3-Commutators Estimates and the Regularity of 1/2-Harmonic Maps into Spheres

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Abstract

We prove the regularity of weak $1/2-$harmonic maps from the real line into a sphere. The key point in our result is first a formulation of the $1/2-$harmonic map equation in the form of a non-local linear Schrödinger type equation with a *3-terms commutators* in the right-hand-side . We then establish a sharp estimate for these *3-commutators.*

Key words. Harmonic maps, nonlinear elliptic PDE's, regularity of solutions, commutator estimates.

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Contents

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1 Introduction

Since the early 50's the analysis of critical points to conformal invariant Lagrangians has raised a special interest, due to the important role they play in physics and geometry.

For a complete overview on this topic we refer the reader to the introduction of [17]. Here we recall some classical examples of conformal invariant variational problems.

The most elementary example of a 2-dimensional conformal invariant Lagrangian is the Dirichlet Energy

$$E(u)=\int_D |\nabla u(x,y)|^2 dxdy\,, \tag{1}$$

where $D\subseteq R^2$ is an open set and $u\colon D\to \mathbb{R}$, ∇u is the gradient of u. We recall that a map $\phi\colon \mathbb{C}\to\mathbb{C}$ is conformal if it satisfies

$$\begin{cases} |\frac{\partial\phi}{\partial x}|=|\frac{\partial\phi}{\partial y}| \\ \langle\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\rangle=0 \\ \det\nabla\phi\geq 0 \quad and \quad \nabla\phi\neq 0 \end{cases} \tag{2}$$

Here $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product in $\mathbb{R}^n$.

For every $u\in W^{1,2}(D,\mathbb{R})$ and every conformal map ϕ, $\deg(\phi)=1$, the following holds

$$E(u)=E(u\circ\phi)=\int_{\phi^{-1}(D)}|(\nabla\circ\phi)u(x,y)|^2dxdy\,.$$

Critical points of this functional are the harmonic functions satisfying

$$\Delta u=0,\quad \text{in } D\,. \tag{3}$$

We can extend E to maps taking values in $\mathbb{R}^m$ as follows

$$E(u)=\int_D|\nabla u(x,y)|^2dxdy=\int_D\sum_{i=1}^m|\nabla u_i(x,y)|^2dxdy\,, \tag{4}$$

where u_i are the components of u. The Lagrangian (4) is still conformally invariant and each component of its critical points satisfies the equation (3).

We can define the Lagrangian (4) also in the set of maps taking values in a compact submanifold $\mathcal{N}\subseteq\mathbb{R}^m$ without boundary. In this case critical points $u\in W^{1,2}(D,\mathcal{N})$ of E satisfy in a weak sense the equation

$$-\Delta u\perp T_u\mathcal{N}\,,$$

where $T_\xi\mathcal{N}$ is the tangent plane a $\mathcal{N}$ at the point $\xi \in \mathcal{N}$, or in a equivalent way

$$-\Delta u = A(u)(\nabla u, \nabla u) := A(u)(\partial_x u, \partial_x u) + A(u)(\partial_y u, \partial_y u), \tag{5}$$

where $A(\xi)$ is the second fundamental form at the point $\xi \in \mathcal{N}$ (see for instance [10]). The equation (5) is called the *harmonic map equation* into $\mathcal{N}$.

In the case when $\mathcal{N}$ is an oriented hypersurface of $I\!R^m$ the harmonic map equation reads as

$$-\Delta u = n\langle \nabla n, \nabla u\rangle\,, \tag{6}$$

where n denotes the composition of u with the unit normal vector field ν to $\mathcal{N}$.

All the above examples belongs to the class of conformal invariant coercive Lagrangians whose corresponding Euler-lagrangian equation is of the form

$$-\Delta u = f(u, \nabla u)\,, \tag{7}$$

where $f\colon I\!R^2 \times (I\!R^m \otimes R^2) \to I\!R^m$ is a continuous function satisfying for some positive constant C

$$C^{-1}|p|^2 \le f(\xi, p)| \le C|p|^2, \quad \forall\, \xi, p\,.$$

One of the main issues related to equation (7) is the regularity of solutions $u \in W^{1,2}(D, \mathcal{N})$. We observe that equation (7) is critical in dimension $n = 2$ for the $W^{1,2}$-norm. Indeed if we plug in the nonlinearity $f(u, \nabla u)$ the information that $u \in W^{1,2}(D, \mathcal{N})$, we get that $\Delta u \in L^1(D)$ and thus $\nabla u \in L^{2,\infty}_{loc}(D)$ the weak L^2 space (see [22]), which has the same homogeneity of L^2. Hence we are back in some sense to the initial situation. This shows that the equation is critical.

In general $W^{1,2}$ solutions to equations (7) are not smooth in dimension greater that 2 (see counter-example in [16]). We refer again the reader to [8] for a more complete presentation of the results concerning the regularity and compactness results for equations (7).

Here we are going to recall the approach introduced by F. Hélein [10] to prove the regularity of harmonic maps from a domain D of $I\!R^2$ into the unit sphere S^{m-1} of $I\!R^m$. In this case the Euler-Lagrange equation is

$$-\Delta u = u|\nabla u|^2\,. \tag{8}$$

It was observed by Shatah [21] that $u \in W^{1,2}(D, S^{m-1})$ is a solution of (8) if and only if the following conservation law holds

$$\operatorname{div}(u_i \nabla u_j - u_j \nabla u_i) = 0, \tag{9}$$

for all $i, j \in \{1, \ldots, m\}$.

Using (9) and the fact that $|u| \equiv 1 \Longrightarrow \sum_{j=1}^m u_j \nabla u_j = 0$, Hélein wrote the equation (8) in the form

$$-\Delta u = \nabla^\perp B \cdot \nabla u, \tag{10}$$

where $\nabla^\perp B = (\nabla^\perp B_{ij})$ with $\nabla^\perp B_{ij} = u_i \nabla u_j - u_j \nabla u_i$, (for every vector field $v\colon I\!\!R^2 \to I\!\!R^n$, $\nabla^\perp v$ denotes the $\pi/2$ rotation of the gradient ∇v, namely $\nabla^\perp v = (-\partial_y v, \partial_x v))$.

The r.h.s of (10) can be written as a sum of jacobians:

$$\nabla^\perp B_{ij} \nabla u_j = \partial_x u_j \partial_y B_{ij} - \partial_y u_j \partial_x B_{ij}\,.$$

This particular structure permits to apply to the equation (8) the following result

Theorem 1.1 *[28] Let D be a smooth bounded domain of $I\!\!R^2$. Let a and b be two measurable functions in D whose gradients are in $L^2(D)$. Then there exists a unique solution $\varphi \in W^{1,2}(D)$ to*

$$\begin{cases} -\Delta\varphi = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x}, & \text{in } D \\ \varphi = 0 & \text{on } \partial D\,. \end{cases} \tag{11}$$

Moreover there exists a constant $C > 0$ independent of a and b such that

$$||\varphi||_\infty + ||\nabla\varphi||_{L^2} \le C ||\nabla a||_{L^2} ||\nabla b||_{L^2}\,.$$

In particular φ is a continuous in D.

Theorem 1.1 applied to equation (10) leads, modulo some standard localization argument in elliptic PDE, to an estimate of the form

$$\|\nabla u\|_{L^2(B_r(x_0))} \le C\ \|\nabla B\|_{L^2(B_r(x_0))}\ \|\nabla u\|_{L^2(B_r(x_0))} + Cr\ \|\nabla u\|_{L^2(\partial B_r(x_0))} \tag{12}$$

for every $x_0 \in D$ and $r > 0$ such that $B_r(x_0) \subset D$. Assuming we are considering radii $r < r_0$ such that $\max_{x_0 \in D} C\ \|\nabla B\|_{L^2(B_r(x_0))} < 1/2$, then (12) implies a Morrey estimate of the form

$$\sup_{x_0, r>0}\ r^{-\beta} \int_{B_r(x_0)} |\nabla u|^2\, dx < +\infty \tag{13}$$

for some $\beta > 0$ which itself implies the Hölder continuity of u by standard embedding result (see [8]). Finally a bootstrap argument implies that u is in fact C^∞ - and even analytic - (see [11] and [14]).

In the present work we are interested in 1 dimensional quadratic Lagrangians which are invariant under the trace of conformal maps that keep invariant the half space $I\!\!R^2_+$: the Moebius group.

A typical example is the following Lagrangian that we will call $L-$*energy* - L stands for "Line" -

$$L(u) = \int_{I\!\!R} |\Delta^{1/4} u(x)|^2 dx\,, \tag{14}$$

where $u\colon I\!\!R \to \mathcal{N}$, $\mathcal{N}$ is a smooth k-dimensional submanifold of $I\!\!R^m$ which is at least C^2, compact and without boundary. We observe that the $L(u)$ in (14) coincides with the semi-norm $||u||^2_{\dot H^{1/2}(I\!\!R)}$ (for the definition of $||\cdot||_{\dot H^{1/2}(I\!\!R)}$ we refer to Section 2). Moreover a more tractable way to look at this norm is given by the following identity

$$\int_{I\!\!R} |\Delta^{1/4}u(x)|^2 dx = \inf\left\{ \int_{I\!\!R^2_+} |\nabla\tilde u|^2 dx :\ \tilde u \in W^{1,2}(I\!\!R^2, I\!\!R^m), \quad \text{trace } \tilde u = u \right\}.$$

The Lagragian L extends to map u in the following function space

$$\dot H^{1/2}(I\!\!R, \mathcal{N}) = \{u \in \dot H^{1/2}(I\!\!R, I\!\!R^m) : \quad u(x) \in \mathcal{N}, \text{a.e,}\}.$$

The operator $\Delta^{1/4}$ on $I\!\!R$ is defined by means of the the Fourier tranform as follows

$$\widehat{\Delta^{1/4}u} = |\xi|^{1/2}\hat u,$$

(given a function f, $\hat f$ denotes the Fourier tranform of f).

Denote $\pi_{\mathcal{N}}$ the orthogonal projection onto $\mathcal{N}$ which happens to be a C^l map in a sufficiently small neighborhood of $\mathcal{N}$ if $\mathcal{N}$ is assumed to be C^{l+1}. We now introduce the notion of 1/2-harmonic map into a manifold.

Definition 1.1 *A map $u \in \dot H^{1/2}(I\!\!R, \mathcal{N})$ is called a weak 1/2-harmonic map into $\mathcal{N}$ if for any $\phi \in C^\infty_0(I\!\!R, I\!\!R^m)$ there holds*

$$\frac{d}{dt} L(\pi_{\mathcal{N}}(u + t\phi)) = 0 \quad .$$

□

In short we say that a weak 1/2−harmonic map is a *critical point of L in $\dot H^{1/2}(I\!\!R, \mathcal{N})$ for perturbations in the target.*

1/2−harmonic maps into the circle S^1 might appear for instance in the asymptotic of equations in phase-field theory for fractional reaction-diffusion such as

$$\epsilon^2\, \Delta^{1/2}u + u(1 - |u|^2) = 0$$

where u is a complex valued "wave function".

In this paper we consider the case $\mathcal{N} = S^{m-1}$. We first write the Euler-Lagrange equation associated to L in $\dot H^{1/2}(I\!\!R, S^{m-1})$ in the following way

Proposition 1.1 *A map u in $\dot H^{1/2}(I\!\!R, S^{m-1})$ is a weak 1/2-harmonic map if and only if it satisfies the following Euler-Lagrange equation*

$$\Delta^{1/4}(u \wedge \Delta^{1/4}u) = T(u\wedge, u), \tag{15}$$

where, in general for an arbitrary integer n*, for every* $Q \in \dot{H}^{1/2}(I\!\!R^n, \mathcal{M}_{\ell\times m}(I\!\!R))$ $\ell \geq 0^{(1)}$ *and* $u \in \dot{H}^{1/2}(I\!\!R^n, I\!\!R^m)$ *,* T *is the operator defined by*

$$T(Q,u) := \Delta^{1/4}(Q\Delta^{1/4}u) - Q\Delta^{1/2}u + \Delta^{1/4}u\Delta^{1/4}Q\,. \tag{16}$$

$\square$

The Euler Lagrange equation (15) will often be completed by the following "structure equation" which a consequence of the fact that $u \in S^{m-1}$ almost everywhere :

Proposition 1.2 *All maps in* $\dot{H}^{1/2}(I\!\!R, S^{m-1})$ *satisfy the following identity*

$$\Delta^{1/4}(u\cdot\Delta^{1/4}u) = S(u\cdot, u) - \mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)\,. \tag{17}$$

where, in general for an arbitrary integer n*, for every* $Q \in \dot{H}^{1/2}(I\!\!R^n, \mathcal{M}_{\ell\times m}(I\!\!R))$*,* $\ell \geq 0$ *and* $u \in \dot{H}^{1/2}(I\!\!R^n, I\!\!R^m)$*,* S *is the operator given by*

$$S(Q,u) := \Delta^{1/4}[Q\Delta^{1/4}u] - \mathcal{R}(Q\nabla u) + \mathcal{R}(\Delta^{1/4}Q\mathcal{R}\Delta^{1/4}u)\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u \tag{18}$$

and $\mathcal{R}$ *is the Fourier multiplier of symbol* $m(\xi) = i\frac{\xi}{|\xi|}$ *.* $\square$

In the present work we will first show that $\dot{H}^{1/2}$ solutions to the 1/2-harmonic map equation (15) are Hölder continuous. This regularity result will be a direct consequence of the following Morrey type estimate that we will establish :

$$\sup_{x_0\in I\!\!R, r>0} r^{-\beta}\int_{B_r(x_0)} |\Delta^{1/4}u|^2\,dx < +\infty \tag{19}$$

To this purpose, in the spirit of what we have just presented regarding Hélein's proof of the regularity of harmonic maps from a 2-dimensional domain into a round sphere, we will take advantage of a "gain of regularity" in the r.h.s of the equations (15) and (17) where the different terms $T(u\wedge, u)$, $S(u\cdot, u)$ and $\mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)$ play more or less the role which was played by $\nabla^\perp B\cdot\nabla u$ in (10). Precisely we will establish the following estimates : for every $u \in \dot{H}^{1/2}(I\!\!R, I\!\!R^m)$ and $Q \in H^{1/2}(I\!\!R, \mathcal{M}_{\ell\times m}(I\!\!R))$ we have

$$\|T(Q,u)\|_{H^{-1/2}} \leq C\,\|Q\|_{\dot{H}^{1/2}(I\!\!R)}\,\|u\|_{\dot{H}^{1/2}(I\!\!R)}\,, \tag{20}$$

$$\|S(Q,u)\|_{H^{-1/2}} \leq C\,\|Q\|_{\dot{H}^{1/2}(I\!\!R)}\,\|u\|_{\dot{H}^{1/2}(I\!\!R)}\,, \tag{21}$$

and

$$\|\mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u))\|_{\dot{H}^{-1/2}} \leq C\,\|u\|^2_{\dot{H}^{1/2}(I\!\!R)}\,. \tag{22}$$

Our denomination "gain of regularity" has been chosen in order to illustrate that, under our assumptions $u \in \dot{H}^{1/2}(I\!\!R, I\!\!R^m)$ and $Q \in \dot{H}^{1/2}(I\!\!R, \mathcal{M}_{\ell\times m}(I\!\!R))$ each term individually

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(1) $\mathcal{M}_{\ell\times m}(I\!\!R)$ denotes, as usual, the space of $\ell \times m$ real matrices.

in T and S - like for instance $\Delta^{1/4}(Q\Delta^{1/4}u)$ or $Q\Delta^{1/2}u$ ... - are not in $H^{-1/2}$ but the special linear combination of them constituting T and S are in $H^{-1/2}$. In a similar way, in dimension 2, $J(a,b) := \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ satisfies, as a direct consequence of Wente's theorem above,

$$\|J(a,b)\|_{\dot{H}^{-1}} \le C\ \|a\|_{\dot{H}^1}\ \|b\|_{\dot{H}^1} \tag{23}$$

whereas, individually, the terms $\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}$ and $\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ are not in H^{-1}.

The estimates (20) and (21) are in fact consequences of the following *3-terms commutator* or simply *3-commutator estimates* which are valid in arbitrary dimension n and which represent two of the main results of the present paper. We recall that BMO denotes the space of *Bounded Mean Oscillations* functions of John and Nirenberg (see for instance [9])

$$\|u\|_{BMO(I\!R^n)} = \sup_{\{x_0\in I\!R^n\ ;\ r>0\}} \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} \left|u(x) - \frac{1}{|B_r(x_0)|}\int u(y)\ dy\right|\ dx \quad .$$

Theorem 1.2 *Let $n \in I\!N^*$ and let $u \in BMO(I\!R^n)$, $Q \in \dot{H}^{1/2}(I\!R^n, \mathcal{M}_{\ell\times m}(I\!R))$. Denote*

$$T(Q,u) := \Delta^{1/4}(Q\Delta^{1/4}u) - Q\Delta^{1/2}u + \Delta^{1/4}u\Delta^{1/4}Q \quad ,$$

then $T(Q,u) \in H^{-1/2}$ and there exists $C > 0$, depending only on n, such that

$$||T(Q,u)||_{H^{-1/2}} \le C\ ||Q||_{\dot{H}^{1/2}(I\!R)}||u||_{BMO} \, . \tag{24}$$

□

Theorem 1.3 *Let $n \in I\!N^*$ and let $u \in BMO(I\!R^n)$, $Q \in \dot{H}^{1/2}(I\!R^n, \mathcal{M}_{\ell\times m}(I\!R))$. Denote*

$$S(Q,u) := \Delta^{1/4}[Q\Delta^{1/4}u] - \mathcal{R}(Q\nabla u) + \mathcal{R}(\Delta^{1/4}Q\mathcal{R}\Delta^{1/4}u)$$

where $\mathcal{R}$ is the Fourier multiplier of symbol $m(\xi) = i\frac{\xi}{|\xi|}$. Then $S(Q,u) \in H^{-1/2}$ and there exists C depending only on n such that

$$||S(Q,u)||_{H^{-1/2}} \le C\ ||Q||_{\dot{H}^{1/2}(I\!R)}||u||_{BMO} \, . \tag{25}$$

□

The fact that Theorem 1.2 and Theorem 1.3 imply estimates (20) and (21) comes from the embedding $\dot{H}^{1/2}(I\!R) \subset BMO(I\!R)$.

The parallel between the structures T and S for $H^{1/2}$ in one hand and the jacobian structure J for H^1 in the other hand can be pushed further as follows. As a consequence of a result of R. Coifman, P.L. Lions, Y. Meyer and S. Semmes [3], Wente estimate (23) can be deduced from a more general one : We denote, for any $i,j \in \{1\cdots n\}$, and $a, b \in \dot{H}^1(I\!R^n)$,

$$J_{ij}(a,b) := \frac{\partial a}{\partial x_i}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial x_i} \, ,$$

and denote $J(a,b) := (J_{ij}(a,b))_{ij=1\cdots n}$. With this notation the main result in [3] implies

$$\|J(a,b)\|_{\dot{H}^{-1}(I\!\!R^n)} \leq C\ \|a\|_{\dot{H}^1(I\!\!R^n)}\ \|b\|_{BMO(I\!\!R^n)} \tag{26}$$

which is reminiscent to (24) and (25). Recall also that (26) is a consequence of a commutator estimate by R. Coifman, R Rochberg and G. Weiss [4].

