Willmore Minmax Surfaces and the Cost of the Sphere Eversion

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Abstract : *We develop a general Minmax procedure in Euclidian spaces for constructing Willmore surfaces of non zero indices. We implement this procedure to the Willmore Minmax Sphere Eversion in the 3 dimensional euclidian space. We compute the cost of the Sphere eversion in terms of Willmore energies of Willmore Spheres in* $\mathbb{R}^3$.

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I Introduction

I.1 The search for Willmore Minmax Surfaces.

Finding optimal shapes is a search probably as old as mathematics and whose motivation goes beyond the exclusive quest for beauty. It is often closely related to the understanding of deep mathematical structures and ultimately to natural phenomena happening to be governed by these pure ideas.

The existence of closed geodesics on arbitrary manifolds as well as its higher dimensional counterpart, the existence question of minimal surfaces, belongs to this search and has been since the XIXth century a very active area of research diffusing in other areas of mathematics and science in general much beyond the field of Differential Geometry.

The theory of Willmore surfaces, introduced by Wilhelm Blaschke around 1920, grew up out of the attempt to merge *minimal surface theory* and *conformal invariance.*

For an arbitrary immersion $\vec{\Phi}$ of a given oriented abstract surface Σ into an euclidian space $\mathbb{R}^m$ Wilhelm Blaschke introduced the lagrangian

$$W(\vec{\Phi}) = \int_\Sigma |\vec{H}_{\vec{\Phi}}|^2 \ dvol_{g_{\vec{\Phi}}}$$

where $\vec{H}_{\vec{\Phi}}$ and $dvol_{g_{\vec{\Phi}}}$ are respectively the mean curvature vector and the volume form of the metric induced by the immersion. He proved in particular that for a closed surface Σ this lagrangian is invariant under conformal transformations.

The critical points to W are called "Willmore Surfaces[1]". The known set of critical points to the Willmore Lagrangian has been for a long time reduced to the minimal surfaces and their conformal transformations. This maybe explains why the study of it's variations has been more or less stopped during several decades following the seminal work of Wilhelm Blaschke which was slowly sinking into oblivion.

After the reviving work of Tom Willmore the first main contribution to "Willmore Surfaces" has been brought by Robert Bryant in [11]. Using algebraic geometric techniques he succeeded in describing

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[1]This denomination has spread, and is now generally used, after the work [51] which relaunched the study of these surfaces that Blaschke originally named "conformal-minimal surfaces".

all the immersed "Willmore Spheres" in $\mathbb{R}^3$ as being given exclusively by the images by inversions of simply connected complete non compact minimal surfaces with planar ends. The Willmore energy of the immersed Willmore spheres was consequently proved to be equal to 4π times the number of planar ends. Due to the non-triviality of the space of holomorphic quartic forms on any other riemann surface this approach has been restricted to the sphere exclusively. Other algebraic geometric approaches for studying critical points to the Willmore lagrangian include “spectral curve methods” and integrable system theory, but these rather abstract methods are addressing issues which are mostly local and, until now, hardly translatable into “back to earth” results exhibiting new complete Willmore surfaces or characterizing the space of Willmore critical points in a decisive way.

Beside algebraic geometric methods a natural strategy for producing new Willmore surfaces would consist in developing the fundamental principles of the calculus of variations applied to the Willmore Lagrangian. Since $\vec{H}_{\vec{\Phi}} = 2^{-1}\,\Delta_{g_{\vec{\Phi}}}\vec{\Phi}$ the Willmore Lagrangian is nothing but 1/4-th of the L^2 norm of the Laplacian of the immersion and is showing in that sense it's 4th order elliptic nature. This coercive structure gives some hope for the success of the variational methods. The pioneered work studying the variations of W has been written by Leon Simon (see [45]) in which he was proving the existence of a torus minimizing the Willmore energy. This existence result was also motivated by the conjecture formulated by Willmore in [51] according to which the torus obtained by rotating around the vertical z axis of the vertical circle of radius 1 centered at $(\sqrt{2},0,0)$ and included in the plane $y = 0$ would be the unique minimizer modulo the conformal transformations. This conjecture has been finally proved some years ago by Fernando Codá Marques and Andre Neves in [31]. In [4] Matthias Bauer and Ernst Kuwert succeeded in proving a succession of strict inequalities excluding possible degeneracies and the splitting of the underlying surface which was still left open in Leon Simon's argument for arbitrary genus. As a consequence the authors proved the existence of a minimizer of W for any arbitrary closed orientable two dimensional manifold Σ.

Leon Simon's approach to the minimization of Willmore is based on energy comparison arguments and local bi-harmonic graph approximation procedures and in that sense is shaped for studying the ground states of index 0. This approach is mostly considering the image of the immersion $\vec{\Phi}(\Sigma)$ and not the immersion *per se* and can be called "ambient" for that reason. In [42] the author of the present work has introduced an alternative proof to Leon Simon's existence result using an approach called “parametric”. In this approach the study of the variations of the immersion is made possible by local extraction of “Coulomb Gauges” (isothermic parametrization) and the use of the conservation laws issued from the application of Noether theorem combined with the *integrability by compensation theory* (see also a systematic presentation of this theory and it's application to Willmore in [40]). Since this approach is not making use of comparison arguments and since it is based on a weak formulation of the Willmore Euler Lagrange equation discovered in [41], it was giving the author good hope to apply it to more diversified calculus of variation arguments than strict minimizations. This is the main achievement of the present work. Precisely, the purpose of the paper is to present a Minmax method for producing critical points to the Willmore energy of non zero indices.

I.2 "Smoothers" based on the second fundamental form.

As already mentioned the Willmore energy is invariant under the action of the Möbius group of conformal transformations of $\mathbb{R}^m$ which is known to be non compact. For that reason in particular it does not satisfy the Palais Smale condition. This is an obstruction for applying Minmax variational principles such as the *mountain path lemma* directly. We shall then adopt a *viscosity approach* and add to the Willmore energy what we call a “smoother” times a small “viscosity parameter” σ^2

$$\text{Full Energy}(\vec{\Phi}) := W(\vec{\Phi}) + \sigma^2\,\text{Smoother}(\vec{\Phi})$$

that makes for the new energy the Palais-Smale condition being satisfied. One can then apply the mountain path arguments to such energy and produce Minmax critical points. In the second part of the procedure one makes σ tend to zero and one studies the converging process hopefully to a Willmore Minmax Surface.

In a first approach, following [44], one could think adding to W a term of the form

$$\text{Smoother}(\vec{\Phi}) = \int_\Sigma (1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2)^2$$

where $\vec{\mathbb{I}}_{\vec{\Phi}}$ is the second fundamental form of the immersion $\vec{\Phi}$. This will make the new Willmore relaxed energy satisfy the *Palais-Smale condition* (as proved in [24], see also [27]) but this will bring us to the study of $p-$harmonic systems which makes the analysis of the convergence rather involved - in particular the energy quantization - when the small viscosity parameter σ tends to zero. From that perspective $p-$harmonic versus harmonic systems, as observed below and as it is also used in [9], replacing the full second fundamental form by it's trace $\vec{H}_{\vec{\Phi}}$ and consider instead

$$\text{Smoother}(\vec{\Phi}) = \int_\Sigma (1+|\vec{H}_{\vec{\Phi}}|^2)^2$$

has the surprising effect to make the highest order term in the Euler Lagrange to be Δ and not $\Delta_2 := div((1+|H|^2)\nabla)$ if one makes use of the various conservation laws issued from Noether theorem, following the main lines of [41]. The drawback however is that $\int_\Sigma (1+|\vec{H}_{\vec{\Phi}}|^2)^2$ fails to satisfy the *Palais-Smale condition* and cannot be a *smoother* by itself and has to be "reinforced".

I.3 Frame Energies

In the portofolio of surface energies, the author, in collaboration with Andrea Mondino, introduced in [35] the notion of *frame energy* for an arbitrary immersion of a torus $\vec{\Phi} : T^2 \longrightarrow \mathbb{R}^m$ equipped by an orthonormal tangent frame $\vec{e} : T^2 \to S^{m-1} \times S^{m-1}$ where $\vec{e}(x)$ is realizing an orthonormal basis of $\vec{\Phi}_* T_x T^2$. The frame energy is then simply given by

$$\mathcal{F}(\vec{\Phi},\vec{e}) := \frac{1}{4}\int_{T^2} |d\vec{e}|^2_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \ge W(\vec{\Phi}) \quad . \tag{I.1}$$

If one considers the "Coulomb frame" associated to the conformal immersion of a fixed flat torus[2], the *frame energy* $F(\vec{\Phi}) := \inf_{\vec{e}} \mathcal{F}(\vec{\Phi},\vec{e})$ is then defining an energy of the immersion $\vec{\Phi}$ that happens to be more coercive than the Willmore energy itself. One could ask whether it can be naturally extended to any other immersion of an arbitrary surface Σ. This happens to be indeed the case as we explain in the sequel.

Let $\vec{\Phi}$ be an arbitrary immersion of a closed surface Σ and denote by g_0 a constant scalar curvature metric of volume one on Σ for which there exists $\alpha : \Sigma \to \mathbb{R}$

$$g_{\vec{\Phi}} = e^{2\alpha}\ g_0 \quad . \tag{I.2}$$

For $\Sigma \neq S^2$ the function α is defined without ambiguity, whereas in the sphere case we have to count with the action of a "gauge group", the space $\mathcal{M}^+(S^2)$ of positive conformal transformations of S^2 and α is uniquely defined modulo the action of this gauge group. For the case of the torus one proves in this paper that

$$F(\vec{\Phi}) = W(\vec{\Phi}) + \frac{1}{2}\int_{T^2} |d\alpha|^2_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \quad .$$

We generalize the frame energy for any surface of genus larger than 1 as follows : $F := W + \mathcal{O}$ where

$$\mathcal{O}(\vec{\Phi}) = \frac{1}{2}\int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} + K_{g_0}\int_\Sigma \alpha\ e^{-2\alpha}\ dvol_{g_{\vec{\Phi}}} - 2^{-1}\,K_{g_0}\,\log\int_\Sigma dvol_{g_{\vec{\Phi}}} \quad , \tag{I.3}$$

[2] By the uniformization theorem such a parametrization always exists

where K_{g_0} is the constant scalar curvature metric of g_0. The reason why adding to the Dirichlet energy of α the term

$$K_{g_0}\int_\Sigma \alpha\ e^{-2\alpha}\ dvol_{g_{\vec{\Phi}}}$$

comes from the fact that the first variation can be expressed miraculously by <u>local</u> quantities ! (although the operation which to g_Φ assigns α is highly non-local - see for instance [50]). Finally the reason why adding the third term is twofold : it makes the energy scaling invariant and non negative as a direct application of Jensen when $K_{g_0} < 0$.

Finally, when $\Sigma = S^2$ we define identically the *Frame energy* to be

$$F(\vec{\Phi}) := W(\vec{\Phi}) + \frac{1}{2}\int_{S^2} |d\alpha|^2_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} + 4\,\pi\ \int_{S^2} \alpha\ e^{-2\alpha}\ dvol_{g_{\vec{\Phi}}} - 2\,\pi\ \log\int_{S^2} dvol_{g_{\vec{\Phi}}}$$

Beside the fact that it is naturally generalizing to S^2 the Dirichlet energy of *Coulomb frames* on tori, there are 4 main reasons why considering this special expression with these particular coefficients

i) $F - W$ is the well known *Onofri Energy* of α (see [37] and a more recent presentation in [14]) and satisfies[3]

$$\mathcal{O}(\vec{\Phi}) := \frac{1}{2}\int_{S^2} |d\alpha|^2_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} + 4\,\pi\ \int_{S^2} \alpha\ e^{-2\alpha}\ dvol_{g_{\vec{\Phi}}} - 2\,\pi\ \log\int_{S^2} dvol_{g_{\vec{\Phi}}} \ge 0 \qquad \text{(I.4)}$$

Observe that the Lagrangian $\mathcal{O}(\vec{\Phi})$ viewed as a functional depending of $\vec{\Phi}$ it self and of $\vec{\Phi}$ is nothing but the main term in the *Polyakov-Alvarez* formula of the *Zeta regularized determinant of the Laplacian* of the underlying riemannian 2 dimensional manifold. This formula is also named *Polyakov-Alvarez conformal anomaly formula* in *Conformal Field Theory* (see [39]).

ii) The first variation of F is explicit and can be expressed using <u>local</u> quantities[4].

iii) The energy $\mathcal{O}(\vec{\Phi})$ is gauge invariant with respect to the action of the Möbius group on S^2 and is independent of the choice of α and g_0 satisfying (I.2) and depends only on $\vec{\Phi}$.

iv) The F-energy is dilation invariant : $F(e^t\ \vec{\Phi}) = F(\vec{\Phi})$ for any $t \in \mathbb{R}$.

