phi}_k}}\ dvol_{\vec{\Phi}_k} = 0 \quad . \tag{A.6}$$

Let V_k be the oriented varifold associated to the immersion of $\vec{\Phi}_k$ with L^2-bounded second fundamental form (see [13]). Using theorem 3.1 and 5.3.2 of [13], modulo extraction of a subsequence V_k varifold converges to an integer oriented varifold V_∞ with generalized second fundamental form equal to zero and without boundary in $B_2(0) \setminus B_1(0)$. V_∞ is then stationary and included in an at most countable union of 2-planes. Using the constancy theorem [33] we deduce that V_∞ is an oriented varifold given by at most countably many intersections of 2-planes with the annulus $B_2(0) \setminus B_1(0)$ with locally constant integer multiplicities. We claim that the intersection between the closed set given by the support of V_∞ and $\partial B_r(0) \times G_2(\mathbb{R}^m)$ is non empty for any $r \in (1,2)$. Indeed, from the assumption (A.5), using Simon's monotonicity formula, for any $r \in (1,2)$ and $0 < \rho < \min\{2-r, r-1\}$, there exists $x_k^r \in \partial B_r(0)$ such that

$$\frac{2\pi}{3}\rho^2 \leq M\left(\vec{\Phi}_k(\Sigma_k) \cap B_\rho^m(x_k^r)\right) + \frac{\rho^2}{2}\int_{\Sigma_k} |\vec{H}_{\vec{\Phi}_k}|^2\ dvol_{g_{\vec{\Phi}_k}}$$

Using (A.6) we deduce that for any $\rho < \min\{2-r, r-1\}$

$$\mu_{V_\infty}(B_{r+\rho}(0) \setminus B_{r-\rho}(0)) \geq \frac{2\pi}{3}\rho^2 \quad .$$

Hence the support of V_∞ intersects all the $\partial B_r(0) \times G_2(\mathbb{R}^m)$ for any $r \in (1,2)$. We consider a sequence of radii $r_i > 1$ and converging to 1. The 2-planes belonging to the support of V_∞ and intersecting $\partial B_{r_i}(0) \times G_2(\mathbb{R}^m)$ has to be constant for i large enough. This implies that the support of V_∞ contains the intersection between the annulus $B_2(0) \setminus B_1(0)$ and a plane touching $\overline{B_1(0)}$. This imposes

$$\mu_{V_\infty}(B_2(0) \setminus B_1(0)) \geq\ 3\,\pi \quad .$$

The later contradicts (A.4) and lemma A.1 is proved. □

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