The two theorems 1.2 and 1.2 will be the consequence of the two following ones which are their "dual versions". Recall first that $\mathcal{H}^1(I\!\!R^n)$ denotes the Hardy space of L^1 functions f on $I\!\!R^n$satisfying

$$\int_{I\!\!R^n} \sup_{t\in I\!\!R} |\phi_t * f|(x)\ dx < +\infty \quad ,$$

where $\phi_t(x) := t^{-n}\ \phi(t^{-1}x)$ and where ϕ is some function in the Schwartz space $\mathcal{S}(I\!\!R^n)$ satisfying $\int_{I\!\!R^n} \phi(x)\ dx = 1$. Recall the famous result by Fefferman saying that the dual space to $\mathcal{H}^1$ is BMO.

In one hand theorem 1.2 is the consequence of the following result.

Theorem 1.4 *Let $u, Q \in \dot{H}^{1/2}(I\!\!R^n)$, denote*

$$R(Q,u) = \Delta^{1/4}(Q\Delta^{1/4}u) - \Delta^{1/2}(Qu) + \Delta^{1/4}((\Delta^{1/4}Q)u)\,.$$

then $R(Q,u) \in \mathcal{H}^1(I\!\!R^n)$ and

$$||R(Q,u)||_{\mathcal{H}^1} \leq C||Q||_{\dot{H}^{1/2}(I\!\!R)}||u||_{\dot{H}^{1/2}(I\!\!R)}\,. \tag{27}$$

In the other hand theorem 1.3 is the consequence this next result.

Theorem 1.5 *Let $u, Q \in H^{1/2}$ and $u \in BMO$.*

$$\tilde{S}(Q,u) = \Delta^{1/4}(Q\Delta^{1/4}u) - \nabla(Q\mathcal{R}u) + \mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q\mathcal{R}u)\,.$$

where $\mathcal{R}$ is the Fourier multiplier of symbol $m(\xi) = i\frac{\xi}{|\xi|}$.Then $\tilde{S}(Q,u) \in \mathcal{H}^1$ and

$$||\tilde{S}(Q,u)||_{\mathcal{H}^1} \leq C||Q||_{\dot{H}^{1/2}(I\!\!R)}||u||_{\dot{H}^{1/2}(I\!\!R)}\,. \tag{28}$$

□

We now say few words on the proof of estimates 27 and 28. The compensations of the 3 different terms in $R(Q,u)$ will be clear from the Littlewood-Paley decomposition of the different products that we present in section 3. Denoting as usual $\Pi_1(fg)$ the high-low contribution - respectively from f and g - denoting $\Pi_2(fg)$ the low-high contribution and $\Pi_3(fg)$ the high-high contribution we shall need the following groupings

- i) For $\Pi_1(R(Q,u))$ we proceed to the following decomposition

$$\Pi_1(R(Q,u)) = \underbrace{\Pi_1(\Delta^{1/4}(Q\Delta^{1/4}u))} + \underbrace{\Pi_1(-\Delta^{1/2}(Qu) + \Delta^{1/4}((\Delta^{1/4}Q)u))} \ .$$

- ii) For $\Pi_2(R(Q,u))$ we decompose as follows

$$\Pi_2(R(Q,u)) = \underbrace{\Pi_2(\Delta^{1/4}(Q\Delta^{1/4}u) - \Delta^{1/2}(Qu))} + \underbrace{\Pi_2(\Delta^{1/4}((\Delta^{1/4}Q)u))} \ .$$

- ii) Finally, for $\Pi_3(R(Q,u))$ we decompose as follows

$$\Pi_3(R(Q,u)) = \underbrace{\Pi_3(\Delta^{1/4}(Q\Delta^{1/4}u))} - \underbrace{\Pi_3(\Delta^{1/2}(Qu))} + \underbrace{\Pi_3(\Delta^{1/4}((\Delta^{1/4}Q)u))} \ .$$

Finally, injecting the Morrey estimate (19) in equations (15) and (17), a classical "elliptic type" bootstrap argument leads to the following result (see [5] for the details of this argument).

Theorem 1.6 *Let u be a weak $1/2$-harmonic map in $\dot{H}^{1/2}(I\!R, S^{m-1})$. Then it belongs to $H^s_{loc}(I\!R, S^{m-1})$ for every $s \in I\!R$ and thus it is C^∞ .* □

The paper is organized as follows.

- In Section 2 we give some preliminary definitions and notations.
- In Section 3 we prove the *3-commutator estimates* Theorems 1.2 and 1.3.
- In section 4 we study geometric localization properties of the $\dot{H}^{1/2}-$ norm on the real line for $\dot{H}^{1/2}-$functions in general
- In Section 5 we prove some $L-$energy decrease control on dyadic annuli for general solutions to some linear non-local systems of equations that will include the systems (15) and (17).
- in Section 6 we derive the Euler-Lagrange equation (15) associated to the Lagrangian (14) - proposition 1.1. We then prove proposition 1.2. We finally use the results of the previous section in order to deduce the Morrey type estimate (19) for $1/2-$harmonic maps into a sphere.

2 Notations and Definitions

In this Section we introduce some notations and definitions we are going to use in the sequel.

For $n \geq 1$, we denote respectively by $\mathcal{S}(I\!\!R^n)$ and $\mathcal{S}'(I\!\!R^n)$ the spaces of Schwartz functions and tempered distributions. Moreover given a function v we will denote either by $\hat{v}$ or by $\mathcal{F}[v]$ the Fourier Transform of v :

$$\hat{v}(\xi) = \mathcal{F}[v](\xi) = \int_{I\!\!R^n} v(x) e^{-i\langle \xi, x\rangle}\, dx\,.$$

Throughout the paper we use the convention that x, y denote variables in the space and ξ, ζ variables in the phase.

We recall the definition of fractional Sobolev space (see for instance [25]).

Definition 2.1 *For a real $s \geq 0$,*

$$H^s(I\!\!R^n) = \{v \in L^2(I\!\!R^n) : \quad |\xi|^s \mathcal{F}[v] \in L^2(I\!\!R^n)\}.$$

For a real $s < 0$,

$$H^s(I\!\!R^n) = \{v \in \mathcal{S}'(I\!\!R^n) : \quad (1 + |\xi|^2)^s \mathcal{F}[v] \in L^2(I\!\!R^n)\}.$$

It is known that $H^{-s}(I\!\!R^n)$ is the dual of $H^s(I\!\!R^n)$.

Another characterization of $H^s(I\!\!R^n)$, with $0 < s < 1$, which does not use the Fourier transform is the following, (see for instance [25]).

Lemma 2.1 *For $0 < s < 1$, $u \in H^s(I\!\!R^n)$ is equivalent to $u \in L^2(I\!\!R^n)$ and*

$$\left(\int_{I\!\!R^n}\int_{I\!\!R^n} \left(\frac{(u(x) - u(y))^2}{|x - y|^{n+2s}}\right) dxdy\right)^{1/2} < +\infty\,.$$

For $s > 0$ we set

$$||u||_{H^s(I\!\!R^n)} = ||u||_{L^2(I\!\!R^n)} + |||\xi|^s \mathcal{F}[v]||_{L^2(I\!\!R^n)}\,,$$

and

$$||u||_{\dot{H}^s(I\!\!R^n)} = |||\xi|^s \mathcal{F}[v]||_{L^2(I\!\!R^n)}\,.$$

For an open set $\Omega \subset I\!\!R^n$, $H^s(\Omega)$ is the space of the restrictions of functions from $H^s(I\!\!R^n)$ and

$$||u||_{\dot{H}^s(\Omega)} = \inf\{||U||_{\dot{H}^s(I\!\!R^n)}, \quad U = u \text{ on } \Omega\}$$

In the case of $0 < s < 1$ then $f \in H^s(\Omega)$ if and only if $f \in L^2(\Omega)$ and

$$\left(\int_{\Omega}\int_{\Omega} \left(\frac{(u(x) - u(y))^2}{|x - y|^{n+2s}}\right) dxdy\right)^{1/2} < +\infty\,.$$

Moreover
$$||u||_{\dot{H}^s(\Omega)} \simeq \left(\int_\Omega\int_\Omega \left(\frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\right) dxdy\right)^{1/2} < +\infty\,,$$
see for instance [25] .

Finally for a submanifold $\mathcal{N}$ of $I\!R^m$ we can define
$$H^s(I\!R,\mathcal{N}) = \{u \in H^s(I\!R, I\!R^m): \quad u(x) \in \mathcal{N}, \text{a.e.}\}\,.$$

We introduce the so-called Littlewood-Paley or dyadic decomposition of unity. Such a decomposition can be obtained as follows . Let $\phi(\xi)$ be a radial Schwartz function supported on $\{\xi: \ |\xi| \le 2\}$, which is equal to 1 on $\{\xi: \ |\xi| \le 1\}$. Let $\psi(\xi)$ be the function $\psi(\xi) := \phi(\xi) - \phi(2\xi)$. ψ is a bump function supported on the annulus $\{\xi: \ 1/2 \le |\xi| \le 2\}$.

We put $\psi_0 = \phi$, $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \neq 0$. The functions ψ_j, for $j \in Z\!\!\!Z$, are supported on $\{\xi: \ 2^{j-1} \le |\xi| \le 2^{j+1}\}$. Moreover $\sum_{j\in Z\!\!\!Z} \psi_j(x) = 1$.

We then set $\phi_j(\xi) := \sum_{k=-\infty}^{j} \psi_k(\xi)$. The function ϕ_j is supported on $\{\xi, \ |\xi| \le 2^{j+1}\}$.

We recall the definition of the homogeneous Besov spaces $\dot{B}^s_{p,q}(I\!R^n)$ and homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{pq}(I\!R^n)$ in terms of the above dyadic decomposition.

Definition 2.2 *Let $s \in I\!R$, $0 < p, q \le \infty$. For $f \in \mathcal{S}'(I\!R^n)$ we set*
$$\begin{array}{ll} ||f||_{\dot{B}^s_{p,q}(I\!R^n)} = \left(\sum_{j=-\infty}^{\infty} 2^{jsq}||\mathcal{F}^{-1}[\psi_j\mathcal{F}[f]]||^q_{L^p(I\!R^n)}\right)^{1/q} & \text{if } q < \infty \\ ||f||_{\dot{B}^s_{p,q}(I\!R^n)} = \sup_{j\in I\!N} 2^{js}||\mathcal{F}^{-1}[\psi_j\mathcal{F}[f]]||_{L^p(I\!R^n)} & \text{if } q = \infty \end{array} \tag{29}$$

When $p, q < \infty$ we also set
$$||f||_{\dot{F}^s_{p,q}(I\!R^n)} = ||\left(\sum_{j=-\infty}^{\infty} 2^{jsq}|\mathcal{F}^{-1}[\psi_j\mathcal{F}[f]]|^q\right)^{1/q}||_{L^p}\,.$$

The space of all tempered distributions f for which the quantity $||f||_{\dot{B}^s_{p,q}(I\!R^n)}$ is finite is called the homogeneous Besov space with indices s, p, q and it is denoted by $\dot{B}^s_{p,q}(I\!R^n)$. The space of all tempered distributions f for which the quantity $||f||_{\dot{F}^s_{p,q}(I\!R^n)}$ is finite is called the homogeneous Triebel-Lizorkin space with indices s, p, q and it is denote by $\dot{F}^s_{p,q}(I\!R^n)$. It is known that $\dot{H}^s(I\!R^n) = \dot{B}^s_{2,2}(I\!R^n) = \dot{F}^s_{2,2}(I\!R^n)$.

Finally we denote $\mathcal{H}^1(I\!R^n)$ the homogeneous Hardy Space in $I\!R^n$. It is known that $\mathcal{H}^1(I\!R^n) \simeq F^0_{2,1}$ thus we have
$$||f||_{\mathcal{H}^1(I\!R^n)} \simeq \int_{I\!R}(\sum_j |\mathcal{F}^{-1}[\psi_j\mathcal{F}[f]]|^2)^{1/2}dx\,.$$

We recall that in dimension $n = 1$, the space $\dot{H}^{1/2}(I\!R)$ is continuously embedded in the Besov space $\dot{B}^0_{\infty,\infty}(I\!R)$. More precisely we have

$$\dot{H}^{1/2}(I\!R) \hookrightarrow BMO(I\!R) \hookrightarrow \dot{B}^0_{\infty,\infty}(I\!R)\,, \tag{30}$$

where (see for instance page 31 in [19], page 129 in [27]).

The s-fractional Laplacian of a function $u\colon I\!R^n \to I\!R$ is defined as a pseudo differential operator of symbol $|\xi|^{2s}$:

$$\widehat{\Delta^s u}(\xi) = |\xi|^{2s}\hat{u}(\xi)\,. \tag{31}$$

In the case where $s = 1/2$, we can write $\Delta^{1/2}u = -\mathcal{R}(\nabla u)$ where $\mathcal{R}$ is Fourier multiplier of symbol $\frac{i}{|\xi|}\sum_{k=1}^{n}\xi_k$:

$$\widehat{\mathcal{R}X}(\xi) = \frac{1}{|\xi|}\sum_{k=1}^{n} i\xi_k \hat{X}_k(\xi)$$

for every $X\colon I\!R^n \to I\!R^n$, namely $\mathcal{R} = -\Delta^{-1/2}\mathrm{div}$.

We denote by $B_r(\bar{x})$ the ball of radius r and centered at $\bar{x}$. If $\bar{x} = 0$ we simply write B_r . If $x, y \in I\!R^n$, $x \cdot y$ denote the scalar product between x, y .

For every function $f\colon I\!R^n \to I\!R$ we denote by $M(f)$ the maximal function of f, namely

$$M(f) = \sup_{r>0, x\in I\!R^n} |B(x,r)|^{-1}\int_{B(x,r)} |f(y)|dy\,. \tag{32}$$

3 3-Commutator Estimates : Proof of theorem 1.2 and theorem 1.3.

In this Section we prove Theorems 1.2 and 1.3 .

We consider the dyadic decomposition introduced in Section 2 . For every $j \in I\!N$ and $f \in \mathcal{S}'(I\!R^n)$ we set

$$f^j = \mathcal{F}^{-1}[\phi_j\mathcal{F}[f]], \quad f_j = \mathcal{F}^{-1}[\psi_j\mathcal{F}[f]]\,.$$

We have $f^j = \sum_{k=0}^{j} f_k$ and $f = \sum_{k=0}^{+\infty} f_k$ (where the convergence is in $\mathcal{S}'(I\!R^n)$) .

Let $f, g \in \mathcal{S}'(I\!R)$. Suppose that fg exists in $\mathcal{S}'(I\!R^n)$. Then we split the product in the following way

$$fg = \Pi_1(fg) + \Pi_2(fg) + \Pi_3(fg),$$

where

$$\begin{aligned}
\Pi_1(fg) &= \sum_{-\infty}^{+\infty} f_j \sum_{k\leq j-4} g_k = \sum_{-\infty}^{+\infty} f_j g^{j-4}\,; \\
\Pi_2(fg) &= \sum_{-\infty}^{+\infty} f_j \sum_{k\geq j+4} g_k \sum_{-\infty}^{+\infty} g_j f^{j-4}\,; \\
\Pi_3(fg) &= \sum_{-\infty}^{+\infty} f_j \sum_{|k-j|<4} g_k \,.
\end{aligned}$$

We observe that for every j we have

$$\begin{aligned}
&\mathrm{supp}\mathcal{F}[f^{j-4}g_j] \subset \{2^{j-2} \leq |\xi| \leq 2^{j+2}\}; \\
&\mathrm{supp}\mathcal{F}[\textstyle\sum_{k=j-3}^{j+3} f_j g_k] \subset \{|\xi| \leq 2^{j+5}\}\,.
\end{aligned}$$

The following Lemma will be often used in the sequel.

Lemma 3.1 *For every $f \in \mathcal{S}'$ we have*

$$\sup_{j\in Z} |f^j| \leq M(f)\,.$$

Proof. We have

$$\begin{aligned}
&f^j = \mathcal{F}^{-1}[\phi_j] \star f = 2^j \int_R \mathcal{F}^{-1}[\phi](2^j(x-y))f(y)dy \\
&= \int_{\mathbb{R}} \mathcal{F}^{-1}[\phi](z) f(x-2^{-j}z)dz \\
&= \sum_{k=-\infty}^{+\infty} \int_{B_{2^k}\setminus B_{2^{k-1}}} \mathcal{F}^{-1}[\phi](z) f(x-2^{-j}z)dz \\
&\leq \sum_{k=-\infty}^{+\infty} \max_{B_{2^k}\setminus B_{2^{k-1}}} |\mathcal{F}^{-1}[\phi](z)| \int_{B_{2^k}\setminus B_{2^{k-1}}} |f(x-2^{-j}z)|dz \\
&\leq \sum_{k=-\infty}^{+\infty} \max_{B_{2^k}\setminus B_{2^{k-1}}} 2^k |\mathcal{F}^{-1}[\phi](z)| 2^{j-k} \int_{B(x,2^{k-j})\setminus B(x,2^{k-1-j})} |f(z)|dz \\
&\leq M(f) \sum_{k=-\infty}^{+\infty} \max_{B_{2^k}\setminus B_{2^{k-1}}} 2^k |\mathcal{F}^{-1}[\phi](z)| \leq CM(f)\,.
\end{aligned}$$

In the last inequality we use the fact $\mathcal{F}^{-1}[\phi]$ is in $\mathcal{S}(\mathbb{R}^n)$ and thus

$$\sum_{k=-\infty}^{+\infty} \max_{B_{2^k}\setminus B_{2^{k-1}}} 2^k |\mathcal{F}^{-1}[\phi](z)| \leq 2 \int_{\mathbb{R}} |\mathcal{F}^{-1}[\phi](z)| d\xi, .$$

We can now start the proof of one of the main result in the paper.