Open Problem 1. *It would be interesting to study the existence of minimizers of the frame energy F in each regular homotopy class of immersions of spheres in $\mathbb{R}^4$. Since the work of Stephen Smale [46] we know that there exists countably many of them given by the $\pi_2(V_{4,2}(\mathbb{R})) = \mathbb{Z}$, the second homotopy class of the Stiefel manifold of 2-frames in $\mathbb{R}^4$. It would also be interesting to study the asymptotic dependence with respect to the class of the infimum of the $F-$energy as the class goes to infinity (if it is linear or sub-linear).*

Finally, for a given immersion $\vec{\Phi}$ it would be interesting to study the minimal Dirichlet energy of any bundle map from $T\Sigma$ into $\mathbb{R}^3 \times \mathbb{R}^3$ which is an isometry from each fiber $(T_x\Sigma, g_{\vec{\Phi}})$ into $\vec{\Phi}_* T_x\Sigma \subset G_2(\mathbb{R}^3)$ and which projects onto $\vec{\Phi}$ - the map $(\vec{\Phi}, d\vec{\Phi})$ is one of such maps of course. Starting from $(\vec{\Phi}, d\vec{\Phi})$ such a bundle map is just given by the choice of an S^1 rotation at each point.

In the case of $\Sigma = T^2$ this coincides with the Dirichlet energy of an optimal global frame and is equal to $F(\vec{\Phi})$.

[3] This inequality is not the direct consequence of Jensen inequality when $K_{g_0} > 0$ and requires more elaborated arguments.

[4] This is a very striking fact which is going to make the analysis simpler in the following sections. It very much depends on the choice of the coefficients in front of each main term of the energy.

I.4 A Viscous Approximation of Willmore.

Inspired by the discussion above we propose to consider the following approximation of the Willmore energy

$$F^\sigma(\vec{\Phi}) := W(\vec{\Phi}) + \sigma^2 \int_\Sigma (1+|\vec{H}|^2)^2 \, dvol_{\vec{\Phi}} + \frac{1}{\log\frac{1}{\sigma}}\mathcal{O}(\vec{\Phi})$$

where $\mathcal{O}(\vec{\Phi})$ is given by (I.4) for $\Sigma = S^2$ or by the expression (I.3) otherwise.

We prove in section V that F^σ satisfies the Palais Smale condition. One can then apply Minmax arguments to F^σ for any admissible family. One of the main achievements of the present work is the proof an $\epsilon-$regularity <u>independent of σ</u> (see lemma IV.1). It is making use of the special choice we made of the logarithmic dependence of the small parameter with respect to the viscosity parameter σ in front of $\mathcal{O}$. This $\epsilon-$regularity permits to pass to the limit in the equation for well chosen sequences of Minmax critical points of F^σ as σ goes to zero. The last main lemma in the paper is an energy quantization result when $\Sigma = S^2$ (see lemma VI.1). It roughly says that no energy can be dissipated in neck regions.

We then have the main tools for performing Minmax procedures for the Willmore energy of spheres. To that aim we introduce the space of $W^{2,4}(S^2,\mathbb{R}^m)$ immersion of S^2 into $\mathbb{R}^m$ that we denote $\mathcal{E}_{\Sigma,2}(\mathbb{R}^m)$. This space is equipped with the $W^{2,4}$ topology. It is proved in [44] that this defines a Banach manifold with a Finsler structure. Before giving the statement of our main result we will recall the definition of admissible families.

Definition I.1. *A family of subsets $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ of a Banach manifold $\mathcal{M}$ is called* **admissible family** *if for every homeomorphism Ξ of $\mathcal{M}$ isotopic to the identity we have*

$$\forall\, A \in \mathcal{A} \qquad \Xi(A) \in \mathcal{A}$$

□

Our main result is the following.

Theorem I.1. [Willmore Minmax Procedures for Spheres] *Let $m \geq 3$ and $k \geq 1$. Let $\mathcal{A}$ be an admissible family of $W^{2,4}$ immersions of the sphere S^2 into $\mathbb{R}^m$. Let*

$$\beta_0 := \inf_{A\in\mathcal{A}} \max_{\vec{\Phi}\in A} W(\vec{\Phi}) \quad .$$

Then there exists finitely many Willmore immersions of S^2 minus finitely many points, $\vec{\xi}_1 \cdots \vec{\xi}_n$, such that

$$\beta_0 = \sum_{i=1}^n W(\vec{\xi}_i) - 4\pi\, N \tag{I.5}$$

where $N \in \mathbb{N}$. □

Remark I.1. *The Theorem I.1 goes with a "bubble tree convergence" of a sequence of Minmax critical points of F^{σ^k} for some well chosen sequence σ^k tending to zero. This convergence produces asymptotically a "bubble tree" of Willmore spheres some of them being shrunk to zero. Among the ones which shrunk to zero there might be non compact (after asymptotic rescaling) simply connected Willmore surfaces with ends at infinity that we have to inverse in order to make them being Willmore sphere. This operation in producing energy given by an integer multiple times 4π. This is why such a quantity is subtracted in (I.5). One of the hard parts in the whole proof is to show that between two successive asymptotic Willmore Spheres in that "bubble tree" no energy is lost at the limit. This is the so called "no neck energy" property.* □

True Willmore Surfaces. *It has been recently proved in [33] by the author in collaboration with Alexis Michelat that the maps $\vec{\xi}_i$, the ones obtained after inversions in the bubble tree, define "true" Willmore possibly branched immersions. By "true" we mean that the first residue*[5] *defined in [41]*

$$\int_\Gamma \partial_\nu \vec{H} - 3\,\pi_{\vec{n}}\left(\partial_\nu \vec{H}\right) + \star\,\partial_\tau \vec{n}\wedge\vec{H}$$

is zero for any closed curve Γ avoiding the center of the inversion. The inversion of the Catenoid is not a "true" Willmore surface in that sense whereas the inversion of the Enneper surface is a "true" Willmore Sphere with a branch point of multiplicity 3 at the origin (see [7]).

Open Problem 2. *Extend the previous result to general surfaces. The "only" obstruction comes from the Energy Quantization result which is missing when the conformal class of the Minmax sequence of F^{σ^k} possibly degenerates. The recent progresses made in [29] and [30] should be of great help in solving this difficulty.* □

One consequence of the previous result is the following corollary. One considers the family $\mathcal{A}$ of loops into $\mathrm{Imm}(S^2,\mathbb{R}^3)\simeq_{hom} SO(3)\times\Omega^2(SO(3))$ realizing a non trivial element of $\pi_1(\mathrm{Imm}(S^2,\mathbb{R}^3))\simeq\mathbb{Z}_2\times\mathbb{Z}$. It is proved in [3] that for instance the Froisart-Morin sphere eversion followed by the mirror image of the time reversed of the same eversion is generating $\pi_1(\mathrm{Imm}(S^2,\mathbb{R}^3))$. In order to avoid uninteresting loops coming from the action of $\mathrm{Diff}(S^2)$ one should rather work modulo the action of reparametrization of the sphere and consider the infinite orbifold[6] $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)$ instead of $\mathrm{Imm}(S^2,\mathbb{R}^3))$ which is an open subspace of the Banach space $W^{2,4}(S^2,\mathbb{R}^3)$.

One can then take $\mathcal{A}$ to be the canonical projections onto $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)$ of paths from $[0,1]$ into $\mathrm{Imm}(S^2,\mathbb{R}^3)$ homotopic to a non trivial element in $\pi_1(\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2))=\mathbb{Z}$. The projection of one time the Froisart-Morin sphere eversion gives such a loop for instance[7].

Corollary I.1. [The Cost of the Sphere Eversion] *Let Ω be the space of continuous paths of C^2 immersions into $\mathbb{R}^3$ joining the standard sphere S^2 with the two opposite orientations and homotopic to the Froisart-Morin Sphere eversion. Define β_0 to be the "cost of the sphere eversion" by*

$$\beta_0 := \inf_{\omega\in\Omega}\max_{\vec{\Phi}\in\omega} W(\vec{\Phi})$$

then there exists finitely many Willmore immersions of S^2 minus finitely many points, $\vec{\xi}_1\cdots\vec{\xi}_n$, such that

$$\beta_0 = \sum_{i=1}^n W(\vec{\xi}_i) - 4\pi\,N$$

where $N\in\mathbb{N}$. □

Remark I.2. *Performing the Willmore Minmax Sphere Eversion has been originally proposed by Robert Kusner.* □

[5] In 3 dimension this residue is also a multiple of the one that can be deduced from the integration of the one form (4.5) in [25].

[6] While $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}^+(S^2)$ the quotient of $\mathrm{Imm}(S^2,\mathbb{R}^3))$ by the group of positive diffeomorphisms of S^2 has a nice bundle structure due to the free action of $\mathrm{Diff}^+(S^2)$ on $\mathrm{Imm}(S^2,\mathbb{R}^3))$, this is not the case anymore for $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)$. The space $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)$ is an infinite orbifold obtained by the quotient of $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}^+(S^2)$ by the map $x\to -x$. This induces a 2 sheets covering away from the subspace of singular orbits which happens to have infinite codimension (see section 3 of [12]). Because of the smallness of the size of singular orbits, using transversality arguments, one can compute homotopy groups of $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)$ as if this covering map would be without singularities.

[7] The canonical projection π of $\mathrm{Imm}(S^2,\mathbb{R}^3)$ onto the infinite orbifold $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)$ induces a morphism between the two first homotopy groups $\pi_*:\pi_1(\mathrm{Imm}(S^2,\mathbb{R}^3))=\mathbb{Z}_2\times\mathbb{Z}\to\pi_1(\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2))=\mathbb{Z}$ equal to the multiplication by 2

Open Problem 3. *Is it true that in corollary I.1*

$$\beta_0 = 16\pi \quad ?$$

A topological result, see [3] (see also the enlightening proof in [22]), asserts that any element in Ω *has to contain at least one immersion with a point of self intersection of order 4 (i.e. a quadruple point). Hence using Li-Yau's result we deduce that* $\beta_0 \geq 16\pi$*. In [17] a candidate for the realization of* β_0 *is proposed. It is the inversion of a simply connected complete minimal surface with 4 planar ends. Hence the Willmore energy of this candidate is* 16π*. Interesting computations reinforcing this conjecture are performed in this work. Establishing upper-bound of the lowest energy Minmax sphere eversion is making difficult by the fact that producing concrete sphere eversions is highly challenging and has been at the origin of many rigorous works, computer simulations and videos too starting from the first example given by Arnold Shapiro (see for instance [36], [16], [38], [2], [18], [49], [1]...)*

Remark I.3. *In a recent work ([33]) the author, in collaboration with Michelat, extends Bryant's classification to "true", possibly branched Willmore spheres.* □

Remark I.4. *An interesting upper-bound of the cost of the Total Curvature Minmax Sphere Eversion is presented in [15]. In this paper it is proved in particular that*

$$\inf_{\omega\in\Omega} \max_{\vec{\Phi}\in\omega} \int_{S^2} |K_{\vec{\Phi}}| \, dvol_{g_{\vec{\Phi}}} \leq 8\pi$$

where $K_{\vec{\Phi}}$ *is the Gauss curvature of* $\vec{\Phi}$*.* □

Open Problem 5. *It would be interesting to study the cost of the Frame Energy* $W + \mathcal{O}$ *Minmax Sphere Eversion. As we saw above this energy is closely related to the minimizing Dirichlet energy among the bundle maps injections induced by* $(\vec{\Phi}, d\vec{\Phi})$ *- which are used by Smale to compute the homotopy type of the space of immersions.*

Open Problem 6. *It would be interesting, beside the study of the Wilmore Minmax Sphere Eversion exclusively, to explore also the Willmore Minmax for* $\mathcal{A}$ *corresponding to other non trivial classes in* $\pi_k(Imm(S^2,\mathbb{R}^3)) = \pi_k(SO(3)\times\Omega^2(SO(3)))$ *for arbitrary k. How would their indices be related to k ?*

Open Problem 7. *Explore the topology of* $Imm(S^2,\mathbb{R}^3)$ *using W as some kind of "quasi*[8] *Morse function" : For instance we could ask if all the Willmore immersions of* S^2 *into* $\mathbb{R}^3$ *- described by Robert Bryant - are related to a Minmax procedure involving the various classes of the various groups* $\pi_k(Imm(S^2,\mathbb{R}^3))$ *? In order to start "slow" this far-reaching question a first certainly instructive step would consist in computing the indices of the Willmore Sphere Immersions in* $\mathbb{R}^3$*. Of course one would have also to complete the space of immersions by considering possibly branched immersions and, up to now, there is no general result known about the extension of Bryant classification of Willmore Immersed Spheres in* $\mathbb{R}^3$ *to Willmore Branched Spheres beside some very special cases treated in [26].*

Most of the proofs below are presented in the particular case $m = 3$ in order to make the presentation of them more accessible. The general case $m \geq 3$ is very similar but requires the use of the conservation laws in arbitrary codimensions introduced in [41] whose formulation involves the use of multi-vectors instead of vectors and are a bit more tedious but do not bring any new fundamental difficulties.

[8] "quasi" because we know that $\partial^2 W$ has at least the null directions given by the action of Möbius transformations.

II The space of immersions into $\mathbb{R}^3$ with L^q bounded second fundamental form.

For $k \in \mathbb{N}$ and $1 \leq q \leq +\infty$, we recall the definition of $W^{k,q}$ Sobolev function on a closed smooth surface Σ (i.e. Σ is compact without boundary). To that aim we take some reference smooth metric g_0 on Σ

$$W^{k,q}(\Sigma, \mathbb{R}) := \left\{ f \quad \text{measurable} \quad s.t \quad \nabla^k_{g_0} f \in L^q(\Sigma, g_0) \right\}$$

where $\nabla^k_{g_0}$ denotes the $k-$th iteration of the Levi-Civita connection associated to Σ. Since the surface is closed the space defined in this way is independent of g_0.

For $p \geq 1$, following [42] in which the case $p = 1$ was considered, we define the space $\mathcal{E}_{\Sigma,p}$ of *weak immersions of* Σ *into* $\mathbb{R}^3$ *with* L^{2p} *bounded second fundamental form as follows.*

$$\mathcal{E}_{\Sigma,p} := \left\{ \begin{array}{l} \vec{\Phi} \in W^{1,\infty}(\Sigma, \mathbb{R}^3) \quad \text{s. t.} \ \exists\, C > 1 \\[2mm] C^{-1}\ g_0 \leq \vec{\Phi}^* g_{\mathbb{R}^3} \leq C\ g_0 \quad \text{a. e.} \\[2mm] \text{the Gauss map } \vec{n}_{\vec{\Phi}} \in W^{1,2p}(\Sigma, Gr_2(\mathbb{R}^3)) \end{array} \right\}$$

For any $\vec{\Phi} \in \mathcal{E}_{\Sigma,p}$, starting from the equation $\Delta_{g_{\vec{\Phi}}} \vec{\Phi} = 2\,\vec{H}$, classical elliptic estimates permit to bootstrap (in the case $p > 1$) and get that $\vec{\Phi}$ is in fact in $W^{2,2p}(\Sigma, \mathbb{R}^3)$ (see [40]). It is then not difficult to observe that $\mathcal{E}_{\Sigma,p}$ is in fact, <u>for $p > 1$</u> an open subset of the Banach space $W^{2,2p}(\Sigma, \mathbb{R}^3)$. The border line case $p = 1$ is more delicate, it was introduced first in [42] as being of prior interest for studying the variations of Willmore and is extensively presented also in [40].

III Frame Energies.

For any weak immersion $\vec{\Phi}$ in $\mathcal{E}_{\Sigma,p}$ we denote $A_{g_{\vec{\Phi}}}$ the connection 1-form - which lives on Σ since the tangent bundle is abelian - equal to the difference between the Levi-Civita connection $\nabla^{g_{\vec{\Phi}}}$ defined by $g_{\vec{\Phi}}$ and the Levi-Civita connection ∇^{g_0}

$$A_{g_{\vec{\Phi}}} := \nabla^{g_{\vec{\Phi}}} - \nabla^{g_0}$$

where g_0 is a constant scalar curvature metric of volume one on Σ for which there exists $\alpha\ : \Sigma\ \rightarrow\ \mathbb{R}$

$$g_{\vec{\Phi}} = e^{2\alpha}\ g_0 \quad .$$

For $\Sigma \neq S^2$ the function α is defined without ambiguity, whereas in the sphere case we have to count with the action of a "gauge group", the space $\mathcal{M}^+(S^2)$ of positive conformal transformations of S^2 and α is uniquely defined modulo the action of this gauge group. We shall now express $|A_{g_{\vec{\Phi}}}|^2_{g_{\vec{\Phi}}}$ locally using moving frames. Let (e_1, e_2) be an orthonormal local frame[9] for the metric $g_{\vec{\Phi}}$. We have

$$|\nabla^{g_{\vec{\Phi}}} - \nabla^{g_0}|^2_{g_{\vec{\Phi}}} = \sum_{i,j=1}^{2} |((\nabla^{g_{\vec{\Phi}}} - \nabla^{g_0})\, e_i \cdot e_j)|^2_{g_{\vec{\Phi}}}$$

[10] Since e_i is a unit vector field for $g_{\vec{\Phi}}$ we have $(\nabla^{g_{\vec{\Phi}}} e_i, e_i) = 0$ and since $f_i := e^{\alpha}\, e_i$ is a unit vector field for g^0 we have

$$(\nabla^{g_0} e_i, e_i) = e^{-2\alpha}\, (\nabla^{g_0} f_i, f_i)_{g_{\vec{\Phi}}} - e^{-2\alpha}\ de^{\lambda}\ (e_i, f_i)_{g_{\vec{\Phi}}} = -\, d\alpha$$

---

[9] We denote by $(\vec{e}_1, \vec{e}_2)$ the push-forward by $\vec{\Phi}$ -it's realization in $\mathbb{R}^3$ - of the "abstract" orthonormal frame (e_1, e_2).

[10] We shall denote by $\cdot$ the scalar product in the tangent space, by , the scalar product in the co-tangent space and by ; the combination of the two scalar products.

Hence

$$\sum_{i=1}^{2}|((\nabla^{g_{\vec{\Phi}}}-\nabla^{g_0})\,e_i\cdot e_i)|^2_{g_{\vec{\Phi}}}\;=\;2\;|d\alpha|^2_{g_{\vec{\Phi}}}$$

In order to compute $|((\nabla^{g_{\vec{\Phi}}}-\nabla^{g_0})\,e_1\cdot e_2)|^2_{g_{\vec{\Phi}}}$ we choose local conformal coordinates (x_1,x_2) for $g_{\vec{\Phi}}$ and we have then the existence of μ locally such that respectively $g_{\vec{\Phi}}=e^{2\,(\alpha+\mu)}\;[dx_1^2+dx_2^2]$ and $g_0=e^{2\mu}\;[dx_1^2+dx_2^2]$. We choose $e_i:=e^{-\alpha-\mu}\;\partial_{x_i}$ and thus $f_i=e^{-\mu}\;\partial_{x_i}$. Using classical computation of the Levi-Civita connection of a metric in conformal charts (see [40]) we have

$$(\nabla^{g_{\vec{\Phi}}}e_1\cdot e_2)_{g_{\vec{\Phi}}}=*\;d(\alpha+\mu)\quad\text{ and }\quad(\nabla^{g_0}f_1\cdot f_2)_{g_0}=*\;d\mu$$

Since e_1 and e_2 are orthogonal to each other with respect to g_0, we have

$$(\nabla^{g_0}e_1\cdot e_2)_{g_{\vec{\Phi}}}=e^{2\alpha}\;(\nabla^{g_0}e_1\cdot e_2)_{g_0}=(\nabla^{g_0}f_1\cdot f_2)_{g_0}=*\;d\mu$$

Hence

$$((\nabla^{g_{\vec{\Phi}}}-\nabla^{g_0})\,e_1\cdot e_2)_{g_{\vec{\Phi}}}=*d\alpha\quad.$$

Combining the previous we obtain

$$|\nabla^{g_{\vec{\Phi}}}-\nabla^{g_0}|^2_{g_{\vec{\Phi}}}=\sum_{i,j=1}^{2}|((\nabla^{g_{\vec{\Phi}}}-\nabla^{g_0})\,e_i,e_j)|^2_{g_{\vec{\Phi}}}=4\,|d\alpha|^2_{g_{\vec{\Phi}}}\quad.\tag{III.1}$$

For any C^1 function f we consider the following "Frame energy".

$$F_f(\vec{\Phi}):=\int_\Sigma\left[f(H_{g_{\vec{\Phi}}})+2^{-3}\,|A_{g_{\vec{\Phi}}}|^2_{g_{\vec{\Phi}}}\right]\;dvol_{g_{\vec{\Phi}}}$$

Observe then that in the case of $\Sigma=T^2$ and $f(t)=t^2$, for some global Coulomb frame $\vec{e}$ (see [35]), we have

$$F_{t^2}(\vec{\Phi})=\int_{T^2}[|H_{g_{\vec{\Phi}}}|^2+2^{-3}\,|A_{g_{\vec{\Phi}}}|^2_{g_{\vec{\Phi}}}]\;dvol_{g_{\vec{\Phi}}}=\frac{1}{4}\int_{T^2}|d\vec{e}|^2\;dvol_{g_{\vec{\Phi}}}\quad.\tag{III.2}$$

which is noting but the Dirichlet energy of the frame and justifies the denomination "Frame energy".

III.1 The first variation of Frame energies.

We shall now compute the first variation of F_f. We shall first concentrate on the second part

$$C(\vec{\Phi}):=\int_\Sigma|A_{g_{\vec{\Phi}}}|^2_{g_{\vec{\Phi}}}\;dvol_{g_{\vec{\Phi}}}\quad,$$

that we call *connection energy*. We observe that locally for any unit vector field e for the metric $g_{\vec{\Phi}}$ one has

$$|d\alpha|^2_{g_{\vec{\Phi}}}=|d\,|e|_{g_0}|^2_{g_0}\quad.$$

We consider a perturbation $\vec{\Phi}_t:=\vec{\Phi}+t\,\vec{w}$. Recall the Liouville equation

$$-\Delta_{g_{\vec{\Phi}_t}}\alpha_t=K_{g_{\vec{\Phi}_t}}-e^{-2\alpha_t}\,K_{g_0}\quad,\tag{III.3}$$

where $\Delta_{\vec{\Phi}_t}$ is the <u>negative</u> Laplace Beltrami operator. Observe that K_{g_0} is independent of t. We have then

$$\frac{d}{dt}\left[\Delta_{g_{\vec{\Phi}_t}}\alpha_t\right]+\;2\,K_{g_0}\;e^{-2\alpha}\frac{d\alpha}{dt}=-\frac{dK_{g_{\vec{\Phi}_t}}}{dt}\quad.\tag{III.4}$$

Hence

$$\Delta_{g_{\vec{\Phi}_t}} \frac{d\alpha}{dt} + 2\,K_{g_0}\ e^{-2\alpha} \frac{d\alpha}{dt} = -\frac{d}{dt}\left[\Delta_{g_{\vec{\Phi}_t}}\right]\alpha - \frac{dK_{g_{\vec{\Phi}_t}}}{dt} \quad . \tag{III.5}$$

In other words, we have

$$\Delta_{g_0} \frac{d\alpha}{dt} + 2\,K_{g_0}\ \frac{d\alpha}{dt} = -\,e^{2\alpha}\ \frac{d}{dt}\left[\Delta_{g_{\vec{\Phi}_t}}\right]\alpha - e^{2\alpha}\ \frac{dK_{g_{\vec{\Phi}_t}}}{dt} \quad . \tag{III.6}$$

In a local chart we have

$$K_{g_{\vec{\Phi}_t}}\ dvol_{g_{\vec{\Phi}_t}} = \vec{n}\cdot\partial_{x_1}\vec{n}\times\partial_{x_2}\vec{n}\ \ dx_1\wedge dx_2 \quad . \tag{III.7}$$

Since

$$\frac{d\vec{n}}{dt}(0) = -\left\langle \vec{n}\cdot d\vec{w}\,,\,d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}} \quad . \tag{III.8}$$

we deduce

$$\begin{aligned}\frac{d(K_{g_{\vec{\Phi}_t}}\ dvol_{g_{\vec{\Phi}_t}})}{dt} &= -\left\langle \vec{n}\cdot d\vec{w}\,,\,d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}}\cdot\partial_{x_1}\vec{n}\times\partial_{x_2}\vec{n}\ \ dx_1\wedge dx_2\\ -\Big[\vec{n}\cdot\partial_{x_1}\left\langle \vec{n}\cdot d\vec{w}\,,\,d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}}&\times\partial_{x_2}\vec{n}+\vec{n}\cdot\partial_{x_1}\vec{n}\times\partial_{x_2}\left\langle \vec{n}\cdot d\vec{w}\,,\,d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}}\Big]\ dx_1\wedge dx_2 \quad .\end{aligned} \tag{III.9}$$

We have

$$-\left\langle \vec{n}\cdot d\vec{w}\,,\,d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}}\cdot\partial_{x_1}\vec{n}\times\partial_{x_2}\vec{n} = 0 \quad . \tag{III.10}$$