Proof of theorem 1.4.
We are going to estimate $\Pi_1(R(Q,u)), \Pi_2(R(Q,u))$ and $\Pi_3(R(Q,u))$.
- Estimate of $||\Pi_1(\Delta^{1/4}(Q\Delta^{1/4}u)||_{\mathcal{H}^1}$.

$$
\begin{aligned}
||\Pi_1(\Delta^{1/4}(Q\Delta^{1/4}u)||_{\mathcal{H}^1} &= \int_{I\!R^n} \left(\sum_{j=-\infty}^{\infty} 2^j Q_j^2 (\Delta^{1/4} u^{j-4} \right)^2)^{1/2} dx \qquad (33)\\
&\leq \int_{I\!R^n} \sup_j |\Delta^{1/4} u^{j-4}| \left(\sum_j 2^j Q_j^2 \right)^{1/2} dx \\
&\leq \left(\int_{I\!R^n} (M(\Delta^{1/4}u))^2 dx \right)^{1/2} \left(\int_R \sum_j 2^j Q_j^2 dx \right)^{1/2} \\
&\leq C ||u||_{\dot{H}^{1/2}} ||Q||_{\dot{H}^{1/2}}.
\end{aligned}
$$

- Estimate of $\Pi_1(\Delta^{1/4}(\Delta^{1/4}Qu) - \Delta^{1/2}(Qu))$. We show that it is in $\dot{B}^0_{1,1}$ $(\mathcal{H}^1 \hookrightarrow \dot{B}^0_{1,1})$. To this purpose we use the "commutator structure of the above term".

$$
\begin{aligned}
&||\Pi_1(\Delta^{1/4}(\Delta^{1/4}Q)u - \Delta^{1/2}(Qu))||_{\dot{B}^0_{1,1}} \qquad (34)\\
&= \sup_{||h||_{\dot{B}^0_{\infty,\infty}} \leq 1} \int_{I\!R^n} \sum_j \sum_{|t-j|\leq 3} [\Delta^{1/4}(u^{j-4}\Delta^{1/4}Q_j) - \Delta^{1/2}(u^{j-4}Q_j)] h_t dx \\
&= \sup_{||h||_{\dot{B}^0_{\infty,\infty}} \leq 1} \int_{I\!R^n} \sum_j \sum_{|t-j|\leq 3} \mathcal{F}[u^{j-4}] \mathcal{F}[\Delta^{1/4}Q_j \Delta^{1/4}h_t - Q_j \Delta^{1/2} h_t] d\xi \\
&= \sup_{||h||_{\dot{B}^0_{\infty,\infty}} \leq 1} \int_{I\!R^n} \sum_j \sum_{|t-j|\leq 3} \mathcal{F}[u^{j-4}](\xi) \\
&\quad \left(\int_{I\!R^n} \mathcal{F}[Q_j](\zeta) \mathcal{F}[\Delta^{1/4}h_t](\xi-\zeta)(|\zeta|^{1/2} - |\xi-\zeta|^{1/2}) d\zeta \right) d\xi .
\end{aligned}
$$

Now we observe that in (34) we have $|\xi| \leq 2^{j-3}$ and $2^{j-2} \leq |\zeta| \leq 2^{j+2}$. Thus $|\frac{\xi}{\zeta}| \leq \frac{1}{2}$. Hence

$$
\begin{aligned}
||\zeta|^{1/2} - |\xi-\zeta|^{1/2} &= |\zeta|^{1/2}[1 - |1 - \frac{\xi}{\zeta}|^{1/2}] \qquad (35)\\
&= |\zeta|^{1/2} \frac{\xi}{\zeta} [1 + |1 - \frac{\xi}{\zeta}|^{1/2}]^{-1} \\
&= |\zeta|^{1/2} \sum_{k=-\infty}^{\infty} \frac{c_k}{k!} (\frac{\xi}{\zeta})^{k+1} .
\end{aligned}
$$

We introduce the following notation: for every $k \geq 0$ and $g \in \mathcal{S}'$ we set

$$S_k g = \mathcal{F}^{-1}[\xi^{-(k+1)}|\xi|^{1/2}\mathcal{F}g].$$

We note that if $h \in B^s_{\infty,\infty}$ then $S_k h \in \dot{B}^{s+1/2+k}_{\infty,\infty}$ and if $h \in \dot{H}^s$ then $S_k h \in \dot{H}^{s+1/2+k}$.

Finally if $Q \in \dot{H}^{1/2}$ then $\nabla^{k+1}(Q) \in \dot{H}^{-k-1/2}$.

We continue the estimate (34).

$$\begin{aligned}
&\sup_{||h||_{\dot{B}^0_{\infty,\infty}} \leq 1} \int_{I\!R^n} \sum_j \sum_{|t-j|\leq 3} \mathcal{F}[u^{j-4}](\xi) \\
&\quad (\int_{I\!R^n} \mathcal{F}[Q_j](\zeta)\mathcal{F}[\Delta^{1/4}h_t](\xi-\zeta)(|\xi-\zeta|^{1/2} - (|\zeta|^{1/2})d\zeta)d\xi \\
&= \sup_{||h||_{\dot{B}^0_{\infty,\infty}} \leq 1} \int_{I\!R^n} \sum_j \sum_{|t-j|\leq 3} \mathcal{F}[u^{j-4}](\xi) \\
&\quad \left[\int_{I\!R^n} |\zeta|^{1/2}\mathcal{F}[Q_j](\zeta)\mathcal{F}[\Delta^{1/4}h_t](\xi-\zeta)\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}(\frac{\xi}{\eta})^{\ell+1}d\zeta\right]d\xi \\
&\leq C \sup_{||h||_{\dot{B}^0_{\infty,\infty}} \leq 1} \sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}\int_{I\!R^n} \sum_j \sum_{|t-j|\leq 3} (i)^{-(\ell+1)}\mathcal{F}[\nabla^{\ell+1}u^{j-4}]\mathcal{F}[S_\ell Q_j \Delta^{1/4}h_t)](\xi)d\xi \\
&\leq C \sup_{||h||_{\dot{B}^0_{\infty,\infty}} \leq 1} ||h||_{\dot{B}^0_{\infty,\infty}} \sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}\int_{I\!R^n} \sum_j 2^{j/2}|\nabla^{\ell+1}u^{j-4}||S_\ell Q_j|dx \\
&\leq \sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}\int_{I\!R^n} \sum_j |2^{-(k+1/2)j}\nabla^{\ell+1}u^{j-4}||2^{(k+1)j}S_\ell Q_j|dx \\
&\leq C\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}(\int_{I\!R^n} \sum_j 2^{-2(\ell+1/2)j}|\nabla^{\ell+1}u^{j-4}|^2dx)^{1/2}(\int_{I\!R^n} \sum_j 2^{2(\ell+1)j}|S_\ell Q_j|^2dx)^{1/2} \\
&\text{by Plancherel Theorem} \\
&= C\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}(\int_{I\!R^n} \sum_j 2^{-2(\ell+1/2)j}|\xi|^{2\ell}|\mathcal{F}[\nabla u^{j-4}]|^2d\xi)^{1/2}(\int_{I\!R^n} \sum_j 2^{2(\ell+1)j}|\xi|^{-2(\ell+1/2)}|\mathcal{F}[Q_j]|^2d\xi)^{1/2} \\
&\leq C\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}2^{-3\ell}(\int_{I\!R^n} \sum_j 2^{-j}|\mathcal{F}[\nabla u^{j-4}]|^2d\xi)^{1/2}(\int_{I\!R^n} \sum_j 2^{j}|\mathcal{F}[Q_j]|^2d\xi)^{1/2} \\
&\leq C\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}2^{-3\ell}||Q||_{\dot{H}^{1/2}}||u||_{\dot{H}^{1/2}}.
\end{aligned}$$

Above we also use the fact that for every vector field X we have

$$\int_{I\!R^n} \sum_{j=-\infty}^{+\infty} 2^{-j}(X^{j-4})^2 dx = \int_{I\!R^n} \sum_{k,\ell} X_k X_\ell \sum_{j-4\ge k, j-4\ge \ell} 2^{-j} dx$$
$$\lesssim \int_{I\!R^n} \sum_{j=-\infty}^{+\infty} 2^{-j}(X_j)^2 dx\,. \tag{36}$$

The estimate of $||\Pi_2(\Delta^{1/4}(Q\Delta^{1/4}u) - \Delta^{1/2}(Qu))||_{\dot{B}^0_{1,1}}$ is analogous to (34).

- Estimate of $||\Pi_2(\Delta^{1/4}(\Delta^{1/4}Qu)||_{\mathcal{H}^1}$. It is as in (33).
- Estimate of $||\Pi_3(\Delta^{1/2}(Qu)||_{\mathcal{H}^1}$.

We show that it is indeed in the smaller space $\dot{B}^0_{1,1}$ (we have $\dot{B}^0_{1,1} \hookrightarrow \mathcal{H}^1$). We first observe that if $h \in \dot{B}^0_{\infty,\infty}$ then $\Delta^{1/2}h \in \dot{B}^{-1}_{\infty,\infty}$ and

$$\Delta^{1/2}h^{j-6} = \sum_{k=-\infty}^{j} \Delta^{1/2}h_k \le \sup_{k\in I\!N} |2^{-k}\Delta^{1/2}h_k| \sum_{k=-\infty}^{j} 2^k \le C2^j ||h||_{\dot{B}^0_{\infty,\infty}}\,.$$

$$||\Pi_3(\Delta^{1/2}(Qu)||_{\dot{B}^0_{1,1}} = \sup_{||h||_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{I\!R^n} \sum_j \sum_{|k-j|\le 3} \Delta^{1/2}(Q_j u_k)h \tag{37}$$
$$= \sup_{||h||_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{I\!R^n} \sum_j \sum_{|k-j|\le 3} \Delta^{1/2}(Q_j u_k)\left[h^{j-6} + \sum_{t=j-5}^{j+6} h_t\right] dx$$
$$= \sup_{||h||_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{I\!R^n} \sum_j \sum_{|k-j|\le 3} (Q_j u_k)\left[\Delta^{1/2}h^{j-6} + \sum_{t=j-5}^{j+6} \Delta^{1/2}h_t\right] dx$$
$$\le C \sup_{||h||_{\dot{B}^0_{\infty,\infty}}\le 1} ||h||_{\dot{B}^0_{\infty,\infty}} \int_{I\!R^n} \sum_j \sum_{|k-j|\le 3} 2^j |Q_j u_k| dx$$
$$\le C \left(\int_{I\!R^n} \sum_j 2^j Q_j^2 dx\right)^{1/2} \left(\int_{I\!R^n} \sum_j 2^j u_j^2 dx\right)^{1/2}$$
$$\le C||Q||_{\dot{H}^{1/2}} ||u||_{\dot{H}^{1/2}}\,.$$

- Estimate of $\Pi_3(\Delta^{1/4}(Q\Delta^{1/4}u))$.

We show that it is in $\dot{B}^0_{1,1}$.

We observe that if $h \in \dot{B}^0_{\infty,\infty}$ then $\Delta^{1/4}h \in B^{-1/2}_{\infty,\infty}$ and thus

$$\begin{aligned}
&||\Pi_3(\Delta^{1/4}(Q,\Delta^{1/4}u))||_{\dot{B}^0_{1,1}} = \sup_{||h||_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j|\le 3} \Delta^{1/4}(Q_j\Delta^{1/4}u_k)h \\
&= \sup_{||h||_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j|\le 3} \Delta^{1/4}(Q_j\Delta^{1/4}u_k)\left[\Delta^{1/4}h^{j-6} + \sum_{t=j-5}^{j+6} \Delta^{1/4}h_t\right] dx \\
&\le C \sup_{||h||_{\dot{B}^0_{\infty,\infty}}\le 1} ||h||_{\dot{B}^0_{\infty,\infty}} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j|\le 3} 2^{j/2}|Q_j\Delta^{1/4}u_k|dx \qquad (38) \\
&\le C(\int_{\mathbb{R}^n} \sum_j 2^j Q_j^2 dx)^{1/2}(\int_{\mathbb{R}^n} \sum_j (\Delta^{1/4}u_j)^2 dx)^{1/2} \\
&\le C||Q||_{\dot{H}^{1/2}}||u||_{\dot{H}^{1/2}}\,.
\end{aligned}$$

The estimate of $\Pi_3(\Delta^{1/4}(\Delta^{1/4}Qu))$ is analogous to (38) . $\square$

From Theorem 1.4 and the duality between BMO and $\mathcal{H}^1$ we get Theorem 1.2.

Proof of Theorem 1.2 .

For all $h, Q \in \dot{H}^{1/2}$ and $u \in BMO$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} [(\Delta^{1/4}(Q\Delta^{1/4}u) - Q\Delta^{1/2}u + \Delta^{1/4}Q\Delta^{1/4}u]hdx \\
&= \int_{\mathbb{R}^n} [(\Delta^{1/4}(Q\Delta^{1/4}h) - \Delta^{1/2}(Qh) + \Delta^{1/4}(h\Delta^{1/4}Q]udx \\
&\quad \text{by Theorem (1.4)} \\
&\le C||u||_{BMO}||R(Q,h)||_{\mathcal{H}^1} \le C||u||_{BMO}||Q||_{\dot{H}^{1/2}}||h||_{\dot{H}^{1/2}}\,.
\end{aligned}$$

Hence

$$||T(Q,u)||_{\dot{H}^{-1/2}} = \sup_{||h||_{\dot{H}^{1/2}}\le 1} \int_{\mathbb{R}^n} T(Q,u)hdx \le C||u||_{BMO}||Q||_{\dot{H}^{1/2}}\,.$$

$\square$

Proof of theorem 1.5. We observe that $\mathcal{R}$ is a Fourier multiplier of order zero thus $\mathcal{R}\colon H^{-1/2} \to H^{-1/2}$, $\mathcal{R}\colon \mathcal{H}^1 \to \mathcal{H}^1$, and $\mathcal{R}\colon \dot{B}^0_{1,1} \to \dot{B}^0_{1,1}$ (see [26] and [20]) .

The estimates are very similar to ones in Theorem 1.4, thus we will make only the following one.

- Estimate of $\Pi_1(\mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q\mathcal{R}u) - \nabla(Q\mathcal{R}u))$.

We observe that $\nabla u = \Delta^{1/4}\mathcal{R}\Delta^{1/4}u$

$$\begin{aligned}
&\sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1}\int_{I\!R^n}\sum_j\sum_{|t-j|\le 3}[\mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q_j\mathcal{R}u^{j-4})-\nabla(Q_j\mathcal{R}u^{j-4})]h_t dx \qquad (39)\\
\simeq &\sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1}\int_{I\!R^n}\sum_j\sum_{|t-j|\le 3}\mathcal{R}u^{j-4}[\mathcal{R}\Delta^{1/4}h_t\Delta^{1/4}Q_j)-\nabla h_t)Q_j]dx\\
\simeq &\sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1}\int_{I\!R^n}\sum_j\sum_{|t-j|\le 3}\mathcal{F}[\mathcal{R}u^{j-4}](\xi)(\int_{I\!R^n}\mathcal{F}[Q_j](\zeta)\mathcal{F}[\mathcal{R}\Delta^{1/4}h_t](\xi-\zeta)\\
&\left(|\zeta|^{1/2}-|\xi-\zeta|^{1/2}\right)d\zeta)d\xi\,.
\end{aligned}$$

Now we can proceed exactly as in (34) and get

$$\begin{aligned}
&\sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1}\int_{I\!R^n}\sum_j\sum_{|t-j|\le 3}[\mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q_j\mathcal{R}u^{j-4})-\nabla(Q_j\mathcal{R}u^{j-4})]h_t dx\\
&\le C\|Q\|_{\dot{H}^{1/2}}\|u\|_{\dot{H}^{1/2}}\,.\Box
\end{aligned}$$

From Theorem 1.5 and the duality between $\mathcal{H}^1$ and BMO we obtain Theorem 1.3 .

Proof of Theorem 1.3. It follows from Theorem 1.5 and the duality between $\mathcal{H}^1$ and BMO . $\Box$

Lemma 3.2 *Let* $u\in\dot{H}^{1/2}(I\!R^n)$*, then* $\mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)\in\mathcal{H}^1$ *and*

$$\|\mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)\|_{\mathcal{H}^1}\le C\|u\|^2_{\dot{H}^{1/2}}\,.$$

Proof of lemma 3.2. Since $\mathcal{R}\colon\mathcal{H}^1\to\mathcal{H}^1$, it is enough to verify that $\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u\in\mathcal{H}^1$.

- Estimate of $\Pi_1(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)$

$$\begin{aligned}
\|\Pi_1(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)\|_{\mathcal{H}^1} &= \int_{I\!R^n}(\sum_{j=-\infty}^{+\infty}[\Delta^{1/4}u_j(\mathcal{R}\Delta^{1/4}u)^{j-4}]^2)^{1/2}dx\\
&\le\int_{I\!R^n}\sup_j|(\mathcal{R}\Delta^{1/4}u)^{j-4})|(\sum_{j=0}^{+\infty}[\Delta^{1/4}u_j]^2)^{1/2}dx \qquad (40)\\
&\le(\int_{I\!R^n}|M(\mathcal{R}\Delta^{1/4}u)|^2dx)^{1/2}(\int_{I\!R^n}\sum_{j=-\infty}^{+\infty}[\Delta^{1/4}u_j]^2dx)^{1/2}\\
&\le C\|u\|^2_{\dot{H}^{1/2}}
\end{aligned}$$

The estimate of the $\mathcal{H}^1$ norm of $\Pi_2(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)$ is similar to (40) .