We choose a chart in which $\vec{\Phi}$ is conformal and we denote

$$g_{\vec{\Phi}} = e^{2\lambda}\ [dx_1^2+dx_2^2] \quad \text{and} \quad g_0 = e^{2\mu}\ [dx_1^2+dx_2^2] \quad ,$$

thus $\alpha = \lambda-\mu$. We have in one hand

$$\begin{aligned}-\vec{n}\cdot\partial_{x_1}\left\langle \vec{n}\cdot d\vec{w}\,,\,d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}}\times\partial_{x_2}\vec{n} &= -\sum_{i=1}^2\partial_{x_1}\left(e^{-2\lambda}\,\vec{n}\cdot\partial_{x_i}\vec{w}\right)\ \vec{n}\cdot\partial_{x_i}\vec{\Phi}\times\partial_{x_2}\vec{n}\\ &\quad -\sum_{i=1}^2 e^{-2\lambda}\,\vec{n}\cdot\partial_{x_i}\vec{w}\ \ \vec{n}\cdot\partial^2_{x_ix_1}\vec{\Phi}\times\partial_{x_2}\vec{n}\\ &= \mathbb{I}_{22}\ \partial_{x_1}\left(e^{-2\lambda}\,\vec{n}\cdot\partial_{x_1}\vec{w}\right) - \mathbb{I}_{21}\ \partial_{x_1}\left(e^{-2\lambda}\,\vec{n}\cdot\partial_{x_2}\vec{w}\right)\\ &\quad -\langle\vec{n}\cdot d\vec{w},d\lambda\rangle_{g_{\vec{\Phi}}}\ \ \vec{n}\cdot\partial_{x_1}\vec{\Phi}\times\partial_{x_2}\vec{n} - \sum_{i=1}^2 e^{-4\lambda}\ \vec{n}\cdot\partial_{x_i}\vec{w}\ \ \partial^2_{x_ix_1}\vec{\Phi}\cdot\partial_{x_2}\vec{\Phi}\ \mathbb{I}_{12} \quad ,\end{aligned} \tag{III.11}$$

and in the other hand

$$\begin{aligned}-\vec{n}\cdot\partial_{x_1}\vec{n}\times\partial_{x_2}\left\langle \vec{n}\cdot d\vec{w}\,,\,d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}} &= \sum_{i=1}^2\partial_{x_2}\left(e^{-2\lambda}\,\vec{n}\cdot\partial_{x_i}\vec{w}\right)\ \vec{n}\cdot\partial_{x_i}\vec{\Phi}\times\partial_{x_1}\vec{n}\\ &\quad +\sum_{i=1}^2 e^{-2\lambda}\,\vec{n}\cdot\partial_{x_i}\vec{w}\ \ \vec{n}\cdot\partial^2_{x_ix_2}\vec{\Phi}\times\partial_{x_1}\vec{n}\\ &= \mathbb{I}_{11}\ \partial_{x_2}\left(e^{-2\lambda}\,\vec{n}\cdot\partial_{x_2}\vec{w}\right) - \mathbb{I}_{21}\ \partial_{x_2}\left(e^{-2\lambda}\,\vec{n}\cdot\partial_{x_1}\vec{w}\right)\\ &\quad +\langle\vec{n}\cdot d\vec{w},d\lambda\rangle_{g_{\vec{\Phi}}}\ \ \vec{n}\cdot\partial_{x_2}\vec{\Phi}\times\partial_{x_1}\vec{n} - \sum_{i=1}^2 e^{-4\lambda}\ \vec{n}\cdot\partial_{x_i}\vec{w}\ \ \partial^2_{x_ix_2}\vec{\Phi}\cdot\partial_{x_1}\vec{\Phi}\ \mathbb{I}_{12} \quad .\end{aligned} \tag{III.12}$$

Summing (III.10), (III.11) and (III.12) gives

$$\begin{aligned}\frac{d(K_{g_{\vec{\Phi}_t}}\ dvol_{g_{\vec{\Phi}_t}})}{dt} &= \left[\mathbb{I}_{22}\ \partial_{x_1}\left(e^{-2\lambda}\,\vec{n}\cdot\partial_{x_1}\vec{w}\right)+\mathbb{I}_{11}\ \partial_{x_2}\left(e^{-2\lambda}\,\vec{n}\cdot\partial_{x_2}\vec{w}\right)\right]dx_1dx_2\\ &-\mathbb{I}_{12}\ \left[\partial_{x_1}\left(e^{-2\lambda}\,\vec{n}\cdot\partial_{x_2}\vec{w}\right)+\partial_{x_2}\left(e^{-2\lambda}\,\vec{n}\cdot\partial_{x_1}\vec{w}\right)\right]\ dx_1dx_2\\ &+2\ H\ e^{2\lambda}\ \ \langle\vec{n}\cdot d\vec{w},d\lambda\rangle_{g_{\vec{\Phi}}}\ \ dx_1dx_2 \quad .\end{aligned} \tag{III.13}$$

Recall Codazzi

$$\left\{\begin{array}{l}\partial_{x_1}\mathbb{I}_{22}-\partial_{x_2}\mathbb{I}_{12}=H\ \partial_{x_1}e^{2\lambda}\\[3mm] \partial_{x_2}\mathbb{I}_{11}-\partial_{x_1}\mathbb{I}_{12}=H\ \partial_{x_2}e^{2\lambda}\quad .\end{array}\right. \tag{III.14}$$

Hence we have proved the following lemma

Lemma III.1. *Under the above notations we have*

$$\begin{aligned}\frac{d(K_{g_{\vec{\Phi}_t}}\ dvol_{g_{\vec{\Phi}_t}})}{dt} &= \left[\partial_{x_1}\left(\mathbb{I}_{22}\ e^{-2\lambda}\,\vec{n}\cdot\partial_{x_1}\vec{w}\right)+\partial_{x_2}\left(\mathbb{I}_{11}\ e^{-2\lambda}\,\vec{n}\cdot\partial_{x_2}\vec{w}\right)\right]dx_1\wedge dx_2\\ &-\left[\partial_{x_1}\left(\mathbb{I}_{12}\ e^{-2\lambda}\,\vec{n}\cdot\partial_{x_2}\vec{w}\right)+\partial_{x_2}\left(\mathbb{I}_{12}\ e^{-2\lambda}\,\vec{n}\cdot\partial_{x_1}\vec{w}\right)\right]\ dx_1\wedge dx_2 \quad .\end{aligned} \tag{III.15}$$

□

Recall

$$\frac{d}{dt}(dvol_{g_{\vec{\Phi}}})(0)=\left[\sum_{i=1}^2\partial_{x_i}\vec{\Phi}\cdot\partial_{x_i}\vec{w}\right]\ dx_1\wedge dx_2=\left\langle d\vec{\Phi}\,;d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \quad . \tag{III.16}$$

Hence

$$\begin{aligned}e^{2\lambda}\ \frac{dK_{g_{\vec{\Phi}_t}}}{dt} &= \partial_{x_1}\left(\mathbb{I}_{22}\ e^{-2\lambda}\,\vec{n}\cdot\partial_{x_1}\vec{w}\right)+\partial_{x_2}\left(\mathbb{I}_{11}\ e^{-2\lambda}\,\vec{n}\cdot\partial_{x_2}\vec{w}\right)\\ &-\partial_{x_1}\left(\mathbb{I}_{12}\ e^{-2\lambda}\,\vec{n}\cdot\partial_{x_2}\vec{w}\right)-\partial_{x_2}\left(\mathbb{I}_{12}\ e^{-2\lambda}\,\vec{n}\cdot\partial_{x_1}\vec{w}\right)-K_{g_{\vec{\Phi}}}\ \sum_{i=1}^2\partial_{x_i}\vec{\Phi}\cdot\partial_{x_i}\vec{w}\end{aligned} \tag{III.17}$$

Recall

$$\Delta_g f:=(det(g_{kl}))^{-1/2}\sum_{i,j=1}^2\partial_{x_i}\left((det(g_{kl}))^{1/2}\,g^{ij}\,\partial_{x_j}f\right) \tag{III.18}$$

and

$$\frac{dg_{ij}}{dt}(0)=\partial_{x_i}\vec{w}\cdot\partial_{x_j}\vec{\Phi}+\partial_{x_j}\vec{w}\cdot\partial_{x_i}\vec{\Phi} \quad . \tag{III.19}$$

Since $\sum_i g_{ki}\,g^{ij}=\delta_{kj}$ and $g_{ki}=e^{2\lambda}\,\delta_{ki}$, we have

$$\frac{dg^{ij}}{dt}(0)=-e^{-4\lambda}\ \left[\partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{w}+\partial_{x_j}\vec{\Phi}\cdot\partial_{x_i}\vec{w}\right] \quad . \tag{III.20}$$

Thus in particular

$$\begin{aligned}\frac{d}{dt}\,(det(g_{ij}))^{1/2} &= 2^{-1}\,(det(g_{ij}))^{-1/2}\ e^{2\lambda}\ \left[\frac{dg_{11}}{dt}+\frac{dg_{22}}{dt}\right]\\ &=\sum_{i=1}^2\partial_{x_i}\vec{\Phi}\cdot\partial_{x_i}\vec{w} \quad ,\end{aligned} \tag{III.21}$$

and

$$\frac{d}{dt}\left(det(g_{ij})\right)^{-1/2} = -e^{-4\lambda}\ \sum_{i=1}^{2} \partial_{x_i}\vec{\Phi}\cdot\partial_{x_i}\vec{w} \quad . \tag{III.22}$$

Combining (III.18)...(III.22) we obtain

Lemma III.2. *Under the previous notations, for any function f independent of t on Σ we have*[11]

$$\frac{d(\Delta_{g_{\vec{\Phi}_t}})f}{dt} = \left\langle d< d\vec{\Phi}; d\vec{w}>_{g_{\vec{\Phi}}}, df\right\rangle_{g_{\vec{\Phi}}} - *_{g_{\vec{\Phi}}} d* \left[[d\vec{\Phi}\dot{\otimes} d\vec{w} + d\vec{w}\dot{\otimes} d\vec{\Phi}]\llcorner_{g_{\vec{\Phi}}} df\right] \quad , \tag{III.23}$$

where we have explicitly in conformal coordinates

$$*_{g_{\vec{\Phi}}} d* \left[[d\vec{\Phi}\dot{\otimes} d\vec{w} + d\vec{w}\dot{\otimes} d\vec{\Phi}]\llcorner_{g_{\vec{\Phi}}} df\right] = e^{-2\lambda}\sum_{i,j=1}^{2}\partial_{x_i}\left(e^{-2\lambda}\left(\partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{w} + \partial_{x_j}\vec{\Phi}\cdot\partial_{x_i}\vec{w}\right)\ \partial_{x_j} f\right) \quad .$$

$\Box$

We have

$$\begin{aligned}\frac{d}{dt}\left[\int_\Sigma |d\alpha_t|^2_{g_{\vec{\Phi}_t}}\ dvol_{g_{\vec{\Phi}_t}}\right] &= \sum_{i,j=1}^{2}\int_\Sigma \frac{dg^{ij}}{dt}\,\partial_{x_i}\alpha\,\partial_{x_j}\alpha\ dvol_{g_{\vec{\Phi}}} + \int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}}\ \frac{d(dvol_{g_{\vec{\Phi}_t}})}{dt}\\ &\quad -2\ \int_\Sigma \alpha\ \Delta_{g_{\vec{\Phi}}}\frac{d\alpha_t}{dt}\ dvol_{g_{\vec{\Phi}}}\end{aligned} \tag{III.24}$$

We first have using (III.20)

$$\sum_{i,j=1}^{2}\int_\Sigma \frac{dg^{ij}}{dt}\,\partial_{x_i}\alpha\,\partial_{x_j}\alpha\ dvol_{g_{\vec{\Phi}}} = -2\ \int_\Sigma \left\langle d\vec{\Phi}, d\alpha\right\rangle_{g_{\vec{\Phi}}}\cdot\langle d\vec{w}, d\alpha\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \quad . \tag{III.25}$$

We then have using (III.16)

$$\int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}}\ \frac{d(dvol_{g_{\vec{\Phi}_t}})}{dt} = \int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}}\ \left\langle d\vec{\Phi}\,; d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \quad . \tag{III.26}$$

Using now (III.5) we have

$$\begin{aligned}-2\ \int_\Sigma \alpha\ \Delta_{g_{\vec{\Phi}}}\frac{d\alpha_t}{dt}\ dvol_{g_{\vec{\Phi}}} &= -2\ \int_\Sigma \alpha\ \Delta_{g_0}\frac{d\alpha_t}{dt}\ dvol_{g_0}\\ &= 4\ K_{g_0}\int_\Sigma \alpha\ \frac{d\alpha}{dt}\ dvol_{g_0} + 2\ \int_\Sigma \alpha\ \frac{d(\Delta_{g_{\vec{\Phi}_t}})\alpha}{dt}\ dvol_{g_{\vec{\Phi}}} + 2\int_\Sigma \alpha\ \frac{dK_{g_{\vec{\Phi}_t}}}{dt}\ dvol_{g_{\vec{\Phi}}} \quad .\end{aligned} \tag{III.27}$$

Using lemma III.2 we obtain

$$\begin{aligned}2\ \int_\Sigma \alpha\ \frac{d(\Delta_{g_{\vec{\Phi}_t}})\alpha}{dt}\ dvol_{g_{\vec{\Phi}}} &= -2\ \int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}}\ \left\langle d\vec{\Phi}\,; d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}}\\ -2\int_\Sigma \alpha\,\Delta_{g_{\vec{\Phi}}}\alpha\ \left\langle d\vec{\Phi}\,; d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} &+ 4\ \int_\Sigma \left\langle d\vec{\Phi}, d\alpha\right\rangle_{g_{\vec{\Phi}}}\cdot\langle d\vec{w}, d\alpha\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \quad .\end{aligned} \tag{III.28}$$

[11] The contraction operator $\llcorner_{g_{\vec{\Phi}}}$ between a quadratic form and a 1-form is defined as follows :

$$a\otimes b\llcorner_{g_{\vec{\Phi}}} c := < b, c>_{g_{\vec{\Phi}}}\ a \quad .$$

Using (III.17) we obtain

$$
\begin{aligned}
2\int_\Sigma \alpha\ \frac{dK_{g_{\vec{\Phi}_t}}}{dt}\ dvol_{g_{\vec{\Phi}}} &= -2\sum_{i=1}^2\int_\Sigma \mathbb{I}_{ii}\ e^{-2\lambda}\ \vec{n}\cdot\partial_{x_{i+1}}\vec{w}\ \partial_{x_{i+1}}\alpha\ dx^2\\
&+2\int_\Sigma \mathbb{I}_{12}\ e^{-2\lambda}\ \vec{n}\cdot[\partial_{x_1}\vec{w}\ \partial_{x_2}\alpha+\partial_{x_2}\vec{w}\ \partial_{x_1}\alpha]\,dx^2\\
&-2\int_\Sigma \alpha\ K_{g_{\vec{\Phi}}}\ \left\langle d\vec{\Phi}\,;d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}}\quad .
\end{aligned}
\tag{III.29}
$$

Combining (III.27), (III.28) and (III.29) gives, using that $\Delta_{g_{\vec{\Phi}}}\alpha+K_{g_{\vec{\Phi}}}=e^{-2\alpha}\,K_{g_0}$

$$
\begin{aligned}
-2\int_\Sigma \alpha\ \Delta_{g_{\vec{\Phi}}}\frac{d\alpha_t}{dt}\ dvol_{g_{\vec{\Phi}}} &= -\,2\int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}}\ \left\langle d\vec{\Phi}\,;d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}}\\
&+4\int_\Sigma \left\langle d\vec{\Phi},d\alpha\right\rangle_{g_{\vec{\Phi}}}\cdot\langle d\vec{w},d\alpha\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}}\\
&-2\sum_{i=1}^2\int_\Sigma \mathbb{I}_{ii}\ e^{-2\lambda}\ \vec{n}\cdot\partial_{x_{i+1}}\vec{w}\ \partial_{x_{i+1}}\alpha\ dx^2\\
&+2\int_\Sigma \mathbb{I}_{12}\ e^{-2\lambda}\ \vec{n}\cdot[\partial_{x_1}\vec{w}\ \partial_{x_2}\alpha+\partial_{x_2}\vec{w}\ \partial_{x_1}\alpha]\,dx^2\\
&-2\ K_{g_0}\int_\Sigma \alpha\ e^{-2\alpha}\ \left\langle d\vec{\Phi}\,;d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}}+4\ K_{g_0}\int_\Sigma \alpha\ \frac{d\alpha}{dt}\ dvol_{g_0}\quad .
\end{aligned}
\tag{III.30}
$$

Observe that

$$
\begin{aligned}
&-2\ K_{g_0}\int_\Sigma \alpha\ e^{-2\alpha}\ \left\langle d\vec{\Phi}\,;d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}}+4\ K_{g_0}\int_\Sigma \alpha\ \frac{d\alpha}{dt}\ dvol_{g_0}\\
&\quad=-2\,K_{g_0}\int_\Sigma \alpha\ \frac{d(dvol_{g_0})}{dt}
\end{aligned}
\tag{III.31}
$$

and since g_0 is normalized to have volume 1, we have

$$
\int_\Sigma \frac{d(dvol_{g_0})}{dt}=0\quad ,
\tag{III.32}
$$

which is consistent with the fact that the addition of constants to α (i.e. dilations of $\vec{\Phi}$) are zero directions of the lagrangian C. Combining now (III.24), (III.25), (III.26) and (III.30) we obtain the following

$$
\begin{aligned}
\frac{d}{dt}\left[\int_\Sigma |d\alpha_t|^2_{g_{\vec{\Phi}_t}}\ dvol_{g_{\vec{\Phi}_t}}\right] &= -\int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}}\ \left\langle d\vec{\Phi}\,;d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}}\\
&+2\int_\Sigma \left\langle d\vec{\Phi},d\alpha\right\rangle_{g_{\vec{\Phi}}}\cdot\langle d\vec{w},d\alpha\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}}+2\int_\Sigma\left(\vec{\mathbb{I}}\llcorner_{g_{\vec{\Phi}}}(*_{g_{\vec{\Phi}}}d\alpha)\right)\dot{\wedge}d\vec{w}\\
&-2\ K_{g_0}\int_\Sigma \alpha\ e^{-2\alpha}\ \left\langle d\vec{\Phi}\,;d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}}+4\ K_{g_0}\int_\Sigma \alpha\ \frac{d\alpha}{dt}\ dvol_{g_0}\quad ,
\end{aligned}
\tag{III.33}
$$

where we have explicitly in positive conformal coordinates

$$
\begin{aligned}
2\,\vec{\mathbb{I}}\llcorner_{g_{\vec{\Phi}}}(*_{g_{\vec{\Phi}}}d\alpha)\dot{\wedge}d\vec{w} &= -2\left[\sum_{i=1}^2\mathbb{I}_{ii}\ e^{-2\lambda}\ \vec{n}\cdot\partial_{x_{i+1}}\vec{w}\ \partial_{x_{i+1}}\alpha\right]\ dx_1\wedge dx_2\\
&+2\,\mathbb{I}_{12}\ e^{-2\lambda}\ \vec{n}\cdot[\partial_{x_1}\vec{w}\ \partial_{x_2}\alpha+\partial_{x_2}\vec{w}\ \partial_{x_1}\alpha]\,dx_1\wedge dx_2
\end{aligned}
\tag{III.34}
$$

where $\dot{wedge}$ is the combination of the exterior product in the domain and the scalar product of vectors in the target. Moreover $d\alpha/dt$ solves the following PDE

$$
\begin{aligned}
\Delta_{g_0}\frac{d\alpha}{dt}+2\,K_{g_0}\,\frac{d\alpha}{dt} &= -\,\left\langle d\left(< d\vec{\Phi}; d\vec{w} >_{g_{\vec{\Phi}}}\right), d\alpha\right\rangle_{g_0}\\
&\quad+*_{g_0}\ d\,*_{g_{\vec{\Phi}}}\left[[d\vec{\Phi}\dot{\otimes}d\vec{w}+d\vec{w}\dot{\otimes}d\vec{\Phi}]\llcorner_{g_{\vec{\Phi}}}\,d\alpha\right]\\
&\quad+*_{g_0}\ d\left[\mathbb{I}\llcorner_{g_{\vec{\Phi}}}(\vec{n}\cdot *_{g_{\vec{\Phi}}}d\vec{w})\right]+K_{g_{\vec{\Phi}}}\ \left\langle d\vec{\Phi}, d\vec{w}\right\rangle_{g_0}\quad .
\end{aligned}
\tag{III.35}
$$

Observe that

$$
\begin{aligned}
\frac{d}{dt}\left[-\frac{1}{4}\int_\Sigma\left[2\alpha_t\ e^{-2\alpha_t}+e^{-2\alpha_t}\right]\ dvol_{\vec{\Phi}_t}\right] &= \int_\Sigma \alpha\,\frac{d\alpha}{dt}\ dvol_{g_0}\\
-\frac{1}{4}\int_\Sigma\left[2\alpha\ e^{-2\alpha}+e^{-2\alpha}\right]\ &< d\vec{\Phi}; d\vec{w} >_{g_{\vec{\Phi}}}\ \ dvol_{g_{\vec{\Phi}}}\quad .
\end{aligned}
\tag{III.36}
$$

Hence

$$
\begin{aligned}
&4\,K_{g_0}\int_\Sigma\alpha\,\frac{d\alpha}{dt}\ dvol_{g_0}-2\,K_{g_0}\int_\Sigma\alpha\ e^{-2\alpha}\ \ < d\vec{\Phi}; d\vec{w} >_{g_{\vec{\Phi}}}\ \ dvol_{g_{\vec{\Phi}}}\\
&=\frac{d}{dt}\left[-\,K_{g_0}\int_\Sigma\left[2\,\alpha_t\ e^{-2\alpha_t}+e^{-2\alpha_t}\right]\ dvol_{\vec{\Phi}_t}\right]\\
&+K_{g_0}\int_\Sigma\alpha\ e^{-2\alpha}\ \ < d\vec{\Phi}; d\vec{w} >_{g_{\vec{\Phi}}}\ \ dvol_{g_{\vec{\Phi}}}\quad .
\end{aligned}
\tag{III.37}
$$

Observe that

$$
\int_\Sigma e^{-\alpha_t}\ dvol_{\vec{\Phi}_t}=\int_\Sigma dvol_{g_{0,t}}\equiv 1\quad .
\tag{III.38}
$$

Combining (III.33), (III.37) and (III.38) we obtain

Lemma III.3. *Under the previous notations we have*

$$
\begin{aligned}
&\frac{d}{dt}\left[\int_\Sigma\left[|d\alpha_t|^2_{g_{\vec{\Phi}_t}}+2\,K_{g_0}\,\alpha_t\ e^{-2\alpha_t}\right]\ dvol_{g_{\vec{\Phi}_t}}\right]\\
&=\int_\Sigma\left(|d\alpha|^2_{g_{\vec{\Phi}}}\ \ *_{g_{\vec{\Phi}}}\,d\vec{\Phi}\right)\dot{\wedge}d\vec{w}-\,2\int_\Sigma\left(\left\langle d\vec{\Phi}, d\alpha\right\rangle_{g_{\vec{\Phi}}}\ \ *_{g_{\vec{\Phi}}}\,d\alpha\right)\dot{\wedge}d\vec{w}\\
&+2\int_\Sigma\left(\vec{\mathbb{I}}\llcorner_{g_{\vec{\Phi}}}(*_{g_{\vec{\Phi}}}d\alpha)\right)\dot{\wedge}d\vec{w}-K_{g_0}\int_\Sigma\left(\alpha\ e^{-2\alpha}\ \ *_{g_{\vec{\Phi}}}\,d\vec{\Phi}\right)\dot{\wedge}d\vec{w}\quad .
\end{aligned}
\tag{III.39}
$$

□

This lemma justifies the introduction of the following modified *frame energy.*

$$
\begin{aligned}
\tilde{F}_f(\vec{\Phi}) &:= \int_\Sigma[f(H_{g_{\vec{\Phi}}})+2^{-3}\,|A_{g_{\vec{\Phi}}}|^2_{g_{\vec{\Phi}}}+K_{g_0}\,\alpha\ e^{-2\alpha}]\ dvol_{g_{\vec{\Phi}}}\\
&=\int_\Sigma[f(H_{g_{\vec{\Phi}}})+2^{-1}\,|d\alpha|^2_{g_{\vec{\Phi}}}+K_{g_0}\,\alpha\ e^{-2\alpha}]\ dvol_{g_{\vec{\Phi}}}
\end{aligned}
$$

Lemma III.4. *Let $\vec{\Phi}$ be an immersion of the sphere S^2 and g_0 be a metric of constant curvature equal to 4π and volume 1 such that there exists a function α satisfying*

$$
g_{\vec{\Phi}}=e^{2\alpha}\,g_0
$$

then the Polyakov-Alvarez Lagrangian

$$L(\vec{\Phi},g_0) := \int_{S^2} 2^{-1}|d\alpha|^2_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} + 4\pi\ \int_{S^2}\alpha\ dvol_{g_0}$$

is independent of the choice of g_0 and in this sense is gauge invariant for the gauge group given by the Möbius group of positive conformal transformations $\mathcal{M}^+(S^2)$. □

Proof of lemma III.4. Let $\alpha(t)$ and $g_0(t)$ be smooth functions such that

$$g_{\vec{\Phi}} = e^{2\alpha(t)}\, g_0(t) \quad .$$

We have

$$\begin{aligned}\frac{d}{dt}L(\vec{\Phi},g_0(t)) &=: -\int_{S^2}\Delta_{g_{\vec{\Phi}}}\alpha(t)\ \frac{d\alpha}{dt}\ dvol_{g_{\vec{\Phi}}} + 4\pi\ \int_{S^2}\frac{d\alpha}{dt}\ [e^{-\,2\,\alpha}-2\,\alpha\ e^{-\,2\,\alpha}]\ dvol_{g_{\vec{\Phi}}}\\ &= -\int_{S^2}\left[\Delta_{g_0(t)}\frac{d\alpha}{dt}+8\pi\ \frac{d\alpha}{dt}\right]\ \alpha(t)\ dvol_{g_0(t)} + 4\pi\ \int_{S^2}\frac{d\alpha}{dt}\ dvol_{g_0(t)} \quad .\end{aligned}$$

Since $\int_{S^2} dvol_{g_0(t)}\equiv 1$ we have in one hand

$$0 = \frac{d}{dt}\int_{S^2} e^{-\,2\,\alpha(t)}\ dvol_{g_{\vec{\Phi}}} = -\,2\ \int_{S^2}\frac{d\alpha}{dt}\ dvol_{g_0(t)} \quad .$$

The Liouville equation gives in the other hand

$$0=\Delta_{g_{\vec{\Phi}}}\alpha(t) +\ K_{g_{\vec{\Phi}}} -\ 4\,\pi\, e^{-2\,\alpha(t)} \quad .$$

Taking the derivative gives

$$0=\Delta_{g_{\vec{\Phi}}}\frac{d\alpha}{dt} +\ 8\pi\, e^{-2\,\alpha(t)}\ \frac{d\alpha}{dt} = e^{-2\,\alpha(t)}\ \left[\Delta_{g_0(t)}\frac{d\alpha}{dt}+8\pi\ \frac{d\alpha}{dt}\right]$$

all the previous says that $L(\vec{\Phi},g_0(t))$ is independent of t and lemma III.4 is proved. □

We have the following definition

Definition III.2. *Let $\vec{\Phi}$ be a weak immersion in $\mathcal{E}_{S^2,1}$ out of all α such that there exists g_0 of constant curvature equal to 4π and volume 1 such that*

$$g_{\vec{\Phi}} = e^{2\alpha}\, g_0 \quad , \tag{III.40}$$

we call a Aubin Gauge a choice of α and $\Psi\in$Diff(S^2) such that

$$\Psi^*g_0 = \frac{g_{S^2}}{4\pi} \qquad and \qquad \forall\ j\in\{1,2,3\} \qquad \int_{S^2} x_j\ e^{2\,\alpha\circ\Psi(x)}\ dvol_{S^2}=0 \quad , \tag{III.41}$$

where g_{S^2} is the standard metric on S^2. □

We have the following theorem by E.Onofri.