- Estimate of $\Pi_3(\Delta^{1/4}u \cdot \mathcal{R}\Delta^{1/4}u)$.

$$
\begin{aligned}
&\sup_{||h||_{\dot B^0_{\infty,\infty}}\le 1}\int_{I\!R^n}\sum_j\sum_{|k-j|\le 3}\Delta^{1/4}u_j\mathcal{R}(\Delta^{1/4}u_k)[h^{j-6}+\sum_{t-=j-5}^{j+6}h_t]dx\\
&=\sup_{||h||_{\dot B^0_{\infty,\infty}}\le 1}\int_{I\!R^n}\sum_j\sum_{|k-j|\le 3}\left[\Delta^{1/4}u_j\mathcal{R}(\Delta^{1/4}u_k)-u_j\nabla u_k+\frac{1}{2}\nabla(u_ju_k)\right]\\
&\qquad [h^{j-6}+\sum_{t-=j-5}^{j+6}h_t]dx \qquad (41)
\end{aligned}
$$

We only estimate the terms with h^{j-6}, being the estimates with h_t similar .

$$
\begin{aligned}
&\sup_{||h||_{\dot B^0_{\infty,\infty}}\le 1}\int_{I\!R^n}\sum_j\sum_{|k-j|\le 3}(\Delta^{1/4}u_j\mathcal{R}(\Delta^{1/4}u_k)-u_j\nabla u_k)h^{j-6}dx\\
&\sup_{||h||_{\dot B^0_{\infty,\infty}}\le 1}\int_{I\!R^n}\sum_j\sum_{|k-j|\le 3}\mathcal{F}[h^{j-6}](x)\left(\int_{I\!R^n}\mathcal{F}[u_j]\mathcal{F}[\mathcal{R}\Delta^{1/4}u_k][|y|^{1/2}-|x-y|^{1/2}]dy\right)dx
\end{aligned}
$$

by arguing as in (34)

$\le C||u||^2_{\dot H^{1/2}}$

Finally we also have

$$
\begin{aligned}
&\sup_{||h||_{\dot B^0_{\infty,\infty}}\le 1}\int_{I\!R^n}\sum_j\sum_{|k-j|\le 3}\frac{1}{2}\nabla(u_ju_k)h^{j-6}dx\\
&=\sup_{||h||_{\dot B^0_{\infty,\infty}}\le 1}\int_{I\!R^n}\sum_j\sum_{|k-j|\le 3}\frac{1}{2}(u_ju_k)\nabla h^{j-6}dx\\
&\le C\sup_{||h||_{\dot B^0_{\infty,\infty}}\le 1}||h||_{\dot B^0_{\infty,\infty}}\int_{I\!R^n}\sum_j\sum_{|k-j|\le 3}2^ju_ju_kdx\\
&\le C(\int_{I\!R^n}\sum_j 2^ju_j^2dx)^{1/2}=C||u||^2_{\dot H^{1/2}}\,. \qquad \Box
\end{aligned}
$$

We get the following result

Corollary 3.1 *Let* $n\in\dot H^{1/2}(I\!R^n,S^{m-1})$. *Then*

$$
\Delta^{1/4}[n\cdot\Delta^{1/4}n]\in\mathcal{H}^1\,. \qquad (42)
$$

Proof. Since $n \cdot \nabla n = 0$ we can write

$$\begin{aligned} \Delta^{1/4}[n \cdot \Delta^{1/4} n] &= \Delta^{1/4}[n \cdot \Delta^{1/4} n] - \mathcal{R}(n \cdot \nabla n) + \mathcal{R}[\Delta^{1/4} n \cdot \mathcal{R}\Delta^{1/4} n] \\ &\quad - \mathcal{R}[\Delta^{1/4} n \cdot \mathcal{R}\Delta^{1/4} n] \\ &= S(n\cdot, n) - -\mathcal{R}[\Delta^{1/4} n \cdot \mathcal{R}\Delta^{1/4} n]\,. \end{aligned}$$

The estimate (42) is a consequence of Theorem 1.5 and Lemma 3.2, which respectively imply that $S(n\cdot, n) \in \mathcal{H}^1$ and $\mathcal{R}(\Delta^{1/4} n \cdot \mathcal{R}\Delta^{1/4} n) \in \mathcal{H}^1$. □

4 Geometric localization properties of the $\dot{H}^{1/2}$−norm on the real line.

In the next Theorem we show that the $\dot{H}^{1.2}([a,b])$ norm $(-\infty \le a < b \le +\infty)$ can be localized in space. This result, besides being of independent interest, will be used in Section 5 for suitable localization estimates. For simplicity we will suppose that $[a,b] = [-1,1]$.

Theorem 4.1 [Localization of $H^{1/2}((-1,1))$ norm] *Let $u \in H^{1/2}((-1,1))$. Then for some $C > 0$ we have*

$$||u||^2_{\dot{H}^{1/2}((-1,1))} \simeq \sum_{j=-\infty}^{0} ||u||^2_{\dot{H}^{1/2}(A_j)}$$

where $A_j = B_{2^{j+1}} \setminus B_{2^{j-1}}$.

Proof. We set for every $i \in \mathbb{Z}$, $A'_i = B_{2^i} \setminus B_{2^{i-1}}$ and $\bar{u}'_i = |A'_i|^{-1} \int_{A'_i} u(x)\, dx$ (i.e. the mean value of u on the annulus A'_i). We have

$$\begin{aligned} ||u||^2_{\dot{H}^{1/2}((-1,1))} &\simeq \int_{[-1,1]} \int_{[-1,1]} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy \qquad (43) \\ &= \sum_{i,j=-\infty}^{0} \int_{A'_i} \int_{A'_j} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy \\ &= \sum_{i=-\infty}^{0} \int_{A'_i} \int_{A'_i} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy \\ &+2 \sum_{j=-\infty}^{0} \sum_{i>j+1} \int_{A'_i} \int_{A'_j} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy \\ &+2 \sum_{j=-\infty}^{0} \int_{A'_j} \int_{A'_{j+1}} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy\,. \end{aligned}$$

We first observe that

$$\sum_{i,j=-\infty}^{0}\int_{A_i'}\int_{A_j'}\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy \le \sum_{i,j=-\infty}^{0}\int_{A_i}\int_{A_j}\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy \tag{44}$$

and

$$\sum_{j=-\infty}^{0}\int_{A_j'}\int_{A_{j+1}'}\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy \le \sum_{j=-\infty}^{0}\int_{A_j}\int_{A_j}\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy\,. \tag{45}$$

It remains to estimate the term $\sum_{j=-\infty}^{0}\sum_{i>j+1}\int_{A_i'}\int_{A_j'}\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy$ in (43).

We have

$$\begin{aligned}
&\sum_{j=-\infty}^{0}\sum_{i>j+1}\int_{A_i'}\int_{A_j'}\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy\\
&\le C\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}\int_{A_i'}\int_{A_j'}|u(x)-u(y)|^2dxdy\\
&\le C(\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}\int_{A_i'}\int_{A_j'}|\bar{u}_i'-\bar{u}_j'|^2dxdy\\
&+\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}\int_{A_i'}\int_{A_j'}|u(x)-\bar{u}_i'|^2dxdy\\
&+\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}\int_{A_i'}\int_{A_j'}|u(y)-\bar{u}_j'|^2dxdy)\\
&\le C(\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}2^{i+j}|\bar{u}_i'-\bar{u}_j'|^2\\
&+\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}2^{j}\int_{A_i'}|u(x)-\bar{u}_i'|^2dx\\
&+\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}2^{i}\int_{A_j'}|u(y)-\bar{u}_j'|^2dy)\,.
\end{aligned}$$

- Estimate of $\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}2^{j}\int_{A_i'}|u(x)-\bar{u}_i'|^2dx\,.$

$$\begin{aligned}
&\sum_{j=-\infty}^{0}\sum_{i\geq j+2}2^{-2i}2^{j}\int_{A_i'}|u(x)-\bar{u}_i'|^2dx \qquad (46)\\
&=\sum_{i=-\infty}^{0}\sum_{j\leq i-2}2^{-2i}2^{j}\int_{A_i'}|u(x)-\bar{u}_i'|^2dx\\
&=\sum_{i=-\infty}^{0}2^{-2i}\int_{A_i'}|u(x)-\bar{u}_i'|^2dx(\sum_{j\leq i-2}2^{j})\\
&\leq C\sum_{i=-\infty}^{0}|A_i'|^{-1}\int_{A_i'}|u(x)-\bar{u}_i'|^2dx\\
&\leq C\sum_{i=-\infty}^{0}\int_{A_i'}\int_{A_i'}\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy\,.
\end{aligned}$$

In the last inequality we use the fact that for every i it holds

$$\begin{aligned}
&|A_i'|^{-1}\int_{A_i'}|u(x)-\bar{u}_i'|^2dx\\
&\leq |A_i'|^{-1}\int_{A_i'}|u(x)-|A_i'|^{-1}\int_{A_i'}u(y)dy|^2dx\\
&\leq |A_i'|^{-2}\int_{A_i'}\int_{A_i'}|u(x)-u(y)|^2dxdy\\
&\leq C\int_{A_i'}\int_{A_i'}\frac{|u(x)-u(y)|^2}{|x-y|^2}dyxdy\,.
\end{aligned}$$

- Estimate of $\sum_{j=-\infty}^{0}\sum_{i\geq j+2}2^{-2i}2^{j}\int_{A_i'}|u(y)-\bar{u}_j'|^2du$

$$\begin{aligned}
&\sum_{j=-\infty}^{0}\sum_{i\geq j+2}2^{-i}\int_{A_j'}|u(y)-\bar{u}_j'|^2dy \qquad (47)\\
&=\sum_{j=-\infty}^{0}\int_{A_j'}|u(x)-\bar{u}_j|^2dx(\sum_{i\leq j+2}2^{-i})\\
&=\frac{1}{2}\sum_{j=-\infty}^{0}2^{-j}\int_{A_j'}|u(x)-\bar{u}_j'|^2dy\\
&\leq C\sum_{j=-\infty}^{0}\int_{A_j'}\int_{A_j'}\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy\,.
\end{aligned}$$

- Estimate of $\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}2^{i+j}|\bar{u}'_i-\bar{u}'_j|^2$. We first observe that

$$|\bar{u}'_i-\bar{u}'_j|^2\le (i-j)\sum_{j}^{i-1}|\bar{u}'_{\ell+1}-\bar{u}'_\ell|^2$$

and

$$|\bar{u}_{\ell+1}-\bar{u}_\ell|^2\le |A_\ell|^{-1}\int_{A_\ell}|u-\bar{u}_\ell|^2dx,$$

where $\bar{u}_\ell=|A_\ell|^{-1}\int_{A_\ell}u(x)\,dx$.

We set $a_\ell=|A_\ell|^{-1}\int_{A_\ell}|u-\bar{u}_\ell|^2dx$. We have

$$\begin{aligned}&\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}2^{i+j}|\bar{u}'_i-\bar{u}'_j|^2\\ &\le\sum_{j=-\infty}^{0}\sum_{i\ge j+2}(i-j)2^{j-i}\sum_{j}^{i-1}a_\ell\le\sum_{\ell=-\infty}^{0}a_\ell\sum_{j=-\infty}^{\ell}\sum_{i-j\ge\ell+1-j}(i-j)2^{j-i}.\end{aligned}$$

We observe that

$$\sum_{i-j\ge\ell+1-j}(i-j)2^{j-i}\le\int_{\ell+1-j}^{+\infty}2^{-x}xdx=2^{-(\ell+1-j)}(\ell+2-j)\qquad(48)$$

and

$$\sum_{j=-\infty}^{\ell}2^{-(\ell+1-j)}(\ell+2-j)\le\int_{1}^{+\infty}2^{-t}(t+1)\,dx\le C,$$

for some constant C independent on ℓ. Therefore we get

$$\begin{aligned}&\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}2^{i+j}|\bar{u}'_i-\bar{u}'_j|^2 &(49)\\ &\le\sum_{j=-\infty}^{0}\sum_{i\ge j+2}(i-j)2^{j-i}\sum_{j}^{i-1}a_\ell\le C\sum_{\ell=-\infty}^{0}a_\ell\le C\sum_{\ell=-\infty}^{0}\int_{A_\ell}\int_{A_\ell}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dxdy.\end{aligned}$$

By combining (44),(45),(46),(47) and (49) we finally obtain

$$||u||^2_{\dot{H}^{1/2}((-1,1))}\lesssim\sum_{\ell=-\infty}^{0}||u||^2_{\dot{H}^{1/2}(A_\ell)}.$$

Next we show that

$$\sum_{\ell=-\infty}^{0}||u||^2_{\dot{H}^{1/2}(A_\ell)}\lesssim||u||^2_{\dot{H}^{1/2}((-1,1))}.\qquad(50)$$

We observe that for every ℓ we have $A_\ell = C_\ell \cup D_\ell$ where $C_\ell = B_{2^{\ell+1}} \setminus B_{2^\ell}$ and $D_\ell = B_{2^\ell} \setminus B_{2^{\ell-1}}$. Thus

$$\begin{aligned}||u||^2_{\dot{H}^{1/2}(A_\ell)} &= \int_{C_\ell}\int_{C_\ell} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy \\ &+ \int_{D_{\ell,h}}\int_{D_\ell} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy + 2\int_{D_{\ell,h}}\int_{C_\ell} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy\,.\end{aligned}$$

Since $\cup_\ell (C_\ell \times C_\ell)$, $\cup_\ell (D_\ell \times C_\ell)$ and $\cup_\ell (D_\ell \times C_\ell)$ are disjoint unions contained in $[0,1]\times[0,1]$ we have

$$\sum_\ell \int_{C_\ell}\int_{C_\ell} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy \le \int_{[-1,1]}\int_{[-1,1]} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy\,;$$

$$\sum_\ell \int_{D_{\ell,h}}\int_{C_\ell} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy \le \int_{[-1,1]}\int_{[-1,1]} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy\,;$$

$$\sum_\ell \int_{D_{\ell,h}}\int_{D_\ell} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy \le \int_{[-1,1]}\int_{[-1,1]} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy\,.$$

It follows that

$$\sum_{\ell=-\infty}^{0} ||u||^2_{\dot{H}^{1/2}(A_\ell)} \le \bar{C} \int_{[-1,1]}\int_{[-1,1]} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy = \bar{C}||u||^2_{\dot{H}^{1/2}((-1,1))}$$

and we can conclude. □

Remark 4.1 By analogous computations one can show that for all $r > 0$ we have

$$||u||^2_{\dot{H}^{1/2}(I\!R)} \simeq \sum_{j=-\infty}^{+\infty} ||u||^2_{\dot{H}^{1/2}(A^r_j)}$$

where $A^r_j = B_{2^{j+1}r} \setminus B_{2^{j-1}r}$, where the equivalence constants do not depend on r.

Next we compare the $\dot{H}^{1/2}$ norm of $\Delta^{-1/4}(M\Delta^{1/4}u)$ with the L^2 norm of $M\Delta^{1/4}u$, where $u \in \dot{H}^{1/2}(I\!R)$ and $M \in \dot{H}^{1/2}(I\!R, \mathcal{M}_{p\times m}(I\!R))$, $p \ge 0$.

Lemma 4.1 *Let $M \in \dot{H}^{1/2}(I\!R, \mathcal{M}_{p\times m}(I\!R))$, $m \ge 1, p \ge 1$, and $u \in \dot{H}^{1/2}(I\!R)$. Then there exist $C_1 > 0$, $C_2 > 0$ and $n_0 \in I\!N$, independent of u and M, such that, for any $r \in (0,1)$, $n > n_0$ and any $x_0 \in I\!R$, we have*

$$
\begin{aligned}
||\Delta^{-1/4}(M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(B_r(x_0))} \quad & \geq C_1\int_{B_{r/2^n}(x_0)}|M\Delta^{1/4}u|^2dx\\
& -C_2\sum_{h=-n}^{+\infty}2^{-h}\int_{B_{2^hr}(x_0)\setminus B_{2^{h-1}r}(x_0)}|M\Delta^{1/4}u|^2\ dx\,.
\end{aligned}
$$

Proof of lemma 4.1. We write

$$
\Delta^{-1/4}(M\Delta^{1/4}u)=\Delta^{-1/4}(1\!\!1_{|x|\leq r/2^n}M\Delta^{1/4}u)+\Delta^{-1/4}((1-1\!\!1_{|x|\leq r/2^n})M\Delta^{1/4}u)\,,
$$

where $n>0$ is large enough(the threshold will be determined later in the proof).

For any $\rho\geq 0$, we denote by $1\!\!1_{|x|\leq\rho}$ and $1\!\!1_{\rho\leq|x|}$ the characteristic functions of the sets of point $x\in I\!\!R$ respectively where $|x|\leq\rho$ and $|x|\geq\rho$. For all $\rho\leq\sigma$ we also denote by $1\!\!1_{\rho\leq|x|\leq\sigma}$ the characteristic function of the set $\{x\in I\!\!R\ ;\ \rho\leq|x|\leq\sigma\}$. We have

$$
\begin{aligned}
&||\Delta^{-1/4}(M\Delta^{1/4}u)||_{\dot{H}^{1/2}(B_r)}\geq||\Delta^{-1/4}(1\!\!1_{r/2^n}M\Delta^{1/4}u)||_{\dot{H}^{1/2}(B_r)}\\
&\qquad -||\Delta^{-1/4}((1-1\!\!1_{|x|\leq r/2^n})M\Delta^{1/4}u)||_{\dot{H}^{1/2}(B_r)}\\
&\geq||\Delta^{-1/4}(1\!\!1_{r/2^n}M\Delta^{1/4}u)||_{\dot{H}^{1/2}(B_r)}-||\Delta^{-1/4}(1\!\!1_{r/2^n\leq|x|\leq 4r}M\Delta^{1/4}u)||_{\dot{H}^{1/2}(B_r)}\\
&\qquad -||\Delta^{-1/4}(1\!\!1_{|x|\geq 4r}M\Delta^{1/4}u)||_{\dot{H}^{1/2}(B_r)}\\
&\geq||\Delta^{-1/4}(1\!\!1_{|x|\leq r/2^n}M\Delta^{1/4}u)||_{\dot{H}^{1/2}(B_r)}-||\Delta^{-1/4}(1\!\!1_{r/2^n\leq|x|\leq 4r}M\Delta^{1/4}u)||_{\dot{H}^{1/2}(I\!\!R)}\\
&\qquad -||\Delta^{-1/4}(1\!\!1_{|x|\geq 4r}M\Delta^{1/4}u)||_{\dot{H}^{1/2}(B_r)}\quad .
\end{aligned}
\tag{51}
$$

We estimate of the last three terms in (51).

- Estimate of $||\Delta^{-1/4}(1\!\!1_{r/2^n\leq|x|\leq 4r}M\Delta^{1/4}u)||_{\dot{H}^{1/2}(I\!\!R)}$.

$$
\begin{aligned}
||\Delta^{-1/4}(1\!\!1_{r/2^n\leq|x|\leq 4r}M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(I\!\!R)}&=\int_{r/2^n\leq|x|\leq 4r}|M\Delta^{1/4}u|^2\ dx\\
=\sum_{h=-n}^{1}\int_{2^hr\leq|x|\leq 2^{h+1}r}|M\Delta^{1/4}u|^2\ dx\,.
\end{aligned}
\tag{52}
$$

- Estimate of $||\Delta^{-1/4}(1\!\!1_{|\xi|\geq 4r}M\Delta^{1/4}u)||_{\dot{H}^{1/2}(B_r)}$. We set

$$
g:=1\!\!1_{|x|\geq 4r}M\Delta^{1/4}u\quad .
$$

With this notation we have

$$\begin{aligned}
&||\Delta^{-1/4}(\mathbb{1}_{|x|\geq 4r}M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(B_r)}\\
&=\int_{B_r}\int_{B_r}\frac{|(\frac{1}{|x|^2}\star g)(t)-(\frac{1}{|x|^2}\star g)(s)|^2}{|x-y|^2}dt\ ds\\
&=\int_{B_r}\int_{B_r}\frac{1}{|t-s|^2}\left(\int_{|x|\geq 4r}g(x)(\frac{1}{|t-x|^{1/2}}-\frac{1}{|s-x|^{1/2}})dx\right)^2 dt\ ds\\
&\quad\text{by Mean Value Theorem}\\
&\leq C\int_{B_r}\int_{B_r}\left(\int_{|x|\geq 4r}|g(x)|\max(\frac{1}{|t-x|^{3/2}},\frac{1}{|s-x|^{3/2}})dx\right)^2 dt\ ds\\
&\leq C\int_{B_r}\int_{B_r}\left(\sum_{h=4}^{+\infty}\int_{2^hr\leq|x|\leq 2^{h+1}r}|g(x)|\max(\frac{1}{|t-x|^{3/2}},\frac{1}{|s-x|^{3/2}})dx\right)^2 dt\ ds\\
&\leq C\int_{B_r}\int_{B_r}\left(\sum_{h=4}^{+\infty}\int_{2^hr\leq|x|\leq 2^{h+1}r}|g(x)|2^{-3/2h}r^{-3/2}d\xi\right)^2 dt\ ds\\
&\quad\text{by Hölder Inequality}\\
&\leq C\int_{B_r}\int_{B_r}\left(\sum_{h=4}^{+\infty}2^{-h}r^{-1}(\int_{2^{h+1}r\leq|x|\leq 2^{h+1}r}|g(x)|^2\ dx)^{1/2}\right)^2 dt\ ds\\
&\quad\text{by Cauchy-Schwartz Inequality}\\
&\leq C(\sum_{h=4}^{+\infty}2^{-h})\left(\sum_{h=4}^{+\infty}2^{-h}\int_{B_{2^{h+1}r}(x_0)\setminus B_{2^hr}(x_0)}|M\Delta^{1/4}u|^2\ dx\right)\\
&\leq C\left(\sum_{h=4}^{+\infty}2^{-h}\int_{B_{2^{h+1}r}(x_0)\setminus B_{2^hr}(x_0)}|M\Delta^{1/4}u|^2\ dx\right)\ .
\end{aligned}\tag{53}$$

- Estimate of $||\Delta^{-1/4}(\mathbb{1}_{|x|\leq r/2^n}M\Delta^{1/4}u)||_{\dot{H}^{1/2}(B_r)}$.