Theorem III.1. *[37] For any weak immersion $\vec{\Phi}$ of S^2 and any α satisfying (III.40) then the following inequality holds*

$$\int_{S^2} 2^{-1}|d\alpha|^2_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} + 4\pi\ \int_{S^2}\alpha\ dvol_{g_0} \ge 2\pi\log\int_{S^2} e^{2\alpha}\ dvol_{g_0} \quad . \tag{III.42}$$

Moreover for any $\vec{\Phi}$ there exists a Aubin Gauge (Ψ,α) satisfying (III.41). □

We are going to use the following result proved by N.Ghoussoub and C.S.Lin.

Theorem III.2. *[19] For any weak immersion $\vec{\Phi}$ of S^2 and any α satisfying (III.40) and (III.41) then the following inequality holds*

$$\int_{S^2} 3^{-1}|d\alpha|^2_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} + 4\pi \int_{S^2} \alpha\ dvol_{g_0} \geq 2\pi \log \int_{S^2} e^{2\alpha}\ dvol_{g_0} \quad . \tag{III.43}$$

□

It is suggested by A.Chang and P.Yang ([13] section 3) that the constant 3^{-1} could be replaced by 4^{-1} in (III.43).

III.2 The variation of the mean curvature

We have

$$\frac{d}{dt}H = \frac{1}{2}\sum_{i,j} \frac{dg^{ij}}{dt}\,\mathbb{I}_{ij} + \frac{1}{2}\sum_{i,j} g^{ij}\,\frac{d\mathbb{I}_{ij}}{dt} \quad .$$

We have

$$\frac{dg_{ij}}{dt} = \frac{d}{dt}\left(\partial_{x_i}\vec{\Phi}_t \cdot \partial_{x_j}\vec{\Phi}_t\right) = \partial_{x_i}\vec{w}\cdot\partial_{x_j}\vec{\Phi} + \partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{w} \quad .$$

Since

$$e^{-2\lambda}\,\frac{dg_{ik}}{dt}\delta_{kj} + e^{2\lambda}\,\frac{dg^{kj}}{dt}\delta_{ik} = 0 \quad ,$$

we have

$$\frac{dg^{ij}}{dt} = -e^{-4\lambda}\,\left[\partial_{x_i}\vec{w}\cdot\partial_{x_j}\vec{\Phi} + \partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{w}\right] \quad .$$

So

$$\frac{1}{2}\sum_{i,j}\frac{dg^{ij}}{dt}\,\mathbb{I}_{ij} = e^{-4\lambda}\,\frac{1}{2}\sum_{i,j}\left[\partial_{x_i}\vec{w}\cdot\partial_{x_j}\vec{\Phi} + \partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{w}\right]\,\partial_{x_i}\vec{n}\cdot\partial_{x_j}\vec{\Phi} = e^{-2\lambda}\nabla\vec{w}\cdot\nabla\vec{\Phi} \quad .$$

We have moreover

$$\frac{d\mathbb{I}_{ij}}{dt} = -\frac{d}{dt}\left(\partial_{x_i}\vec{n}_t\cdot\partial_{x_j}\vec{\Phi}_t\right) = -\partial_{x_i}\frac{d\vec{n}_t}{dt}\cdot\partial_{x_j}\vec{\Phi} - \partial_{x_i}\vec{n}\cdot\partial_{x_j}\vec{w} \quad .$$

So combining the previous assertions we have

$$\frac{dH}{dt} = -\frac{e^{-2\lambda}}{2}\nabla\vec{n}\cdot\nabla\vec{w} - \frac{e^{-2\lambda}}{2}\nabla\frac{d\vec{n}_t}{dt}\cdot\nabla\vec{\Phi} + e^{-2\lambda}\nabla\vec{n}\cdot\nabla\vec{\Phi} \quad .$$

Since

$$\frac{d\vec{n}}{dt}\cdot\Delta\vec{\Phi} = 0 \quad ,$$

we obtain

$$\frac{dH}{dt} = \frac{e^{-2\lambda}}{2}\nabla\vec{n}\cdot\nabla\vec{w} - \frac{e^{-2\lambda}}{2}\mathrm{div}\left(\frac{d\vec{n}_t}{dt}\cdot\nabla\vec{\Phi}\right) \quad .$$

One has

$$\frac{d\vec{n}}{dt} = -\ e^{-2\lambda}\ \vec{n}\cdot\nabla\vec{w}\ \nabla\vec{\Phi} \quad .$$

Hence

$$\mathrm{div}\left(\frac{d\vec{n}_t}{dt}\cdot\nabla\vec{\Phi}\right) = -\mathrm{div}\left(\vec{n}\cdot\nabla\vec{w}\right) \quad .$$

Combining the previous gives

$$\begin{aligned}\frac{dH}{dt} &= \frac{e^{-2\lambda}}{2}\ \left[\mathrm{div}\,(\vec{n}\cdot\nabla\vec{w}) + \nabla\vec{n}\cdot\nabla\vec{w}\right]\\ &= -\,2^{-1} d^{*_{g_{\vec{\Phi}}}}\ (\vec{n}\cdot d\vec{w}) + 2^{-1}\ \langle d\vec{w}; d\vec{n}\rangle_{g_{\vec{\Phi}}} \quad .\end{aligned} \tag{III.44}$$

We then have for any C^1 function f

$$\begin{aligned}\frac{d}{dt}\left[\int_\Sigma f(H_{\vec{\Phi}_t})\ dvol_{\vec{\Phi}_t}\right] &= -\,2^{-1}\int_\Sigma f'(H)\ d^{*_{g_{\vec{\Phi}}}}\ (\vec{n}\cdot d\vec{w})\ dvol_{g_{\vec{\Phi}}}\\ &+ 2^{-1}\int_\Sigma f'(H)\ \langle d\vec{w}; d\vec{n}\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} + \int_\Sigma f(H)\ \left\langle d\vec{w}; d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \quad .\end{aligned} \tag{III.45}$$

We can then deduce the following result.

Lemma III.5. *Let f be a C^1 function, under the previous notations we have*

$$\begin{aligned}\frac{d}{dt}\left[\int_\Sigma f(H_{g_{\vec{\Phi}_t}})\ dvol_{g_{\vec{\Phi}_t}}\right] &= 2^{-1}\int_\Sigma \left(*_{g_{\vec{\Phi}}} d[f'(H)\,\vec{n}]\right)\dot{\wedge} d\vec{w}\\ &- \int_\Sigma \left(f'(H)\ *_{g_{\vec{\Phi}}}\ d\vec{n}\right)\dot{\wedge} d\vec{w} - \int_\Sigma \left(f(H)\ \ *_{g_{\vec{\Phi}}}\ d\vec{\Phi}\right)\dot{\wedge} d\vec{w} \quad .\end{aligned} \tag{III.46}$$

□

Observe moreover that from (III.21) we can also deduce the following elementary lemma.

Lemma III.6. *Under the previous notations we have*

$$\begin{aligned}\frac{d}{dt}\log\left[\int_\Sigma e^{2\alpha}\ dvol_{g_0}\right] &= [A_{\vec{\Phi}}(\Sigma)]^{-1}\int_\Sigma < d\vec{\Phi}, d\vec{w} >_{g_{\vec{\Phi}}}\ \ dvol_{g_{\vec{\Phi}}}\\ &= -[A_{\vec{\Phi}}(\Sigma)]^{-1}\int_\Sigma \left[*_{g_{\vec{\Phi}}} d\vec{\Phi}\right]\dot{\wedge} d\vec{w} \quad ,\end{aligned} \tag{III.47}$$

where $A_{\vec{\Phi}}(\Sigma) := \int_\Sigma e^{2\alpha}\ dvol_{g_0} = \int_\Sigma dvol_{g_{\vec{\Phi}}}$ is the area of the immersion $\vec{\Phi}$. □

III.3 The first variation of the Frame energies and conservation laws

Combining lemma III.3 and lemma III.5 we obtain

Lemma III.7. *Let Σ be a closed oriented two dimensional manifold. Let f be a C^1 function on $\mathbb{R}$, let $\vec{\Phi}$ be an immersion into $\mathbb{R}^3$, let $g_{\vec{\Phi}}$ be the induced metric on Σ. Let g_0 be a constant Gauss curvature metric*[12] *of volume 1 on Σ such that there exists α with $g_{\vec{\Phi}} = e^{2\alpha}\, g_0$. The immersion is a critical point of*

$$\tilde{F}_f^\Lambda(\vec{\Phi}) := \int_\Sigma \left[f(H) +\ 2^{-1}\,|d\alpha|^2_{g_{\vec{\Phi}}} + K_{g_0}\ \alpha\ e^{-2\alpha}\right]\ dvol_{g_{\vec{\Phi}}} - \Lambda\ \log\left(\int_\Sigma e^{2\alpha}\ dvol_{g_0}\right) \tag{III.48}$$

[12]Observe that g_0 is unique if Σ has non zero genus. When $\Sigma \simeq S^2$ it is unique modulo the action of the Möbius group $\mathcal{M}^+(S^2)$. Nevertheless, because of lemma III.4, the Lagrangian, and hence the Euler Lagrange Equation is insensitive to the gauge action.

if and only if the following conservation law (i.e. closeness of a one form) holds

$$\begin{aligned} d\Big[*_{g_{\vec{\Phi}}} d[f'(H)\,\vec{n}] - 2\,f'(H)\ *_{g_{\vec{\Phi}}} d\vec{n} \\ + \Big[-2\,f(H) + |d\alpha|^2_{g_{\vec{\Phi}}} - K_{g_0}\,\alpha\,e^{-2\alpha} + 2\,\Lambda\,[A_{\vec{\Phi}}(\Sigma)]^{-1}\Big]\ *_{g_{\vec{\Phi}}} d\vec{\Phi} \\ -2\,\left\langle d\vec{\Phi}, d\alpha\right\rangle_{g_{\vec{\Phi}}}\ *_{g_{\vec{\Phi}}} d\alpha + 2\,\vec{\mathbb{I}}\llcorner_{g_{\vec{\Phi}}}(*_{g_{\vec{\Phi}}} d\alpha)\Big] = 0 \quad . \end{aligned} \tag{III.49}$$

□

Assume $\vec{\Phi}$ is a critical point of $\tilde{F}^{\Lambda}_f$ given by (III.48) and denote locally

$$\begin{aligned} d\vec{L} := *_{g_{\vec{\Phi}}} d[f'(H)\,\vec{n}] - 2\,f'(H)\ *_{g_{\vec{\Phi}}} d\vec{n} \\ + \Big[-2\,f(H) + |d\alpha|^2_{g_{\vec{\Phi}}} - K_{g_0}\,\alpha\,e^{-2\alpha} + 2\,\Lambda\,[A_{\vec{\Phi}}(S^2)]^{-1}\Big]\ *_{g_{\vec{\Phi}}} d\vec{\Phi} \\ -2\,\left\langle d\vec{\Phi}, d\alpha\right\rangle_{g_{\vec{\Phi}}}\ *_{g_{\vec{\Phi}}} d\alpha + 2\,\vec{\mathbb{I}}\llcorner_{g_{\vec{\Phi}}}(*_{g_{\vec{\Phi}}} d\alpha) \quad . \end{aligned} \tag{III.50}$$

In conformal coordinates this gives

$$\begin{aligned} \partial_{x_1}\vec{L} = -\partial_{x_2} f'(H)\,\vec{n} + f'(H)\,\partial_{x_2}\vec{n} \\ -[-2\,f(H) + |d\alpha|^2_{g_{\vec{\Phi}}} - K_{g_0}\ \alpha\ e^{-2\alpha} + 2\,\Lambda\,[A_{\vec{\Phi}}(S^2)]^{-1}]\ \partial_{x_2}\vec{\Phi} \\ +2\ \left\langle d\vec{\Phi}, d\alpha\right\rangle_{g_{\vec{\Phi}}}\ \partial_{x_2}\alpha + 2\,e^{-2\lambda}\sum_{i=1}^2 \vec{\mathbb{I}}_{1i}(*_{g_{\vec{\Phi}}} d\alpha)_i \quad , \end{aligned} \tag{III.51}$$

and

$$\begin{aligned} \partial_{x_2}\vec{L} = \partial_{x_1} f'(H)\,\vec{n} - f'(H)\,\partial_{x_1}\vec{n} \\ +[-2\,f(H) + |d\alpha|^2_{g_{\vec{\Phi}}} - K_{g_0}\ \alpha\ e^{-2\alpha} + 2\,\Lambda\,[A_{\vec{\Phi}}(S^2)]^{-1}]\ \partial_{x_1}\vec{\Phi} \\ -2\ \left\langle d\vec{\Phi}, d\alpha\right\rangle_{g_{\vec{\Phi}}}\ \partial_{x_1}\alpha + 2\,e^{-2\lambda}\sum_{i=1}^2 \vec{\mathbb{I}}_{2i}(*_{g_{\vec{\Phi}}} d\alpha)_i \quad . \end{aligned} \tag{III.52}$$

We have

$$\begin{aligned} d\vec{\Phi}\dot{\wedge} d\vec{L} := \Big[\partial_{x_1}\vec{\Phi}\cdot\partial_{x_2}\vec{L} - \partial_{x_2}\vec{\Phi}\cdot\partial_{x_1}\vec{L}\Big]\ dx_1\wedge dx_2 \\ = 2\,(f'(H)\,H - 2\,f(H) - K_{g_0}\ \alpha\ e^{-2\alpha} + 2\,\Lambda\,[A_{\vec{\Phi}}(S^2)]^{-1})\ dvol_{g_{\vec{\Phi}}} \quad . \end{aligned} \tag{III.53}$$

We have also

$$
\begin{aligned}
d\vec{\Phi}\wedge d\vec{L} &:= \left[\partial_{x_1}\vec{\Phi}\times\partial_{x_2}\vec{L}-\partial_{x_2}\vec{\Phi}\times\partial_{x_1}\vec{L}\right]\ dx_1\wedge dx_2\\
&= \partial_{x_1}\vec{\Phi}\times\left[\partial_{x_1}f'(H)\,\vec{n}-f'(H)\,\partial_{x_1}\vec{n}-\ 2\ \left\langle d\vec{\Phi},d\alpha\right\rangle_{g_{\vec{\Phi}}}\ \partial_{x_1}\alpha\right]\ dx_1\wedge dx_2\\
&+2\,e^{-2\lambda}\ \sum_{i=1}^{2}\partial_{x_1}\vec{\Phi}\times\vec{\mathbb{I}}_{2i}(*_{g_{\vec{\Phi}}}d\alpha)_i\ dx_1\wedge dx_2\\
&+\partial_{x_2}\vec{\Phi}\times\left[\partial_{x_2}f'(H)\,\vec{n}-f'(H)\,\partial_{x_2}\vec{n}-\ 2\ \left\langle d\vec{\Phi},d\alpha\right\rangle_{g_{\vec{\Phi}}}\ \partial_{x_2}\alpha\right]\ dx_1\wedge dx_2\\
&-2\,e^{-2\lambda}\ \sum_{i=1}^{2}\partial_{x_2}\vec{\Phi}\times\vec{\mathbb{I}}_{1i}(*_{g_{\vec{\Phi}}}d\alpha)_i\ dx_1\wedge dx_2 \quad .
\end{aligned}
\tag{III.54}
$$

This gives

$$
\begin{aligned}
d\vec{\Phi}\wedge d\vec{L} &:= [\partial_{x_1}\vec{\Phi}\,\partial_{x_2}f'(H)-\partial_{x_2}\vec{\Phi}\,\partial_{x_1}f'(H)]\ dx_1\wedge dx_2\\
&-2\,e^{-2\lambda}\ \sum_{i,j=1}^{2}\mathbb{I}_{ij}(*_{g_{\vec{\Phi}}}d\alpha)_j\ \partial_{x_i}\vec{\Phi}\ dx_1\wedge dx_2 \quad .
\end{aligned}
\tag{III.55}
$$

We have

$$
\begin{aligned}
-2\,e^{-2\lambda}\ \sum_{i,j=1}^{2}\mathbb{I}_{ij}(*_{g_{\vec{\Phi}}}d\alpha)_j\ \partial_{x_i}\vec{\Phi}\ &= 2\,\partial_{x_1}\alpha\ \left[e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_1}\vec{\Phi}+e^{-2\lambda}\,\mathbb{I}_{22}\,\partial_{x_2}\vec{\Phi}\right]\\
&-2\,\partial_{x_2}\alpha\ \left[e^{-2\lambda}\,\mathbb{I}_{11}\,\partial_{x_1}\vec{\Phi}+e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_2}\vec{\Phi}\right] \quad .
\end{aligned}
\tag{III.56}
$$

We compute, using Codazzi identity (III.14)

$$
\begin{aligned}
&e^{2\lambda}\,\partial_{x_1}\left[e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_1}\vec{\Phi}+e^{-2\lambda}\,\mathbb{I}_{22}\,\partial_{x_2}\vec{\Phi}\right]-\ e^{2\lambda}\,\partial_{x_2}\left[e^{-2\lambda}\,\mathbb{I}_{11}\,\partial_{x_1}\vec{\Phi}+e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_2}\vec{\Phi}\right]\\
&=\partial_{x_1}\vec{\Phi}\ \left[-2\,\partial_{x_1}\lambda\ \mathbb{I}_{12}+\partial_{x_1}\mathbb{I}_{12}+e^{-2\lambda}\,\mathbb{I}_{12}\,\partial^2_{x_1^2}\vec{\Phi}\cdot\partial_{x_1}\vec{\Phi}+e^{-2\lambda}\,\mathbb{I}_{22}\,\partial^2_{x_1x_2}\vec{\Phi}\cdot\partial_{x_1}\vec{\Phi}\right]\\
&+\partial_{x_2}\vec{\Phi}\ \left[-2\,\partial_{x_1}\lambda\ \mathbb{I}_{22}+\partial_{x_1}\mathbb{I}_{22}+e^{-2\lambda}\,\mathbb{I}_{22}\,\partial^2_{x_1x_2}\vec{\Phi}\cdot\partial_{x_2}\vec{\Phi}+e^{-2\lambda}\,\mathbb{I}_{12}\,\partial^2_{x_1^2}\vec{\Phi}\cdot\partial_{x_2}\vec{\Phi}\right]\\
&-\partial_{x_1}\vec{\Phi}\ \left[-2\,\partial_{x_2}\lambda\ \mathbb{I}_{11}+\partial_{x_2}\mathbb{I}_{11}+e^{-2\lambda}\,\mathbb{I}_{11}\,\partial^2_{x_1x_2}\vec{\Phi}\cdot\partial_{x_1}\vec{\Phi}+e^{-2\lambda}\,\mathbb{I}_{12}\,\partial^2_{x_2^2}\vec{\Phi}\cdot\partial_{x_1}\vec{\Phi}\right]\\
&-\partial_{x_2}\vec{\Phi}\ \left[-2\,\partial_{x_2}\lambda\ \mathbb{I}_{12}+\partial_{x_2}\mathbb{I}_{12}+e^{-2\lambda}\,\mathbb{I}_{12}\,\partial^2_{x_2^2}\vec{\Phi}\cdot\partial_{x_2}\vec{\Phi}+e^{-2\lambda}\,\mathbb{I}_{11}\,\partial^2_{x_1x_2}\vec{\Phi}\cdot\partial_{x_2}\vec{\Phi}\right] \quad .
\end{aligned}
\tag{III.57}
$$

Making use of Codazzi identity (III.14) we finally obtain

$$
\begin{aligned}
&e^{2\lambda}\,\partial_{x_1}\left[e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_1}\vec{\Phi}+e^{-2\lambda}\,\mathbb{I}_{22}\,\partial_{x_2}\vec{\Phi}\right]-e^{2\lambda}\,\partial_{x_2}\left[e^{-2\lambda}\,\mathbb{I}_{11}\,\partial_{x_1}\vec{\Phi}+e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_2}\vec{\Phi}\right]\\
&=\partial_{x_1}\vec{\Phi}\ \left[-H\,\partial_{x_2}e^{2\lambda}+\partial_{x_2}\lambda\ [\mathbb{I}_{22}+\mathbb{I}_{11}]\right]+\partial_{x_2}\vec{\Phi}\ \left[H\,\partial_{x_1}e^{2\lambda}-\partial_{x_1}\lambda\ [\mathbb{I}_{22}+\mathbb{I}_{11}]\right]\\
&=0 \quad .
\end{aligned}
\tag{III.58}
$$

Hence there exists locally $\vec{D}$ such that (see also lemma III.2 in [35])

$$\left\{\begin{array}{l}\partial_{x_1}\vec{D} := \left[e^{-2\lambda}\,\mathbb{I}_{11}\,\partial_{x_1}\vec{\Phi} + e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_2}\vec{\Phi}\right]\\[2mm] \partial_{x_2}\vec{D} := \left[e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_1}\vec{\Phi} + e^{-2\lambda}\,\mathbb{I}_{22}\,\partial_{x_2}\vec{\Phi}\right] \quad .\end{array}\right. \tag{III.59}$$

Combining all the previous we obtain the following lemma

Lemma III.8. *Let Σ be a closed two dimensional manifold. Let f be a C^1 function on $\mathbb{R}$, let $\vec{\Phi}$ be an immersion into $\mathbb{R}^3$, let $g_{\vec{\Phi}}$ be the induced metric on Σ. Let g_0 be a constant Gauss curvature metric of volume 1 on Σ such that there exists α with $g_{\vec{\Phi}} = e^{2\alpha}\,g_0$. Assume the immersion $\vec{\Phi}$ is a critical point of*

$$\tilde{F}_f^{\Lambda}(\vec{\Phi}) := \int_\Sigma \left[f(H) + 2^{-1}\,|d\alpha|^2_{g_{\vec{\Phi}}} + K_{g_0}\ \alpha\ e^{-2\alpha}\right]\ dvol_{g_{\vec{\Phi}}} - \Lambda\,\log\left(\int_\Sigma e^{2\alpha}\ dvol_{g_0}\right) \quad , \tag{III.60}$$

and following lemma III.7 introduce locally $\vec{L}$

$$\begin{array}{l} d\vec{L} := *_{g_{\vec{\Phi}}} d[f'(H)\,\vec{n}] - 2\,f'(H)\ *_{g_{\vec{\Phi}}} d\vec{n}\\[2mm] \quad + \left[-2\,f(H) + |d\alpha|^2_{g_{\vec{\Phi}}} - K_{g_0}\,\alpha\,e^{-2\alpha} + 2\,\Lambda\,[A_{\vec{\Phi}}(S^2)]^{-1}\right]\ *_{g_{\vec{\Phi}}} d\vec{\Phi}\\[2mm] \quad - 2\left\langle d\vec{\Phi}, d\alpha\right\rangle_{g_{\vec{\Phi}}}\ *_{g_{\vec{\Phi}}} d\alpha + 2\,\vec{\mathbb{I}}\llcorner_{g_{\vec{\Phi}}}(*_{g_{\vec{\Phi}}} d\alpha) \quad , \end{array} \tag{III.61}$$

then the following almost conservation law holds

$$d\vec{\Phi}\dot{\wedge}d\vec{L} = 2\,(f'(H)\,H - 2\,f(H) - K_{g_0}\ \alpha\ e^{-2\alpha} + 2\,\Lambda\,[A_{\vec{\Phi}}(S^2)]^{-1})\ dvol_{g_{\vec{\Phi}}} \tag{III.62}$$

and the following exact conservation law holds

$$d\vec{\Phi}\wedge d\vec{L} = d\vec{\Phi}\wedge df'(H) + 2\,d\alpha\wedge d\vec{D} \quad , \tag{III.63}$$

where $\vec{D}$ satisfies

$$d\vec{D} = \mathbb{I}\llcorner_g\, d\vec{\Phi} \quad . \tag{III.64}$$

□

Remark III.1. *The three conservation laws or almost conservation laws (III.61), (III.62) and (III.63) can be deduced from Noether theorem (see [5]). Precisely the existence of $\vec{L}$ satisfying the first conservation law (III.61) is due to the translation invariance of the Lagrangian $\tilde{F}_f^{\Lambda}$, (III.62) instead is related to the lack of invariance of the lagrangian under dilation whereas (III.63) is related to the rotation invariance of the Lagrangian.* □

III.4 Various Bounds involving the Frame Energies.

First of all we establish the following lemma.