We set

$$A_h^r:=\{x\ :\ 2^{h-1}r\leq|x|\leq 2^{h+1}r\}\quad .$$

By Localization Theorem 4.1 there exists a constant $\tilde{C} > 0$ (independent on r) such that

$$\begin{aligned}
||\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(\mathbb{R})} &\le \tilde{C} \sum_{h=-\infty}^{+\infty} ||\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(A^r_h)} \\
&\le \tilde{C}\ ||\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(B_r)} \\
&+\tilde{C} \sum_{h=0}^{+\infty} ||\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(A^r_h)}\,.
\end{aligned} \tag{54}$$

- Estimate of $\sum_{h=0}^{+\infty} ||\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(A^r_h)}$. We set now

$$f(x) := \mathbb{1}_{|x|\le r/2^n}\ (M\Delta^{1/4}u)\,.$$

Using this notation we have

$$\begin{aligned}
&\sum_{h=0}^{+\infty} ||\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(A^r_h)} \\
&\le \sum_{h=0}^{+\infty} \int_{A^r_h}\int_{A^r_h} \left(\int_{|x|\le r/2^n} |f(x)|(|\frac{1}{|t-x|^{1/2}} - \frac{1}{|s-x|^{1/2}}|)dx\right)^2 dt\ ds \\
&\quad \text{by Mean Value Theorem} \\
&\le C\sum_{h=0}^{+\infty} \int_{A^r_h}\int_{A^r_h} \left(\int_{|x|\le r/2^n} |f(x)| \max(\frac{1}{|t-x|^{3/2}}, \frac{1}{|s-x|^{3/2}})d\xi\right)^2 dt\ ds \\
&\le C\sum_{h=0}^{+\infty} \int_{A^r_h}\int_{A^r_h} \max(\frac{1}{|t|^3}, \frac{1}{|s|^3})\frac{r}{2^n}(\int_{|x|\le r/2^n} |f(x)|^2\ dx)dt\ ds \\
&= \frac{C}{2^n}\sum_{h=0}^{+\infty} 2^{-h}(\int_{|x|\le r/2^n} |f(x)|^2\ dx) \le \frac{C}{2^n}\int_{B_{r/2^n}(x_0)} |M\Delta^{1/4}u|^2\ dx
\end{aligned} \tag{55}$$

If n is large enough in such a way that $C\,\tilde{C}/2^n < 1/2$, we get, combining (51), (52), (53), (54) and (55), for some C_1, C_2 positive,

$$\begin{aligned}
&||\Delta^{-1/4}(M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(B_r)} \\
&\quad \ge C_1 \int_{B_{r/2^n}} |M\Delta^{1/4}u|^2\ dx \\
&\quad -C_2 \sum_{h=-n}^{+\infty} 2^{-h} \int_{B_{2^{h+1}r}\setminus B_{2^h r}} |M\Delta^{1/4}u|^2\ dx\,,
\end{aligned} \tag{56}$$

which ends the proof of the lemma. $\Box$

In the following Lemma we compare the $\dot{H}^{1/2}$ norm of $w = \Delta^{-1/4}(M\Delta^{1/4}u)$ in the annuli $A_h = B_{2^{h+1}}(x_0) \setminus B_{2^{h-1}}(x_0)$ with the L^2 norm in the same annuli of $M\Delta^{1/4}u$. Such a result will be used in the following Section for suitable *localization* estimates.

Lemma 4.2 *Let $M \in \dot{H}^{1/2}(I\!R, \mathcal{M}_{p\times m}(I\!R))$, $m \geq 1, p \geq 1$, and $u \in \dot{H}^{1/2}(I\!R)$. Then there exists $C > 0$ such that for every $\gamma \in (0,1)$, for all $n \geq n_0 \in I\!N$ (n_0 dependent on γ and independent of u and M), for every $k \in Z\!\!Z$, and any $x_0 \in I\!R$, we have*

$$\sum_{h=k}^{+\infty} 2^{k-h} ||\Delta^{-1/4}(M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(B_{2^{h+1}}(x_0)\setminus B_{2^{h-1}}(x_0))} \leq \gamma \int_{B_{2^{k-n}}(x_0)} |M\Delta^{1/4}u|^2 d\xi$$
$$+ \sum_{h=k-n}^{+\infty} 2^{\frac{k-h}{2}} \int_{B_{2^{h+1}}(x_0)\setminus B_{2^{h-1}}(x_0)} |M\Delta^{1/4}u|^2 d\xi\,.$$

Proof. Given $h \in Z\!\!Z$ and $\ell \geq 3$ we set $A_h = B_{2^{h+1}}(x_0) \setminus B_{2^{h-1}}(x_0)$ and $D_{\ell,h} = B_{2^{h+\ell}}(x_0) \setminus B_{2^{h-\ell}}(x_0)$. For simplicity of notations we suppose that $x_0 = 0$ but all the following estimates will be independent on x_0.

We fix $\gamma \in (0,1)$.

We have

$$||w||^2_{\dot{H}^{1/2}(A_h)} = \int_{A_h}\int_{A_h} \frac{|w(x)-w(y)|^2}{|x-y|^2} dxdy$$

$$\leq 2||\Delta^{-1/4} 1\!\!1_{D_{\ell,h}} M\Delta^{1/4}u||^2_{\dot{H}^{1/2}(A_h)} + 2||\Delta^{-1/4}(1 - 1\!\!1_{D_{\ell,h}})M\Delta^{1/4}u||^2_{\dot{H}^{1/2}(A_h)}$$

$$\leq 2||\Delta^{-1/4} 1\!\!1_{D_{\ell,h}} M\Delta^{1/4}u||^2_{\dot{H}^{1/2}(I\!R)} + 2||\Delta^{-1/4}(1 - 1\!\!1_{D_{\ell,h}})M\Delta^{1/4}u||^2_{\dot{H}^{1/2}(A_h)}\,.$$

The constant ℓ will be determined later.

- Estimate of $||\Delta^{-1/4} 1\!\!1_{D_{\ell,h}} M\Delta^{1/4}u||^2_{\dot{H}^{1/2}(I\!R)}$.

$$||\Delta^{-1/4} 1\!\!1_{D_{\ell,h}} M\Delta^{1/4}u||^2_{\dot{H}^{1/2}(I\!R)} = \int_{D_{\ell,h}} |M\Delta^{1/4}u|^2 dx$$
$$= \sum_{s=h-\ell}^{h+\ell-1} \int_{B_{2^{s+1}}\setminus B_{2^s}} |M\Delta^{1/4}u|^2 dx\,. \tag{57}$$

We multiply (57) by 2^{k-h} and we sum up from $h = k$ to $+\infty$ and get

$$\sum_{h=k}^{+\infty} 2^{k-h} ||\Delta^{-1/4} 1\!\!1_{D_{\ell,h}} M\Delta^{1/4}u||^2_{\dot{H}^{1/2}(I\!R)} \leq C 2^{\ell} \sum_{h=k-\ell}^{+\infty} \int_{B_{2^{h+1}}\setminus B_{2^{h-1}}} |M\Delta^{1/4}u|^2 dx\,. \tag{58}$$

- Estimate of $||\Delta^{-1/4}(1-1\!\!1_{D_{\ell,h}})M\Delta^{1/4}u||^2_{\dot H^{1/2}(A_h)}$.

We set $g=(1-1\!\!1_{D_{\ell,h}})M\Delta^{1/4}u\,.$

$$
\begin{aligned}
&||\Delta^{-1/4}(1-1\!\!1_{D_{\ell,h}})M\Delta^{1/4}u||^2_{\dot H^{1/2}(A_h)}=\int_{A_h}\int_{A_h}\frac{|(\frac{1}{|x|^2}\star g)(t)-(\frac{1}{|x|^2}\star g)(s)|^2}{|t-s|^2}dtds\\
&\le 2\int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|>2^{\ell+h}}g(x)(\frac{1}{|x-t|^{1/2}}-\frac{1}{|x-s|^{1/2}})dx\right)^2dtds \qquad (59)\\
&+2\int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|<2^{h-\ell}}g(x)(\frac{1}{|x-t|^{1/2}}-\frac{1}{|x-s|^{1/2}})dx\right)^2dtds\,.
\end{aligned}
$$

We estimate the last two terms in (59).

1. Estimate of $\int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|>2^{\ell+h}}g(x)(\frac{1}{|x-t|^{1/2}}-\frac{1}{|x-s|^{1/2}})dx\right)^2dtds\,.$

$$
\begin{aligned}
&\int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|>2^{\ell+h}}g(x)(\frac{1}{|x-t|^{1/2}}-\frac{1}{|x-s|^{1/2}})dx\right)^2dtds\\
&\le C\int_{A_h}\int_{A_h}\left(\sum_{s=h+\ell}^{\infty}\int_{2^s\le|x|\le2^{s+1}}g(x)\max(\frac{1}{|x-t|^{3/2}},\frac{1}{|x-s|^{3/2}})dx\right)^2dtds\\
&\text{by Hölder Inequality} \qquad (60)\\
&\le C\int_{A_h}\int_{A_h}\left(\sum_{s=h+\ell}^{\infty}2^{-s}\int_{2^s\le|x|\le2^{s+1}}|g(x)|^2dx)^{1/2}\right)^2dtds\\
&\quad\text{by Cauchy-Schwartz Inequality}\\
&\le C2^{2h}(\sum_{s=h+\ell}^{\infty}2^{-s})\left(\sum_{s=h+\ell}^{\infty}2^{-s}(\int_{2^s\le|x|\le2^{s+1}}|g(x)|^2dx\right)\\
&\le C2^{h-\ell}\left(\sum_{s=h+\ell}^{\infty}2^{-s}\int_{2^s\le|x|\le2^{s+1}}|g(x)|^2dx\right)\,.
\end{aligned}
$$

We observe that in (60) we use the fact that, since $\ell\ge3$ then $|x-t|,|x-s|\ge2^{s-1}$ for every $x,y\in A_h$ and $2^s\le|\xi|\le2^{s+1}$.

We multiply the last term in (60) by 2^{k-h}, where $k \in \mathbb{Z}$, and we sum up from $h = k$ to $+\infty$. We get

$$\begin{aligned}
&\sum_{h=k}^{+\infty} 2^{k-h} 2^{h-\ell} \left(\sum_{s=h+\ell}^{\infty} 2^{-s} \int_{2^s \le |x| \le 2^{s+1}} |M\Delta^{1/4}u|^2 dx \right) \\
&= 2^{-\ell} \sum_{s=k+\ell}^{+\infty} 2^{k-s}(s-\ell-k) \left(\int_{2^s \le |x| \le 2^{s+1}} |M\Delta^{1/4}u|^2 dx \right) \\
&\le C2^{-\ell} \sum_{s=k+\ell}^{+\infty} 2^{\frac{k-s}{2}} \left(\int_{2^s \le |x| \le 2^{s+1}} |M\Delta^{1/4}u|^2 dx \right) .
\end{aligned} \tag{61}$$

2. Estimate of $\int_{A_h} \int_{A_h} \frac{1}{|t-s|^2} \left(\int_{|x|<2^{h-\ell}} g(x)(\frac{1}{|x-t|^{1/2}} - \frac{1}{|x-s|^{1/2}})dx \right)^2 dtds$.
For $h \ge k$ we have

$$\begin{aligned}
&\int_{A_h} \int_{A_h} \frac{1}{|t-s|^2} \left(\int_{|x|<2^{h-\ell}} g(x)(\frac{1}{|x-s|^{1/2}} - \frac{1}{|x-t|^{1/2}})dx \right)^2 dtds \\
&\text{ny Mean Value Theorem} \\
&\le C \int_{A_h} \int_{A_h} \left(\int_{|x|<2^{h-\ell}} g(x) \max(\frac{1}{|x-t|^{3/2}}, \frac{1}{|x-s|^{3/2}})dx \right)^2 dtds \\
&\le C \int_{A_h} \int_{A_h} 2^{-3h} 2^{h-\ell} \left(\int_{|x|<2^{h-\ell}} |g(x)|^2 dx \right) dtds \\
&= C2^{-\ell} \int_{|x|<2^{h-\ell}} |M\Delta^{1/4}u|^2 dx \\
&= C2^{-\ell} \left(\int_{|x|<2^{k-\ell}} |M\Delta^{1/4}u|^2 dx + \sum_{s=k-\ell}^{h-\ell} \int_{2^s \le |\xi| < 2^{s+1}} |M\Delta^{1/4}u|^2 dx \right) .
\end{aligned} \tag{62}$$

In (62) we use the fact that since $\ell \ge 3$, $t, s \in A_h$ and $|x| < 2^{h-\ell}$ we have $|x-s|, |x-t| \ge 2^{h-2}$.

We multiply (62) by 2^{k-h}, and we sum up from $h = k$ to $+\infty$. We get

$$\begin{aligned}
&\int_{A_h} \int_{A_h} \frac{1}{|x-y|^2} \left(\int_{|x|<2^{h-\ell}} g(x)(\frac{1}{|x-s|^{1/2}} - \frac{1}{|x-t|^{1/2}})dx \right)^2 dtds \\
&\le C2^{-\ell} \int_{|x|<2^{k-\ell}} |M\Delta^{1/4}u|^2 dx + C2^{-2\ell} \sum_{h=k-\ell}^{+\infty} 2^{k-h} \int_{2^h \le |x| \le 2^{h+1}} |M\Delta^{1/4}u|^2 dx .
\end{aligned} \tag{63}$$

We choose ℓ so that $C2^{-\ell} < \gamma$ and let $n_0 \geq \ell$. Then for all $n \geq n_0$ we obtain

$$\sum_{h=k}^{+\infty} 2^{k-h} \left[C2^{-\ell} \int_{|x|<2^{k-\ell}} |M\Delta^{1/4}u|^2 dx + C2^{-2\ell} \sum_{s=k-\ell}^{h-\ell} \int_{2^s \leq |x| \leq 2^{s+1}} |M\Delta^{1/4}u|^2 dx \right]$$
$$\leq \gamma \int_{|x|<2^{k-n}} |M\Delta^{1/4}u|^2 dx + \sum_{h=k-n}^{+\infty} 2^{k-h} \int_{2^h \leq |x| \leq 2^{h+1}} |M\Delta^{1/4}u|^2 dx\,.$$

By combining (58), (61), (63), for $n \geq n_0$ we finally get

$$\sum_{h=k}^{+\infty} 2^{k-h} ||\Delta^{-1/4}(M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(A_h)}$$
$$\leq \gamma \int_{|x|<2^{k-n}} |M\Delta^{1/4}u|^2 dx + \sum_{h=k-n}^{+\infty} \int_{2^{h-1} \leq |x| \leq 2^{h+1}} 2^{k-h} |M\Delta^{1/4}u|^2 dx\,.$$

and we conclude the proof. □

Next we show a sort of Poincaré Inequality for functions in $\dot{H}^{1/2}(I\!\!R)$ having compact support. We remark that in general the extension by zero of a function in $H_0^{1/2}(\Omega) = \overline{C_0^\infty(\Omega)}^{H^{1/2}}$, Ω open subset of $I\!\!R$ is not in $H^{1/2}(I\!\!R)$. This is the reason why Lions and Magenes [12] introduced the set $H_{00}^{1/2}(\Omega)$ for which Poincaré Inequality holds.

Theorem 4.2 *Let $v \in \dot{H}^{1/2}(I\!\!R)$ be such that supp$(v) \subset (-1,1)$.*
Then $v \in L^2([-1,1])$ and

$$\int_{[-1,1]} |v(x)|^2 dx \leq C||v||^2_{\dot{H}^{1/2}}((-2,2))\,.$$

Proof. We have

$$\int_{[-1,1]} |v(x)|^2 dx \leq 9 \int_{1 \leq |y| \leq 2|} \int_{|x| \leq 1} \frac{|v(x)|^2}{|x-y|^2} dx dy$$
$$\leq C \int_{1 \leq |y| \leq 2} \int_{|x| \leq 1} \frac{|v(x)|^2}{|x-y|^2} dx dy$$
$$\leq C \int_{1 \leq |y| \leq 2|} \int_{|x| \leq 1} \frac{|v(x)-v(y)|^2}{|x-y|^2} dx dy$$
$$\leq C \int_{|y| \leq 2|} \int_{|x| \leq 2} \frac{|v(x)-v(y)|^2}{|x-y|^2} dx dy = C||v||^2_{\dot{H}^{1/2}}([-2,2])\,.$$

We can conclude. □

From Lemma 4.2 it follows that

$$||v||_{L^2((-r,r))} \leq Cr^{1/2}||v||_{\dot{H}^{1/2}(I\!R)} \, .$$

We conclude this Section with the following technical result.

Proposition 4.1 *Let* $(a_k)_k$ *be a sequence of positive real numbers satisfying* $\sum_{k=-\infty}^{+\infty} a_k^2 < +\infty$ *and for every* $n \leq 0$

$$\sum_{-\infty}^{n} a_k^2 \leq C \left(\sum_{k=n+1}^{+\infty} 2^{\frac{n+1-k}{2}} a_k^2 \right) . \tag{64}$$

Then there are $0 < \beta < 1$, $C > 0$ *and* $\bar{n} < 0$ *such that for* $n \leq \bar{n}$ *we have*

$$\sum_{-\infty}^{n} a_k^2 \leq C(2^n)^{\beta} \, .$$

Proof. For $n < 0$, we set $A_n = \sum_{-\infty}^{n} a_k^2$. We have $a_k^2 = A_k - A_{k-1}$ and thus

$$A_n \leq C \sum_{n+1}^{+\infty} 2^{\frac{n+1-k}{2}} (A_k - A_{k-1}) \leq C(1 - 1/\sqrt{2}) \sum_{n+1}^{+\infty} 2^{\frac{n+1-k}{2}} A_k - CA_n \, .$$

Therefore

$$A_n \leq \tau \sum_{n+1}^{+\infty} 2^{\frac{n+1-k}{2}} A_k \, , \tag{65}$$

$\tau = \frac{C}{(C+1)}(1 - 1/\sqrt{2}) < 1 - 1/\sqrt{2}\, .$

The relation (65) implies the following estimate

$$
\begin{aligned}
& A_n \le \tau A_{n+1} + \tau \sum_{n+2}^{+\infty} 2^{\frac{n+1-k}{2}} A_k \\
& \text{by induction} \\
& \le \tau^2 \left(\sum_{n+2}^{+\infty} 2^{\frac{n+2-k}{2}} A_k \right) + \frac{\tau}{\sqrt{2}} \left(\sum_{n+2}^{+\infty} 2^{\frac{n+2-k}{2}} A_k \right) \\
& = \tau(\tau + 1/\sqrt{2}) \left(\sum_{n+2}^{+\infty} 2^{\frac{n+2-k}{2}} A_k \right) \\
& = \tau(\tau + 1/\sqrt{2}) \left[A_{n+2} + 1/\sqrt{2} \sum_{n+3}^{+\infty} 2^{\frac{n+3-k}{2}} A_k \right] \\
& \text{again by induction} \\
& \le \tau(\tau + 1/\sqrt{2})^2 \sum_{n+3}^{+\infty} 2^{\frac{n+3-k}{2}} A_k \\
& \le \dots \\
& \le \tau(\tau + 1/\sqrt{2})^{-n} \sum_{k=0}^{+\infty} 2^{-k} A_k \\
& \le \tau(\tau + 1/\sqrt{2})^{-n} \sum_{h=-\infty}^{+\infty} a_h^2 \, .
\end{aligned}
$$

Therefore for some $\beta \in (0,1)$ and for all $n < 0$ we have

$$A_n \le C(2^n)^\beta \, . \qquad \Box$$

5 L-Energy Decrease Controls.

In this Section we provide some *localization estimates* of solutions to the following equations

$$\Delta^{1/4}(M\Delta^{1/4}u) = T(Q,u)\, ; \tag{66}$$

and

$$\Delta^{1/4}(M\Delta^{1/4}u) = S(Q,u) - \mathcal{R}(\Delta^{1/4}u \cdot \mathcal{R}\Delta^{1/4}u)\, , \tag{67}$$

where $Q \in \dot{H}^{1/2}(I\!R, \mathcal{M}_{\ell\times m}(I\!R))$, $M \in \dot{H}^{1/2}(I\!R, \mathcal{M}_{p\times m}(I\!R))$, $\ell, p \ge 1$.

We will consider a dyadic decomposition of the unity φ_j such that

$$\operatorname{supp}(\varphi_j) \subset B_{2^{j+1}} \setminus B_{2^{j-1}}, \quad \sum_{-\infty}^{+\infty} \varphi_j = 1\, .$$

For every $k, h \in \mathbb{Z}$, we set

$$\chi_k := \sum_{-\infty}^{k-1} \varphi_j \,,\,, \ \ \bar{u}_k = |B_{2^k}|^{-1} \int_{B_{2^k}} u(x)\, dx,$$

$$A_h = B_{2^{h+1}} \setminus B_{2^{h-1}} \ \text{ and } \bar{u}^h = |A_h|^{-1} \int_{A_h} u(x)\, dx\,,$$

$$A'_h = B_{2^h} \setminus B_{2^{h-1}} \ \text{ and } \bar{u}'^{,h} = |A'_h|^{-1} \int_{A'_h} u(x) dx\,.$$

We prove the following results.

Lemma 5.1 *Let* $Q \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{\ell\times m}(\mathbb{R}))$, $M \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{p\times m}(\mathbb{R}))$, $\ell, p \geq 0$ *and let* $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ *be a solution of (66). Then for* $k < 0$ *with* $|k|$ *large enough we have*

$$\begin{aligned} ||M\Delta^{1/4}u||^2_{L^2(B_{2^k})} - \frac{1}{4}||\Delta^{1/4}u||^2_{L^2(B_{2^k})} \ &\leq \ C\left[\sum_{h=k}^{\infty}(2^{\frac{k-h}{2}})||M\Delta^{1/4}u||^2_{L^2(A_h)}\right. \\ &\left. \quad + \sum_{h=k}^{\infty}(2^{\frac{k-h}{2}})||\Delta^{1/4}u||^2_{L^2(A_h)}\right]. \end{aligned} \tag{68}$$

Lemma 5.2 *Let* $Q \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{\ell\times m})$, $M \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{p\times m}(\mathbb{R}))$, $\ell, p \geq 1$ *and let* $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ *be a solution of (67). Then for* $k < 0$ *with* $|k|$ *large enough the estimate (68) holds.*

In the next Section we will see that weak 1/2-harmonic maps u satisfy both the equations (15) and (17) which are (66) and (67) with (M, Q) given respectively by $(u\wedge, u\wedge)$ and $(u\cdot, u\cdot)$.

We premise some estimates.

Lemma 5.3 *Let* $u \in \dot{H}^{1/2}(\mathbb{R})$. *Then for all* $k \in \mathbb{Z}$ *the following estimate holds*

$$\sum_{h=k}^{+\infty} 2^{k-h}||\varphi_h(u-\bar{u}_k)||_{\dot{H}^{1/2}(\mathbb{R})} \leq C\left[\sum_{s\leq k} 2^{s-k}||u||_{\dot{H}^{1/2}(A_s)} + \sum_{s\geq k} 2^{k-s}||u||_{\dot{H}^{1/2}(A_s)}\right]. \tag{69}$$

Proof of Lemma 5.3 . We have first

$$||\varphi_h(u-\bar{u}_k)||_{\dot{H}^{1/2}(\mathbb{R})} \leq ||\varphi_h(u-\bar{u}^h)||_{\dot{H}^{1/2}(\mathbb{R})} + ||\varphi_h||_{\dot{H}^{1/2}(\mathbb{R})}|\bar{u}_k - \bar{u}^h|\,. \tag{70}$$

We estimate the r.h.s of (70). We have

$$
\begin{aligned}
&||\varphi_h(u-\bar u^h)||_{\dot H^{1/2}(I\!R)} && (71)\\
&=\int_{A_h}\int_{A_h}\frac{|\varphi_h(u-\bar u^h)(x)-\varphi_h(u-\bar u^h)(y)|^2}{|x-y|^2}dxdy\\
&\le 2\left[\int_{A_h}\int_{A_h}\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy+||\nabla\varphi_h||^2_\infty\int_{A_h}\int_{A_h}|u-\bar u^h|^2dxdy\right]\\
&\le C\left[||u||^2_{\dot H^{1/2}(A_h)}+2^{-h}\int_{A_h}|u-\bar u^h|^2dx\right]\\
&\le C||u||^2_{\dot H^{1/2}(A_h)}\,.
\end{aligned}
$$

where we use the fact $||\nabla\varphi_h||_\infty\le C2^{-h}$.

Now we estimate $|\bar u_k-\bar u^h|$. We can write $\bar u_k=\sum_{\ell=-\infty}^{k-1}2^{\ell-k}\bar u^{\prime,\ell}$.

Moreover

$$
\begin{aligned}
&|\bar u_k-\bar u^h|\le|\bar u^h-\bar u^{\prime,h}|+|\bar u_k-\bar u^{\prime,h}|\\
&\le C|A_h|^{-1}\int_{A_h}|u-\bar u^h|\,dx+\sum_{\ell=-\infty}^{k-1}2^{\ell-k}\sum_{s=\ell}^{h-1}|\bar u^{\prime,s+1}-\bar u^{\prime,s}|\\
&\le C|A_h|^{-1}\int_{A_h}|u-\bar u^h|\,dx+\sum_{\ell=-\infty}^{k-1}2^{\ell-k}\sum_{s=\ell}^{h-1}|A_{s+1}|^{-1}\int_{A_{s+1}}|u-\bar u^{s+1}|\,dx && (72)\\
&\le C\left[||u||_{\dot H^{1/2}(A_h)}+\sum_{\ell=-\infty}^{k-1}2^{\ell-k}\sum_{s=\ell}^{h-1}||u||_{\dot H^{1/2}(A_{s+1})}\right].
\end{aligned}
$$

Thus combining (71) and (72) we get

$$
\begin{aligned}
&||\varphi_h(u-\bar u^h)||_{\dot H^{1/2}(I\!R)}\le[||\varphi_h(u-\bar u^h)||_{\dot H^{1/2}(I\!R)}+||\varphi_h||_{\dot H^{1/2}(I\!R)}|\bar u_k-\bar u^h|]\\
&\le C\left[||u||_{\dot H^{1/2}(A_h)}+\sum_{\ell=-\infty}^{k-1}2^{\ell-k}\sum_{s=\ell}^{h-1}||u||_{\dot H^{1/2}(A_{s+1})}\right]. && (73)
\end{aligned}
$$

Multiplying both sides of (73) by 2^{k-h} and summing uo from $h=k$ to $+\infty$ we get

$$
\begin{aligned}
&\sum_{h=k}^{+\infty}2^{k-h}\sum_{\ell=-\infty}^{k-1}2^{\ell-k}\sum_{s=\ell+1}^{h}||u||_{\dot H^{1/2}(A_s)} && (74)\\
&\le C\sum_{s\le k}||u||_{\dot H^{1/2}(A_s)}\sum_{h\ge k}\sum_{\ell\le s}2^{\ell-h}+\sum_{s\ge k}||u||_{\dot H^{1/2}(A_s)}\sum_{h\ge s}\sum_{\ell\le k}2^{\ell-h}\\
&\le C\sum_{s\le k}2^{s-k}||u||_{\dot H^{1/2}(A_s)}+\sum_{s\ge k}2^{k-s}||u||_{\dot H^{1/2}(A_s)}\,.
\end{aligned}
$$

This ends the proof of Lemma 5.3.. $\square$

Now we recall the value of the Fourier transform of some functions that will be used in the sequel.

We have $\mathcal{F}[|x|^{-1/2}](\xi) = |\xi|^{-1/2}$. The Fourier transforms of $|x|$, $x|x|^{-1/2}$, $|x|^{1/2}$ are the tempered distributions defined, for every $\varphi \in \mathcal{S}(I\!R)$, respectively by

$$\begin{aligned}\langle \mathcal{F}[|x|], \varphi\rangle &= \langle \mathcal{F}[\frac{x}{|x|}] \star \mathcal{F}[x], \varphi\rangle = \langle p.v.(\frac{1}{x}) \star (\delta)_0'(x), \varphi\rangle \qquad (75)\\ &= p.v. \int_{I\!R} \frac{\varphi(x) - \varphi(0)}{x^2} dx\,;\end{aligned}$$

$$\begin{aligned}\langle \mathcal{F}[x|x|^{-1/2}], \varphi\rangle &= \langle \mathcal{F}[x] \star \mathcal{F}[|x|^{-1/2}], \varphi\rangle = \langle p.v.(\frac{1}{x}) \star (\delta)_0'(x), \varphi\rangle \qquad (76)\\ &= p.v. \int_{I\!R} [\varphi(x) - \varphi(0)] \frac{x}{|x|} \frac{1}{|x|^{3/2}} dx \qquad (77)\end{aligned}$$

and

$$\langle \mathcal{F}[|x|^{1/2}], \varphi\rangle = p.v. \int_{I\!R} \frac{\varphi(x) - \varphi(0)}{|x|^{3/2}} dx\,.$$

Next we set

$$F(Q, a) = \Delta^{1/4}(Qa) - Q\Delta^{1/4}a + \Delta^{1/4}Qa\,,$$

and

$$G(Q, a) = \mathcal{R}\Delta^{1/4}(Qa) - Q\Delta^{1/4}\mathcal{R}a + \Delta^{1/4}Q\mathcal{R}a\,.$$

We observe that $T(Q, u) = F(Q, \Delta^{1/4}u)$ and $S(Q, u) = \mathcal{R}G(Q, \Delta^{1/4}u)$.

Lemma 5.4 *Let $Q \in \dot{H}^{1/2}(I\!R, \mathcal{M}_{\ell\times m}(I\!R))$, $M \in \dot{H}^{1/2}(I\!R, \mathcal{M}_{p\times m}(I\!R))$ and let $u \in \dot{H}^{1/2}(I\!R)$ be a solution of (66). Then there exist $C > 0$, $\bar{n} > 0$ (independent of u and M) such that for all $\eta \in (0, 1/4)$ for all $k < k_0$ (k_0 depending on η) and $n \geq \bar{n}$, we have*

$$\begin{aligned}||\chi_{k-4}(w - \bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)} &\leq \eta||\chi_{k-4}\Delta^{1/4}u||_{L^2} \qquad (78)\\ &+C\left(\sum_{h=k}^{\infty} 2^{\frac{k-h}{2}}||\Delta^{1/4}u||_{L^2(A_h)} + \sum_{h=k-n}^{+\infty} 2^{k-h}||w||_{\dot{H}^{1/2}(A_h)}\right)\end{aligned}$$

where $w = \Delta^{-1/4}(M\Delta^{1/4}u)$ and we recall that $\chi_{k-4} \equiv 1$ on $B_{2^{k-5}}$ and $\chi_{k-4} \equiv 0$ on $B^c_{2^{k-4}}$.

Lemma 5.5 *Let* $Q \in \dot{H}^{1/2}(I\!R, \mathcal{M}_{\ell\times m}(I\!R))$, $M \in \dot{H}^{1/2}(I\!R, \mathcal{M}_{p\times m}(I\!R))$ *and let* $u \in \dot{H}^{1/2}(I\!R)$ *be a solution of (67). Then there exist* $C > 0$, $\bar{n} > 0$ *(independent of* u *and* M*) such that for all* $\eta \in (0, 1/4)$, *for all* $k < k_0$ *(*k_0 *depending on* η*) and* $n \geq \bar{n}$ *, we have*

$$\begin{aligned}||\chi_{k-4}(w - \bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)} &\leq \eta||\chi_{k-4}\Delta^{1/4}u||_{L^2(I\!R)} \\ &+C\left(\sum_{h=k}^{\infty}(2^{\frac{k-h}{2}}||\Delta^{1/4}u||_{L^2(A_h)} + \sum_{h=k-n}^{k-3} 2^{h-k}||w||_{\dot{H}^{1/2}(A_h)}\right)\end{aligned} \tag{79}$$

where $w = \Delta^{-1/4}(M\Delta^{1/4}u)$ *.*

Proof of Lemma 5.4 .

We fix $\eta \in (0, 1/4)$.

Let $k < 0$ be large enough so that $||\chi_k(Q - \bar{Q}_k)||_{\dot{H}^{1/2}(I\!R)} \leq \varepsilon$, where $\varepsilon \in (0, 1)$ will be determined later.

We write

$$F(Q, \Delta^{1/4}u) = F(Q_1, \Delta^{1/4}u) + F(Q_2, \Delta^{1/4}u)\,,$$

where $Q_1 = \chi_k(Q - \bar{Q}_k)$ and $Q_2 = (1 - \chi_k)(Q - \bar{Q}_k)$. We observe that, by construction, we have $supp(Q_2) \subseteq B^c_{2^{k-1}}$, $||Q_1||_{\dot{H}^{1/2}(I\!R)} \leq \varepsilon$ and $||Q_2||_{\dot{H}^{1/2}(I\!R)} \leq ||Q||_{\dot{H}^{1/2}(I\!R)}$.

We rewrite the equation (66) as follows:

$$\begin{aligned}\Delta^{1/2}(\chi_{k-4}(w - \bar{w}_{k-4})) &= -\sum_{h=k-4}^{+\infty} \Delta^{1/2}(\varphi_h(w - \bar{w}_{k-4})) \\ &+ F(Q_1, \Delta^{1/4}u) + F(Q_2, \Delta^{1/4}u)\,.\end{aligned} \tag{80}$$

We multiply the equation (80) by $\chi_{k-4}(w - \bar{w}_{k-4})$ and integrate over $I\!R$. We get

$$\begin{aligned}\int_{I\!R} |\Delta^{1/4}(\chi_{k-4}(w - \bar{w}_{k-4}))|^2 dx &= -\sum_{h=k-4}^{+\infty} \int_{I\!R} \Delta^{1/2}(\varphi_h(w - \bar{w}_{k-4}))(\chi_{k-4}(w - \bar{w}_{k-4}))dx \\ + \int_{I\!R} F(Q_1, \Delta^{1/4}u)(\chi_{k-4}(w - \bar{w}_{k-4}))dx &+ \int_{I\!R} F(Q_2, \Delta^{1/4}u)(\chi_{k-4}(w - \bar{w}_{k-4}))dx\,.\end{aligned} \tag{81}$$

We estimate the last three terms in (81) .

- Estimate of $-\sum_{h=k-4}^{+\infty}\int_{I\!R} \Delta^{1/2}(\varphi_h(w - \bar{w}_{k-4}))(\chi_{k-4}(w - \bar{w}_{k-4}))$.

Case $k-4 \leq h \leq k-3$ **.**

$$\begin{aligned}
&\sum_{h=k-4}^{k-3} \int_{I\!R} \Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(\chi_{k-4}(w-\bar{w}_{k-4}))dx \\
&\leq ||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)} \left[\sum_{h=k-4}^{k-3} ||(\varphi_h(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)}\right] \\
&\text{by Lemma 5.3} \qquad\qquad (82) \\
&\leq ||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)} \left(\sum_{h=k-4}^{k-3}\left[||w||_{\dot{H}^{1/2}(A_h)} + \sum_{\ell=-\infty}^{k-5} 2^{\ell-(k-4)} \sum_{s=\ell+1}^{h} ||w||_{\dot{H}^{1/2}(A_s)}\right]\right) \\
&\leq C||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)} \left[\sum_{h=-\infty}^{k-3} 2^{h-k}||w||_{\dot{H}^{1/2}(A_h)}\right]
\end{aligned}$$

From Localization Theorem 4.1 it follows that

$$\sum_{h=-\infty}^{k-6} ||w||^2_{\dot{H}^{1/2}(A_h)} \leq C||(\chi_{k-4}(w-\bar{w}_{k-4}))||^2_{\dot{H}^{1/2}(I\!R)},$$

where $C>0$ is independent of k and w. Thus there exists $n_1 \geq 6$ such that $n \geq n_1$ we have

$$C\sum_{h=-\infty}^{k-n} 2^{h-k}||w||_{\dot{H}^{1/2}(A_h)} \leq \frac{1}{8}||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)}.$$

Thus for $n \geq n_1$ we have

$$\begin{aligned}
&\sum_{h=k-4}^{k-3} \int_{I\!R} \Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(\chi_{k-4}(w-\bar{w}_{k-4})) \\
&\leq \frac{1}{8}||(\chi_{k-4}(w-\bar{w}_{k-4}))||^2_{\dot{H}^{1/2}(I\!R)} + C\sum_{h=k-n}^{k-3} 2^{h-k}||w||_{\dot{H}^{1/2}(A_h)}. \qquad (83)
\end{aligned}$$

Case $k-2 \leq h < +\infty$ **.**

In this case we use the fact that the supports of φ_h and of χ_{k-4} are disjoint and in particular $0 \notin (\varphi_h(w-\bar{w}_{k-4})) \star (\chi_{k-4}(w-\bar{w}_{k-4}))$.

$$
\begin{aligned}
&\sum_{h=k-2}^{+\infty}\int_{I\!R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(\chi_{k-4}(w-\bar{w}_{k-4}))dx\\
&\sum_{h=k-2}^{+\infty}\int_{I\!R}\mathcal{F}^{-1}(|\xi|)(x)(\varphi_h(w-\bar{w}_{k-4}))\star(\chi_{k-4}(w-\bar{w}_{k-4}))dx\\
&\le\sum_{h=k-2}^{+\infty}||\mathcal{F}^{-1}(|\xi|)||_{L^\infty(B_{2^{h+2}}\setminus B_{2^{h-2}})}||\varphi_h(w-\bar{w}_{k-4})||_{L^1}||\chi_{k-4}(w-\bar{w}_{k-4}|)|_{L^1}\\
&\le C\sum_{h=k-2}^{+\infty}2^{-2h}2^{h/2}||\varphi_h(w-\bar{w}_{k-4})||_{L^2(I\!R)}2^{k/2}||\chi_{k-4}(w-\bar{w}_{k-4})||_{L^2(I\!R)}\\
&\text{by Theorem 4.2} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (84)\\
&\le C\sum_{h=k-2}^{+\infty}2^{k-h}||\varphi_h(w-\bar{w}_{k-4}||_{\dot{H}^{1/2}(I\!R)}||\chi_{k-4}(w-\bar{w}_{k-4}||_{\dot{H}^{1/2}(I\!R)}\\
&\text{by Lemma 5.3}\\
&\le C\sum_{h=k-2}^{+\infty}2^{k-4-h}\left[||w||_{\dot{H}^{1/2}(A_h)}+\sum_{\ell=-\infty}^{k-5}2^{\ell-(k-4)}\sum_{s=\ell+1}^{h}||w||_{\dot{H}^{1/2}(A_s)}\right]\\
&\qquad ||\chi_{k-4}(w-\bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)}\\
&\le C\left[\sum_{h=k-2}^{+\infty}2^{k-4-h}||w||_{\dot{H}^{1/2}(A_h)}+\sum_{s\le k-4}||w||_{\dot{H}^{1/2}(A_s)}\left(\sum_{h\ge k-4}\sum_{\ell\le s-1}2^{\ell-h}\right)\right.\\
&\left.+\sum_{s\ge k-4}||w||_{\dot{H}^{1/2}(A_s)}\left(\sum_{h\ge s-1}\sum_{\ell\le k-4}2^{\ell-h}\right)\right]||\chi_{k-4}(w-\bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)}\\
&\le C\left[\sum_{h=k-4}^{+\infty}2^{k-4-h}||w||_{\dot{H}^{1/2}(A_h)}+\sum_{h=-\infty}^{k-5}2^{h-(k-4)}||w||_{\dot{H}^{1/2}(A_h)}\right]||\chi_{k-4}(w-\bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)}
\end{aligned}
$$

Let $n_2\ge 6$ be such that if $n\ge n_2$ we have

$$
C\sum_{h=-\infty}^{k-n}2^{h-(k-4))}||w||_{\dot{H}^{1/2}(A_h)}\le\frac{1}{8}||\chi_{k-4}(w-\bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)}\,.
$$

Thus if $n>\bar{n}=\max(n_1,n_2)$, then from (83) and (84) it follows

$$\sum_{h=k-4}^{+\infty}\int_{I\!R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(\chi_{k-4}(w-\bar{w}_{k-4})) \tag{85}$$

$$\leq \frac{1}{4}||\chi_{k-4}(w-\bar{w}_{k-4})||^2_{\dot{H}^{1/2}(I\!R)}+C\sum_{h=k-n}^{+\infty}2^{k-h}||w||_{\dot{H}^{1/2}(A_h)}\,.$$

• Estimate of $\int_{I\!R}F(Q_1,\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx\,.$
We write

$$F(Q_1,\Delta^{1/4}u)=F(Q_1,\chi_{k-4}\Delta^{1/4}u)+\sum_{h=k-4}^{+\infty}F(Q_1,\varphi_h\Delta^{1/4}u)\,.$$

The following estimate holds

$$\int_{I\!R}F(Q_1,\chi_{k-4}\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx \tag{86}$$

by Theorem 1.4

$$\leq C||Q_1||_{\dot{H}^{1/2}(I\!R)}||\chi_{k-4}\Delta^{1/4}u||_{L^2}||\chi_{k-4}(w-\bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)}$$

$$C\varepsilon||\chi_{k-4}\Delta^{1/4}u||_{L^2}||\chi_{k-4}(w-\bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)}\,.$$

By choosing $\varepsilon>0$ small enough, we may assume that $C\varepsilon<\frac{\eta}{4}<\frac{1}{16}$.

Case $k-4\leq h\leq k+1$.

We use again Theorem 1.4.

$$\sum_{h=k-4}^{k+1}\int_{I\!R}F(Q_1,\varphi_h\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx \tag{87}$$

by Theorem 1.4

$$\leq C\sum_{h=k-4}^{k+1}||Q_1||_{\dot{H}^{1/2}(I\!R)}||\varphi_h\Delta^{1/4}u||_{L^2(I\!R)}||\chi_{k-4}(w-\bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)}$$

Case $h\geq k+2$.

We estimate the single terms of $F(Q_1,\varphi_h\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))$.

We observe that if $h\geq k+2$ then the supports of Q_1 and φ_h and those of χ_{k-4} and φ_h are disjoint. Therefore

$$F(Q_1,\varphi_h\Delta^{1/4}u)\chi_{k-4}(w-\bar{w}_{k-4})=Q_1\Delta^{1/4}(\varphi_h\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))\,.$$

$$
\begin{aligned}
&\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} F(Q_1,\varphi_h\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx \\
&=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} Q_1\Delta^{1/4}(\varphi_h\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx \\
&=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} \mathcal{F}^{-1}(|\xi|^{1/2})(x)\left([\varphi_h\Delta^{1/4}u]\star[Q_1(\chi_{k-4}(w-\bar{w}_{k-4}))]\right) \\
&\sum_{h=k+2}^{+\infty}||\mathcal{F}^{-1}(|\xi|^{1/2})||_{L^\infty(B_{2^{h+2}}\setminus B_{2^{h-2}})}||\varphi_h\Delta^{1/4}u||_{L^1}||Q_1(\chi_{k-4}(w-\bar{w}_{k-4}))||_{L^1} \\
&\le C\sum_{h=k+2}^{+\infty}2^{-3/2h}||\varphi_h\Delta^{1/4}u||_{L^1}||Q_1(\chi_{k-4}(w-\bar{w}_{k-4}))||_{L^1} \\
&\text{by Theorem 4.2} \\
&\le C\sum_{h=k+2}^{+\infty}2^{k-h}||Q_1||_{\dot{H}^{1/2}(\mathbb{R})}||\,||\varphi_h\Delta^{1/4}u||_{L^2(\mathbb{R})}||\chi_{k-4}(w-\bar{w}_{k-4})||_{\dot{H}^{1/2}(\mathbb{R})} \\
&\le C\sum_{h=k+2}^{+\infty}2^{k-h}||\varphi_h\Delta^{1/4}u||_{L^2}||\chi_{k-4}(w-\bar{w}_{k-4})||_{\dot{H}^{1/2}(\mathbb{R})}
\end{aligned}
\tag{88}
$$

- Estimate of $\int_{\mathbb{R}} F(Q_2,\Delta^{1/4}u)\,(\chi_{k-4}(w-\bar{w}_{k-4}))dx$.

As above we write

$$
F(Q_2,\Delta^{1/4}u)=F(Q_2,\chi_{k-4}\Delta^{1/4}u)+\sum_{h=k-4}^{+\infty}F(Q_2,\varphi_h\Delta^{1/4}u)\,.
$$

Since the support of Q_2 is included in $B^c_{2^{k-1}}$, we have

$$
F(Q_2,\chi_{k-4}\Delta^{1/4}u)\,(\chi_{k-4}(w-\bar{w}_{k-4}))=\Delta^{1/4}Q_2(\chi_{k-4}\Delta^{1/4}u)\,(\chi_{k-4}(w-\bar{w}_{k-4}))\,.
$$

We can write $Q_2 = \sum_{h=k-1}^{+\infty} \varphi_h(Q_2 - \bar{Q}_{2k-1})$, $(\bar{Q}_{2k-1} = 0)$.

$$\begin{aligned}
&\int_{I\!R} F(Q_2, \chi_{k-4}\Delta^{1/4}u)(\chi_{k-4}(w - \bar{w}_{k-4}))dx \qquad (89)\\
&= \sum_{h=k-1}^{+\infty} \int_{I\!R} \Delta^{1/4}(\varphi_h(Q_2 - \bar{Q}_{2k-1}))(\chi_{k-4}\Delta^{1/4}u)(\chi_{k-4}(w - \bar{w}_{k-4}))\\
&\leq C \sum_{h=k-1}^{+\infty} \int_{I\!R} \mathcal{F}^{-1}(|\xi|^{1/2}) \left([\varphi_h(Q_2 - \bar{Q}_{2k-1})] \star [(\chi_{k-4}\Delta^{1/4}u)(\chi_{k-4}(w - \bar{w}_{k-4}))]\right)\\
&\leq C \sum_{h=k-1}^{+\infty} 2^{-h/2}2^{k/2}||\varphi_h(Q_2 - \bar{Q}_{2k-1})||_{\dot{H}^{1/2}(I\!R)}||\chi_{k-4}\Delta^{1/4}u||_{L^2}||\chi_{k-4}(w - \bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)}\,.
\end{aligned}$$

From Lemma 5.3, by choosing possibly k smaller, it follows that

$$C \sum_{h=k-1}^{+\infty} 2^{\frac{k-h}{2}}||\varphi_h(Q_2 - \bar{Q}_{2k-1})||_{\dot{H}^{1/2}(I\!R)} \leq \frac{\eta}{4} < \frac{1}{16}\,.$$

- Estimate of $\sum_{h=k-4}^{+\infty} \int_{I\!R} F(Q_2, \varphi_h\Delta^{1/4}u)(\chi_{k-4}(w - \bar{w}_{k-4}))dx$.
The following estimates holds.

$$\begin{aligned}
&\sum_{h=k-4}^{k+1} \int_{I\!R} F(Q_2, \varphi_h\Delta^{1/4}u)(\chi_{k-4}(w - \bar{w}_{k-4}))dx \qquad (90)\\
&\leq C \sum_{h=k-4}^{k+1} ||Q||_{\dot{H}^{1/2}(I\!R)}||\varphi_h\Delta^{1/4}u||_{L^2}||\chi_{k-4}(w - \bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)}\,.
\end{aligned}$$

On another hand since the support of Q_2 is included in $B^c_{2^{k-1}}$, if $h \geq k+1$, we have

$$\begin{aligned}
&F(Q_2, \varphi_h\Delta^{1/4}u)(\chi_{k-4}(w - \bar{w}_{k-4})) = (\chi_{k-4}(w - \bar{w}_{k-4}))\\
&\left[\Delta^{1/4}(Q_2\,\varphi_h\Delta^{1/4}u) - Q_2\Delta^{1/4}(\varphi_h\Delta^{1/4}u) + \Delta^{1/4}Q_2\varphi_h\Delta^{1/4}u\right]\\
&= (\chi_{k-4}(w - \bar{w}_{k-4}))\Delta^{1/4}(Q_2\varphi_h\Delta^{1/4}u)\,.
\end{aligned}$$

Let $\psi_h \in C_0^\infty(I\!R)$, $\psi_h \equiv 1$ in $B_{2^{h+1}} \setminus B_{2^{h-1}}$ and $supp(\psi) \subset B_{2^{h+2}} \setminus B_{2^{h-2}}$.

$$\sum_{h=k+1}^{+\infty} \int_{I\!R} F(Q_2, \varphi_h \Delta^{1/4} u)(\chi_{k-4}(w - \bar{w}_{k-4}))dx \tag{91}$$

$$\begin{aligned}
&\sum_{h=k+1}^{+\infty} \int_{I\!R} \Delta^{1/4}(Q_2 \varphi_h \Delta^{1/4} u)(\chi_{k-4}(w - \bar{w}_{k-4}))dx \\
&= \sum_{h=k+1}^{+\infty} \int_{I\!R} \mathcal{F}^{-1}(|\xi|^{1/2}) \left([\varphi_h \Delta^{1/4} u(Q_2 - \bar{Q}_{2k-1})] \star [\chi_{k-4}(w - \bar{w}_{k-4})] \right) dx \\
&\le C \sum_{h=k+1}^{+\infty} 2^{-3/2h} ||\varphi_h \Delta^{1/4} u||_{L^2(I\!R)} ||\psi_h(Q_2 - \bar{Q}_{2k-1})||_{L^2(I\!R)} ||\chi_{k-4}(w - \bar{w}_{k-4})||_{L^2(I\!R)} \\
&\le C \sum_{h=k+1}^{+\infty} 2^{k-h} ||\varphi_h^{1/2}(Q_2 - \bar{Q}_{2k-1})||_{\dot{H}^{1/2}(I\!R)} ||\varphi_h^{1/2} \Delta^{1/4} u||_{L^2(I\!R)} ||\chi_{k-4}(w - \bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)}
\end{aligned}$$

by Cauchy-Schwartz Inequality

$$\begin{aligned}
&\le C \left(\sum_{h=k+1}^{+\infty} 2^{k-h} ||\psi_h(Q_2 - \bar{Q}_{2k-1})||^2_{\dot{H}^{1/2}(I\!R)} \right)^{1/2} \left(\sum_{h=k+1}^{+\infty} 2^{k-h} ||\varphi_h \Delta^{1/4} u||^2_{L^2} \right)^{1/2} \\
&\qquad ||\chi_{k-4}(w - \bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)} .
\end{aligned}$$

From Lemma 5.3 (with φ replaced by ψ) and Theorem 4.1 we deduce that

$$\Big(\sum_{h=k+1}^{+\infty} 2^{k-h} ||\psi_h(Q_2 - \bar{Q}_{2k-1})||^2_{\dot{H}^{1/2}(I\!R)}\Big)^{1/2} \le C ||Q||_{\dot{H}^{1/2}(I\!R)} .$$

Thus

$$\begin{aligned}
\sum_{h=k+1}^{+\infty} \int_{I\!R} F(Q_2, \varphi_h \Delta^{1/4} u)(\chi_{k-4}(w - \bar{w}_{k-4}))dx &\le C ||\chi_{k-4}(w - \bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)} \\
&\quad \left(\sum_{h=k+1}^{+\infty} 2^{k-h} ||\varphi_h \Delta^{1/4} u||^2_{L^2} \right)^{1/2} .
\end{aligned}$$

By combining (87),(88), (90) and (91) we obtain (for some constant C depending on Q)

$$\begin{aligned}
\int_{I\!R} F(Q, \Delta^{1/4} u)(\chi_{k-4}(w - \bar{w}_{k-4}))dx &\le \frac{\eta}{2} ||\chi_{k-4} \Delta^{1/4} u||_{L^2} ||\chi_{k-4}(w - \bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)} \\
&+ C \sum_{h=k-4}^{+\infty} 2^{\frac{k-h}{2}} ||\Delta^{1/4} u||_{L^2(A_h)} ||\chi_{k-4}(w - \bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)} .
\end{aligned} \tag{92}$$

Finally for all $n \geq \bar{n}$ we have

$$||\chi_{k-4}(w - \bar{w}_{k-4})||_{\dot{H}^{1/2}(I\!R)} \leq \eta ||\chi_{k-4}\Delta^{1/4}u||_{\dot{H}^{1/2}(I\!R)} + C \sum_{h=k-n}^{+\infty} 2^{k-h}||w||_{\dot{H}^{1/2}(A_s)} \quad (93)$$

$$+ C \sum_{h=k-4}^{+\infty} 2^{\frac{k-h}{2}}||\Delta^{1/4}u||_{L^2(A_h)}\,.$$

and we can conclude. □

We prove Lemma 5.5.

Proof of Lemma 5.5 . The proof is similar to that of Lemma 5.4 thus we just sketch it.

We observe that equation (67) is equivalent to

$$\mathcal{R}\Delta^{1/4}(M\Delta^{1/4}u) = G(Q, \Delta^{1/4}u) - \Delta^{1/4}u \cdot (\mathcal{R}\Delta^{1/4}u)\,. \quad (94)$$

We fix $\eta \in (0, 1/4)$.

Let $k < 0$ be such that $||\chi_k(Q - \bar{Q}_k)||_{\dot{H}^{1/2}(I\!R)} \leq \varepsilon$ and $||\chi_k\Delta^{1/4}u||_{\dot{H}^{1/2}(I\!R)} \leq \varepsilon$, with $\varepsilon > 0$ to be determined later.

We write

$$G(Q, \Delta^{1/4}u) = G(Q_1, \Delta^{1/4}u) + G(Q_2, \Delta^{1/4}u)\,,$$

where $Q_1 = \chi_k(Q - \bar{Q}_k)$ and $Q_2 = (1 - \chi_k)(Q - \bar{Q}_k)$. We observe that $supp(Q_2) \subseteq B^c_{2^{k-1}}$ and $||Q_1||_{\dot{H}^{1/2}(I\!R)} \leq \varepsilon$.

We also set $u_1 = \chi_k\Delta^{1/4}u$ and $u_2 = (1 - \chi_k)\Delta^{1/4}u$ and $w = \Delta^{-1/4}(M\Delta^{1/4}u)$.

We rewrite the equation (94) as follows:

$$\mathcal{R}\Delta^{1/2}(\chi_{k-4}(w - \bar{w}_{k-4})) = -\sum_{h=k-4}^{+\infty} \mathcal{R}\Delta^{1/2}(\varphi_h(w - \bar{w}_{k-4})) \quad (95)$$

$$+G(Q_1, \Delta^{1/4}u) + G(Q_2, \Delta^{1/4}u) + u_1(\mathcal{R}\Delta^{1/4}u) + u_2(\mathcal{R}\Delta^{1/4}u)\,.$$

We multiply the equation (95) by $\chi_{k-4}(w-\bar{w}_{k-4})$ and integrate over $I\!R$. We get

$$\begin{aligned}
&\int_{I\!R}|\Delta^{1/4}(\chi_{k-4}(w-\bar{w}_{k-4}))|^2dx\\
&=-\sum_{h=k-4}^{+\infty}\int_{I\!R}\mathcal{R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(\chi_{k-4}(w-\bar{w}_{k-4}))dx\\
&+\int_{I\!R}G(Q_1,\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx\\
&+\int_{I\!R}G(Q_2,\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx\\
&+\int_{I\!R}u_1(\mathcal{R}\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx+\int_{I\!R}u_2(\mathcal{R}\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx\,.
\end{aligned}\tag{96}$$

We observe that $\int_{I\!R}u_2(\mathcal{R}\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx=0$, having u_2 and χ_{k-4} supports disjoint.

- Estimate of $-\sum_{h=k-4}^{+\infty}\int_{I\!R}\mathcal{R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(\chi_{k-4}(w-\bar{w}_{k-4}))dx$.

Case $k-4\le h\le k-3$**.**

$$\begin{aligned}
&\sum_{h=k-4}^{k-3}\int_{I\!R}\mathcal{R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(\chi_{k-4}(w-\bar{w}_{k-4}))\\
&\le\sum_{h=k-4}^{k-3}\int_{I\!R}||\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))||_{\dot{H}^{-1/2}(I\!R)}||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)}\\
&\text{by Lemma 5.3}\\
&\le\sum_{h=k-4}^{k-3}\left[||w||_{\dot{H}^{1/2}(A_h)}+\sum_{\ell=-\infty}^{k-5}2^{\ell-(k-4)}\sum_{s=\ell+1}^{h}||w||_{\dot{H}^{1/2}(A_s)}\right]\\
&\qquad||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)}
\end{aligned}\tag{97}$$

Let $n_1\ge 6$ be such that

$$C\sum_{h=-\infty}^{k-n_1}2^{h-k}||w||_{\dot{H}^{1/2}(A_h)}\le\frac{1}{8}||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)}\,.$$

Thus if $n\ge n_1$ we have

$$(97)\le\quad\frac{1}{8}||(\chi_{k-4}(w-\bar{w}_{k-4}))||^2_{\dot{H}^{1/2}(I\!R)}+C\sum_{h=k-n}^{k-3}2^{h-k}||w||_{\dot{H}^{1/2}(A_h)}\,.\tag{98}$$

Case $k-2\le h<+\infty$**.**

In this case we use the fact that

$$supp((\varphi_h(w-\bar{w}_{k-4}))\star(\chi_{k-4}(w-\bar{w}_{k-4})))\subseteq B_{2^{h+2}}\setminus B_{2^{h-2}}\,,$$

and in particular $0\notin((\varphi_h(w-\bar{w}_{k-4}))\star(\chi_{k-4}(w-\bar{w}_{k-4})))$.

$$\begin{aligned}
&\sum_{h=k-2}^{+\infty}\int_{I\!R}\mathcal{R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(\chi_{k-4}(w-\bar{w}_{k-4}))dx \qquad (99)\\
&\sum_{h=k-2}^{+\infty}\int_{I\!R}\mathcal{F}^{-1}(\xi)(x)\left([\varphi_h(w-\bar{w}_{k-4}))]\star[\chi_{k-4}(w-\bar{w}_{k-4})]\right)dx\\
&=\sum_{h=k-2}^{+\infty}\int_{I\!R}\delta_0'(x)\left(\varphi_h(w-\bar{w}_{k-4}))\star(\chi_{k-4}(w-\bar{w}_{k-4}))\right)dx=0\,.
\end{aligned}$$

- Estimate of $\int_{I\!R}u_1(\mathcal{R}\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx$.

We have

$$\begin{aligned}
&\int_{I\!R}u_1(\mathcal{R}\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx\\
&=\int_{I\!R}u_1(\mathcal{R}u_1)(\chi_{k-4}(w-\bar{w}_{k-4}))dx+\sum_{h=k}^{+\infty}\int_{I\!R}u_1(\mathcal{R}\varphi_h\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx\,.
\end{aligned}$$

By applying Lemma 3.2 we get

$$\begin{aligned}
&\int_{I\!R}u_1(\mathcal{R}u_1)(\chi_{k-4}(w-\bar{w}_{k-4}))dx\\
&C||u_1(\mathcal{R}u_1)||_{\mathcal{H}}||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)}\\
&\le C||u_1||^2_{L^2}||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)}\\
&\le C\varepsilon||\chi_k\Delta^{1/4}u||_{L^2}||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)}\,.
\end{aligned}$$

By choosing $\varepsilon>0$ smaller we may suppose that $C\varepsilon<\frac{\eta}{4}$.

Now we observe that for $h\ge k$ the supports of φ_h and χ_{k-4} are disjoint. Thus we have

$$
\begin{aligned}
&\sum_{h=k}^{+\infty}\int_{I\!R} u_1(\mathcal{R}\varphi_h\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx\\
&\sum_{h=k}^{+\infty}\int_{I\!R} \mathcal{F}^{-1}(\frac{\xi}{|\xi|})\left([\varphi_h\Delta^{1/4}u]\star[u_1(\chi_{k-4}(w-\bar{w}_{k-4})]\right)dx\\
&\le C\sum_{h=k}^{+\infty}|||x|^{-1}||_{L^\infty(B_{2^{h+2}}\setminus B_{2^{h-2}})}||\varphi_h\Delta^{1/4}u\star\Delta^{1/4}u_1(\chi_{k-4}(w-\bar{w}_{k-4}))||_{L^1}\\
&\le C\sum_{h=k}^{+\infty}2^{-h}2^{h/2}2^{k/2}||\varphi_h\Delta^{1/4}u||_{L^2}||u_1||_{L^2}||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)}\\
&\le C\varepsilon\sum_{h=k}^{+\infty}2^{\frac{k-2}{2}}||\varphi_h\Delta^{1/4}u||_{L^2}||(\chi_{k-4}(w-\bar{w}_{k-4}))||_{\dot{H}^{1/2}(I\!R)}\,.
\end{aligned}
$$

The estimate of the terms

$$\int_{I\!R} G(Q_1,\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx \text{ and } \int_{I\!R} G(Q_2,\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx$$

are analogous of those of

$$\int_{I\!R} F(Q_1,\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx \text{ and } \int_{I\!R} F(Q_2,\Delta^{1/4}u)(\chi_{k-4}(w-\bar{w}_{k-4}))dx$$

and so we omit them. □

Now we can prove Theorem 5.1 and 5.2 .

Proof of Theorem 5.1 .

From Lemma 5.4, it follows that there exist $C>0$ and $\bar{n}>0$ such that for all $n>\bar{n}$, $0<\eta<1/4$, $k<k_0$ (k_0 depending on η), every solution to (66) satisfies for some constant $C>0$ (93) and thus

$$
\begin{aligned}
||\chi_{k-4}(w-\bar{w}_{k-4})||^2_{\dot{H}^{1/2}(I\!R)} &\le \eta^2||\chi_{k-4}\Delta^{1/4}u||^2_{L^2}+C\sum_{h=k-n}^{+\infty}2^{k-h}||w||^2_{\dot{H}^{1/2}(A_h)} \quad (100)\\
&+C\sum_{h=k-4}^{+\infty}2^{\frac{k-h}{2}}||\Delta^{1/4}u||^2_{L^2}\,.
\end{aligned}
$$

We fix $n\ge\bar{n}$

From Lemma 4.1 it follows that there exist $C_1, C_2 > 0$ and $m_1 > 0$ (independent on n, k) such that if $m \geq m_1$ we have

$$\begin{aligned}
&||\chi_{k-4}(w - \bar{w}_{k-4})||^2_{\dot{H}^{1/2}(I\!R)} \geq \\
&\quad \geq C_1 \int_{B_{2^{k-n-m}}} |M\Delta^{1/4}u|^2 dx - C_2 \sum_{h=k-n-m}^{+\infty} 2^{k-h} \int_{B_{2^h} \setminus B_{2^{h-1}}} |M\Delta^{1/4}u|^2 dx\,.
\end{aligned} \tag{101}$$

Finally from Lemma 4.2 it follows that there is $C > 0$ such that for all $\gamma \in (0,1)$ there exists $m_2 > 0$ such that if $m \geq m_2$ we have

$$\begin{aligned}
&\sum_{h=k-n}^{+\infty} 2^{k-h} ||w||^2_{\dot{H}^{1/2}(A_h)} = \sum_{h=k-n}^{+\infty} 2^{k-h} ||\Delta^{-1/4}(M\Delta^{1/4}u)||^2_{\dot{H}^{1/2}(A_h)} \\
&\leq \gamma \int_{|\xi| \leq 2^{k-n-m}} |M\Delta^{1/4}u|^2 dx + \sum_{h=k-n-m}^{+\infty} 2^{\frac{k-h}{2}} \int_{2^h \leq |\xi| \leq 2^{h+1}} |M\Delta^{1/4}u|^2 dx\,.
\end{aligned} \tag{102}$$

By combining (100), (101) and (102) we get

$$\begin{aligned}
C_1 ||M\Delta^{1/4}u||^2_{L^2(B_{2^{k-n-m}})} \;\leq\; & C \sum_{h=k-n-m}^{\infty} (2^{\frac{k-h}{2}}) ||M\Delta^{1/4}u||^2_{L^2(A_h)} + C_2 \sum_{h=k-n-m}^{+\infty} 2^{\frac{k-h}{2}} ||\Delta^{1/4}u||_{L^2(A_h)} \\
& + \eta^2 ||\chi_{k-4}\Delta^{1/4}u||^2_{L^2(I\!R)} + C\gamma ||M\Delta^{1/4}u||^2_{L^2(B_{2^{k-n-m}})}\,.
\end{aligned}$$

We choose $\gamma, \eta > 0$ so that $C_1^{-1} C\gamma < 1/4$ and $C_1^{-1}\eta^2 < 1/4$.

We get for some constant $C > 0$

$$\begin{aligned}
&||M\Delta^{1/4}u||^2_{L^2(B_{2^{k-n-m}})} - \frac{1}{4} ||\Delta^{1/4}u||^2_{L^2(B_{2^{k-n-m}})} \\
&\leq C \left[\sum_{h=k-n-m}^{\infty} (2^{\frac{k-h}{2}}) ||M\Delta^{1/4}u||^2_{L^2(A_h)} + \sum_{h=k-n-m}^{+\infty} 2^{\frac{k-h}{2}} ||\Delta^{1/4}u||_{L^2(A_h)} \right]
\end{aligned} \tag{103}$$

Thus by replacing in (103) $k - n - m$ by k we get (68) and we conclude the proof. $\square$

The **proof of Theorem 5.2** is analogous to that of Theorem 5.1 and thus we omit it.

6 Morrey estimates and Hölder continuity of 1/2-Harmonic Maps into the Sphere

We consider the $m-1$-dimensional sphere $S^{m-1} \subset I\!\!R^m$. Let $\Pi_{S^{m-1}}$ be the orthogonal projection on S^{m-1}. We also consider the Dirichlet energy

$$\mathcal{L}(u) = \int_{I\!\!R} |\Delta^{1/4} u(x)|^2 dx\,. \tag{104}$$

where $u\colon I\!\!R \to S^{m-1}$.

The weak 1/2-harmonic maps are defined as critical points of the functional (104) with respect to perturbation of the form $\Pi_{S^{m-1}}(u+t\phi)$, where ϕ is an arbitrary compacted supported smooth map from $I\!\!R$ into $I\!\!R^m$.

Definition 6.1 *We say that $u \in H^{1/2}(I\!\!R, S^{m-1})$ is a weak 1/2-harmonic map if and only if, for every arbitrary compacted supported smooth maps ϕ from $I\!\!R$ into $I\!\!R^m$ we have*

$$\frac{d}{dt}\mathcal{L}(\Pi_{S^{m-1}}(u+t\phi))_{|t=0}\,.$$

We introduce some notations. We denote by $\bigwedge(I\!\!R^m)$ the exterior algebra (or Grassmann Algebra) of $I\!\!R^m$ and by the symbol $\wedge$ the *exterior or wedge product*. For every $p = 1,\ldots,m$, $\bigwedge_p(I\!\!R^m)$ is the vector space of p-vectors

If $(e_i)_{i=1,\ldots,m}$ is the canonical orthonormal basis of $I\!\!R^m$ then every element $v \in \bigwedge_p(I\!\!R^m)$ is written as $v = \sum_I v_I e_I$ where $I = \{i_1,\ldots,i_p\}$ with $1 \le i_1 \le \ldots \le i_p \le m$, $v_I := v_{i_1,\ldots,i_p}$ and $e_I =:= e_{i_1} \wedge \ldots \wedge e_{i_p}$.

By the symbol $\llcorner$ we denote the interior multiplication $\llcorner\colon \bigwedge_p(I\!\!R^m) \times \bigwedge_q(I\!\!R^m) \to \bigwedge_{q-p}(I\!\!R^m)$ defined as follows.

Let $e_I = e_{i_1} \wedge \ldots \wedge e_{i_p}$, $e_J = e_{j_1} \wedge \ldots \wedge e_{j_q}$, with $q \ge p$. Then $e_I \llcorner e_J = 0$ if $I \not\subset J$, otherwise $e_I \llcorner e_J = (-1)^M e_K$ where e_K is a $q-p$ vector and M is the number of pairs $(i,j) \in I \times J$ with $j > i$.

By the symbol $\bullet$ we denote the first order contraction between multivectors. We recall that it satisfies $\alpha\bullet\beta = \alpha\llcorner\beta$ if β is a 1-vector and $\alpha\bullet(\beta\wedge\gamma) = (\alpha\bullet\beta)\wedge\gamma + (-1)^{pq}(\alpha\bullet\gamma)\wedge\beta$, if β and γ are respectively a p-vector and a q-vector.

Finally by the symbol $*$ we denote the Hodge-star operator, $*\colon \bigwedge_p(I\!\!R^m) \to \bigwedge_{m-p}(I\!\!R^m)$, defined by $*\beta = (e_1 \wedge \ldots \wedge e_n) \bullet \beta$. For an introduction of the Grassmann Algebra we refer the reader to the first Chapter of the book by Federer[7].

Next we write the Euler equation associated to the functional (104).

Theorem 6.1 *All weak 1/2-harmonic maps $u \in H^{1/2}(I\!\!R, \mathcal{N})$ satisfy in a weak sense*

i) the equation

$$\int_{I\!\!R} \Delta^{1/2} u v\, dx = 0, \tag{105}$$

for every $v \in C_0^\infty(I\!\!R, I\!\!R^m)$ *and* $v \in T_{u(x)}\mathcal{N}$ *almost everywhere, or in a equivalent way*

ii) the equation

$$\Delta^{1/2}u \wedge u = 0, \quad in\ \mathcal{D}' , \tag{106}$$

or

iii) the equation

$$\Delta^{1/4}(u \wedge \Delta^{1/4}u) = T(Q, u)\,; \tag{107}$$

with $Q = u\wedge$ *.*

Proof of Theorem 6.1

The proof of (105) is the same of the one of Lemma 1.4.10 in [10] , so we omit it.

We prove (106). We take $\varphi \in C_0^\infty(I\!\!R, \bigwedge_{m-2}(I\!\!R^m))$. The following holds

$$\int_{I\!\!R} \varphi \wedge u \wedge \Delta^{1/2}u\, dx = \left(\int_{I\!\!R} *(\varphi \wedge u) \cdot \Delta^{1/2}u\, dx\right) e_1 \wedge \ldots \wedge e_m \tag{108}$$

Claim : $v = *(\varphi \wedge u) \in \dot{H}^{1/2}(I\!\!R, I\!\!R^m)$ and $v(x) \in T_{u(x)}S^{m-1}$ a.e.

Proof of the claim.

The fact that $v \in \dot{H}^{1/2}(I\!\!R, I\!\!R^m)$ follows form the fact that its components are the product of two functions which are in $\dot{H}^{1/2}(I\!\!R, I\!\!R^m) \cap L^\infty(I\!\!R, I\!\!R^m)$.

We have

$$v \cdot u = *(u \wedge \varphi) \cdot u = *(u \wedge \varphi \wedge u) = 0\,. \tag{109}$$

It follows from (105) and (108) that

$$\int_{I\!\!R} \varphi \wedge u \wedge \Delta^{1/2}u dx = 0\,.$$

This shows that $(\Delta)^{1/2}u \wedge u = 0$, in $\mathcal{D}'$, and we can conclude.

As far as equation (107) is concerned it is enough to observe that $\Delta^{1/2}u \wedge u = 0$ and $\Delta^{1/4}u \wedge \Delta^{1/4}u = 0$. □

By combing Theorem 6.1, Proposition 1.2 and the results of the previous Section we get the Hölder regularity of weak 1/2-harmonic maps.

Theorem 6.2 *Let* $u \in \dot{H}^{1/2}(I\!\!R, S^{m-1})$ *be a harmonic map. Then* $u \in C^{0,\alpha}(I\!\!R, S^{m-1})$ *.*

Proof of 6.2. From Theorem 6.1 it follows that u satisfies equation (107). Moreover, since $|u| = 1$, Proposition 1.2 implies that u satisfies (17) as well. Theorem 5.1 and Theorem 5.2 yield respectively that for $k < 0$, with $|k|$ large enough

$$||u \wedge \Delta^{1/4}u||^2_{L^2(B_{2^k})} \le C \sum_{h=k}^{\infty} (2^{\frac{k-h}{2}}) ||\Delta^{1/4}u||^2_{L^2(A_h)} + \frac{1}{4}||\Delta^{1/4}u||^2_{L^2(B_{2^k})}\,. \tag{110}$$

and

$$||u\cdot\Delta^{1/4}u||^2_{L^2(B_{2^k})}\le C\sum_{h=k}^{\infty}(2^{\frac{k-h}{2}})||\Delta^{1/4}u||^2_{L^2(A_h)}+\frac{1}{4}||\Delta^{1/4}u||^2_{L^2(B_{2^k})}\,. \tag{111}$$

Since

$$||\Delta^{1/4}u||^2_{L^2(B_{2^k})}=||u\cdot\Delta^{1/4}u||^2_{L^2(B_{2^k})}+||u\wedge\Delta^{1/4}u||^2_{L^2(B_{2^k})}\,,$$

we get

$$||\Delta^{1/4}u||^2_{L^2(B_{2^k})}\le C\sum_{h=k}^{\infty}(2^{\frac{k-h}{2}})||\Delta^{1/4}u||^2_{L^2(A_h)}\,. \tag{112}$$

Now observe that for some $C>0$ (independent on k) we have

$$C^{-1}\sum_{h=-\infty}^{k-1}||\Delta^{1/4}u||^2_{L^2(A_h)}\le||\Delta^{1/4}u||^2_{L^2(B_{2^k})}\le C\sum_{h=-\infty}^{k}||\Delta^{1/4}u||^2_{L^2(A_h)}\,.$$

Thus from (113) and (111) it follows

$$\sum_{h=-\infty}^{k-1}||\Delta^{1/4}u||^2_{L^2(A_h)}\le C\sum_{h=k}^{\infty}(2^{\frac{k-h}{2}})||\Delta^{1/4}u||^2_{L^2(A_h)}\,.$$

By applying Proposition 4.1 and using again (112) we get for $r>0$ small enough and some $\beta\in(0,1)$

$$\int_{B_r}|\Delta^{1/4}u|^2dx\le Cr^{\beta}\,. \tag{113}$$

Condition (113) yields that u belongs to the Morrey-Campanato Space $\mathcal{L}^{2,-\beta}$ (see [1]), and thus $u\in C^{0,\beta/2}(I\!R)$, (see for instance [1, 8]). $\square$

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