Lemma III.9. *Under the previous notations we have for any $\sigma > 0$*

$$\left|\frac{\log\left(\displaystyle\int_{S^2} e^{2\alpha}\ dvol_{g_0}\right)}{\log\left(\dfrac{1}{\sigma}\right)}\right| \le 2 + \frac{\log\left(\sigma^2\displaystyle\int_{S^2}(1+H^2)^2\ dvol_{g_{\vec{\Phi}}}\right)}{\log\left(\dfrac{1}{\sigma}\right)} \tag{III.65}$$

Proof of lemma III.9. We have obviously

$$\log\left(\int_{S^2} e^{2\alpha}\ dvol_{g_0}\right) \le \log\left(\sigma^2\int_{S^2}(1+H^2)^2\ dvol_{g_{\vec{\Phi}}}\right) + 2\ \log\left(\frac{1}{\sigma}\right) \tag{III.66}$$

We have also

$$16\,\pi^2 \le \left(\int_{S^2} H^2_{\vec{\Phi}}\ dvol_{g_{\vec{\Phi}}}\right)^2 \le \int_{S^2} e^{2\alpha}\ dvol_{g_0}\ \int_{S^2} H^4_{\vec{\Phi}}\ dvol_{g_{\vec{\Phi}}} \tag{III.67}$$

Hence

$$2\ \log\left(\frac{1}{\sigma}\right) + \log\left(\sigma^2\int_{S^2}(1+H^2)^2\ dvol_{g_{\vec{\Phi}}}\right) \ge -\log\left(\int_{S^2} e^{2\alpha}\ dvol_{g_0}\right) + \log 16\pi^2 \tag{III.68}$$

Combining (III.66) and (III.68) gives (III.65) and lemma III.9 is proved. □

A useful lemma is the following which is a direct consequence of theorem III.2

Lemma III.10. *Let $\vec{\Phi}$ be a weak immersion of S^2 and (α,Ψ) be an Aubin Gauge satisfying (III.41) then the following inequality holds*

$$\begin{aligned} &6^{-1}\int_{S^2}|d\alpha|^2_{g_0}\,dvol_{g_0}\\ &\le \int_{S^2} 2^{-1}|d\alpha|^2_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} + 4\pi\ \int_{S^2}\alpha\ dvol_{g_0} - 2\pi\log\int_{S^2} e^{2\alpha}\ dvol_{g_0} \end{aligned} \tag{III.69}$$

□

For Σ being an arbitrary closed surface we denote

$$\begin{aligned} F^\sigma(\vec{\Phi}) &:= \left(\log\frac{1}{\sigma}\right)^{-1}\ \tilde{F}^{K_{g_0}}_{f_\sigma}(\vec{\Phi})\\ &:= \left(\log\frac{1}{\sigma}\right)^{-1}\ \int_\Sigma\left[f_\sigma(H_{g_{\vec{\Phi}}}) +\ 2^{-1}\,|d\alpha|^2_{g_{\vec{\Phi}}} + K_{g_0}\ \alpha\ e^{-2\alpha}\right]\ dvol_{g_{\vec{\Phi}}}\\ &-2^{-1}\,K_{g_0}\ \left(\log\frac{1}{\sigma}\right)^{-1}\ \log\left(\int_\Sigma e^{2\alpha}\ dvol_{g_0}\right) \end{aligned} \tag{III.70}$$

where $f_\sigma(t) = \log\sigma^{-1}[t^2+\sigma^2(1+t^2)^2]$. First, in the particular case $\Sigma = S^2$ we have

Lemma III.11. *Let $\vec{\Phi}$ be a weak immersion of S^2 in $\mathcal{E}_{S^2,2}$ and g_0 be a constant Gauss curvature metric on S^2 of volume 1 such that*

$$g_{\vec{\Phi}} = e^{2\alpha}\,g_0$$

We have for $\sigma\in(0,1)$

$$W(\vec{\Phi}) + \sigma^2\int_{S^2}(1+H^2)^2\ dvol_{g_{\vec{\Phi}}} \le F^\sigma(\vec{\Phi}) \tag{III.71}$$

and for any Aubin gauge we have

$$\left(\log\frac{1}{\sigma}\right)^{-1}\ \int_{S^2}|d\alpha|^2_{g_{\vec{\Phi}}} \le 6\ \left[F^\sigma(\vec{\Phi}) - W(\vec{\Phi})\right] \tag{III.72}$$

We have also for any gauge

$$\inf_{x\in S^2}\alpha\ge\frac{1}{2}\log A_{\vec{\Phi}}(S^2)-\frac{1}{2\pi}\int_{S^2}|d\alpha|^2_{g_0}\,dvol_{g_0}\ \ -C\ \left[1+W(\vec{\Phi})\right]\tag{III.73}$$

and

$$\left(\log\frac{1}{\sigma}\right)^{-1}\ \|\alpha\|_{L^\infty(S^2)}\le C\ \left(\log\frac{1}{\sigma}\right)^{-1}\ \left[\int_{S^2}|d\alpha|^2_{g_{\vec{\Phi}}}+W(\vec{\Phi})\right]+\left|\frac{\log A_{\vec{\Phi}}(S^2)}{2\,\log\frac{1}{\sigma}}\right|\tag{III.74}$$

moreover

$$\frac{\log A_{\vec{\Phi}}(S^2)}{2\,\log\frac{1}{\sigma}}\le 1+\left(\log\frac{1}{\sigma}\right)^{-1}\ \log\left[F^\sigma(\vec{\Phi})-W(\vec{\Phi})\right]\tag{III.75}$$

where the constant C is a positive universal constant. □

Proof of lemma III.11. First of all, Onofri inequality (III.42) implies

$$F^\sigma(\vec{\Phi})\ge W(\vec{\Phi})+\sigma^2\int_{S^2}(1+H^2)^2\ dvol_{g_{\vec{\Phi}}}\tag{III.76}$$

We have also

$$\begin{aligned}F^\sigma(\vec{\Phi})-W(\vec{\Phi})\ge&\left(\log\frac{1}{\sigma}\right)^{-1}\ \int_{S^2}\left[2^{-1}\,|d\alpha|^2_{g_{\vec{\Phi}}}+K_{g_0}\ \alpha\ e^{-2\alpha}\right]\ dvol_{g_{\vec{\Phi}}}\\&-2\pi\ \left(\log\frac{1}{\sigma}\right)^{-1}\ \log\left(\int_{S^2}e^{2\alpha}\ dvol_{g_0}\right)\end{aligned}\tag{III.77}$$

Then we obtain (III.72) directly from (III.69).

By the uniformization theorem on S^2, modulo the action of a conformal diffeomorphism, we can assume $g_0=g_{S^2}/4\pi$ where g_{S^2} is the standard metric on S^2. The immersion $\vec{\Phi}$ is then conformal from the standard sphere into $\mathbb{R}^3$.

The Liouville equation reads in this gauge

$$-\Delta_{g_0}\alpha=e^{2\alpha}\,K_{\vec{\Phi}}-4\pi\quad .$$

Standard elliptic estimate give on the standard sphere S^2

$$\|d\alpha\|_{L^{2,\infty}_{g_0}(S^2)}\le C(S^2)\ \|\Delta_{g_0}\alpha\|_{L^1_{g_0}(S^2)}\quad .$$

Hence we have the existence of a constant $C(S^2)$ such that

$$\|d\alpha\|_{L^{2,\infty}_{g_0}(S^2)}\le C(S^2)\ \left[1+W(\vec{\Phi})\right]\tag{III.78}$$

where the norm is taken with respect to the metric g_0. We cover S^2 by a finite, controlled family of geodesic convex balls for the metric g_0 and each of these balls we choose a conformal chart $\Psi_i\ :\ D^2\ \to\Sigma$ such that $\Psi_i(D^2_{1/2})$ still cover S^2. We consider any of these ball and we omit to write explicitly the composition of $\vec{\Phi}$ with Ψ_i. Denote $\vec{e}_j:=e^{-\lambda}\,\partial_{x_j}\vec{\Phi}$ where $e^\lambda=\log|\partial_{x_j}\vec{\Phi}|$. $(\vec{e}_1,\vec{e}_2)$ realizes a moving frame and

$$|\nabla\vec{e}_j|^2=|\vec{n}\cdot\nabla\vec{e}_j|^2+|\nabla\lambda|^2\quad .$$

We have

$$-\Delta\alpha = \Delta\mu + (\nabla\vec{e}_1, \nabla^\perp\vec{e}_2)$$

Using Wente together with classical elliptic estimates we get the existence of $\overline{\alpha} \in \mathbb{R}$ such that

$$\|\alpha - \overline{\alpha}\|_{L^\infty(D^2_{3/4})} \le C\ \left[\|\nabla\alpha\|_{L^{2,\infty}(D^2)} + \|\nabla\mu\|_{L^\infty(D^2)} + \|\nabla\vec{n}\|^2_{L^2(D^2)}\right] + \frac{1}{2\pi}\int_{S^2} |d\alpha|^2_{g_0}\, dvol_{g_0} \quad \text{(III.79)}$$

Since $\int_\Sigma dvol_{g_0} = 1$, there exists $x \in \Sigma$ such that

$$\alpha(x) := \frac{1}{2}\log \text{Area}(\vec{\Phi}(S^2)) \quad \text{(III.80)}$$

Combining (III.79) and (III.80) we get (III.73) and lemma III.11 is proved. □

First we shall need the following lemma

Lemma III.12. *Let Σ be a closed oriented surface and $\vec{\Phi}$ be a weak immersion in $\mathcal{E}_{\Sigma,2}$. Let g_0 be a constant Gauss curvature metric of volume 1 on Σ such that there exists α for which*

$$g_{\vec{\Phi}} = e^{2\alpha}\, g_0$$

Then, if genus$(\Sigma) \ge 1$ we have

$$K_{g_0}\ \int_\Sigma \alpha\ dvol_{g_0} \ge 2^{-1}\, K_{g_0}\ \log\int_\Sigma\ dvol_{g_{\vec{\Phi}}} \quad . \quad \text{(III.81)}$$

Proof of lemma III.12 We have to bound from bellow the following quantity

$$4\pi\,(1 - \text{genus}(\Sigma))\ \int_\Sigma\ \alpha\ dvol_{g_0} \quad .$$

Since we fixed $\int_\Sigma dvol_{g_0} = 1$, using the convexity of exp and Jensen inequality we have

$$\exp\left(2\int_\Sigma\ \alpha\ dvol_{g_0}\right) \le \int_\Sigma\ e^{2\alpha}\ dvol_{g_0} = \int_\Sigma\ dvol_{g_{\vec{\Phi}}} \quad . \quad \text{(III.82)}$$

We then have for $K_{g_0} < 0$

$$K_{g_0}\ \int_\Sigma \alpha\ dvol_{g_0} \ge 2\pi\ (1 - \text{genus}(\Sigma))\ \log\int_\Sigma\ dvol_{g_{\vec{\Phi}}} \quad . \quad \text{(III.83)}$$

This concludes the proof of lemma III.12. □

Combining (III.70), (III.74) and (III.81) we obtain the following lemma

Lemma III.13. *Let Σ be a closed surface of non zero genus and $\vec{\Phi}$ be a weak immersion of Σ in $\mathcal{E}_{\Sigma,2}$ and g_0 be the constant Gauss curvature metric on Σ of volume 1 such that*

$$g_{\vec{\Phi}} = e^{2\alpha}\, g_0$$

We have for $\sigma \in (0,1)$

$$W(\vec{\Phi}) + \sigma^2\int_\Sigma (1 + H^2)^2\ dvol_{g_{\vec{\Phi}}} + \left(2\,\log\frac{1}{\sigma}\right)^{-1}\ \int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}} \le F^\sigma(\vec{\Phi}) \quad \text{(III.84)}$$

and

$$\left(\log\frac{1}{\sigma}\right)^{-1}\ \|\alpha\|_{L^\infty(\Sigma)} \le C\ \left(\log\frac{1}{\sigma}\right)^{-1}\ \left[\int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}} + W(\vec{\Phi})\right] + \left|\frac{\log A_{\vec{\Phi}}(\Sigma)}{2\,\log\frac{1}{\sigma}}\right| \quad , \quad \text{(III.85)}$$

where the constant C is a positive universal constant. □

IV Uniform regularity for critical points to the Frame Energies approximating the Willmore Energy

IV.1 Some Banach Spaces relevant to the proof.

In the sequel we shall make use of some Banach spaces for which we recall the definitions. For any domain $\Omega \in \mathbb{R}^2$ the *weak L^p space* $L^{p,\infty}(\Omega)$ is given by

$$L^{p,\infty}(\Omega) := \left\{ f \quad \text{measurable s. t. } |f|_{p,\infty} := \sup_{t>0} t\, |\{x \in \Omega\ ;\ |f(x)| > t\}|^{1/p} < +\infty \right\} \quad .$$

For $p \in (1, +\infty)$ the quasi-norm $|\cdot|_{p,\infty}$ is equivalent to a norm and the normed vector space $L^{p,\infty}(\Omega)$ is complete (see for instance [20]). It is the dual space to $L^{p',1}(\Omega)$ given by the set of measurable functions f such that

$$|f|_{p',1} := \int_0^{+\infty} |\{x\ ;\ |f(x)| > t\}|^{1/p'} \ dt < +\infty \quad .$$

The quasi-norm $|\cdot|_{p',1}$ is equivalent to a norm denoted $\|\cdot\|_{p',1}$ and the space $L^{p',1}(\Omega)$ equipped with this norm is complete (see for instance [20]).

We shall also make use of the space

$$L^{2,\infty}{+}\sigma^{-1/2}L^{4/3}(D^2) := \left\{ f \quad \text{measurable s. t. } \ \|f\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}} := \inf_{f=f_1+f_2} \|f_1\|_{2,\infty} + \sigma^{-1/2}\|f_2\|_{4/3} < +\infty \right\} \quad ,$$

which is dual to

$$L^{2,1} \cap \sigma^{1/2}L^4(D^2) := \left\{ f \quad \text{measurable s. t. } \ \|f\|_{L^{2,1}\cap\sigma^{1/2}L^4} := \|f\|_{2,1} + \sigma^{1/2}\|f\|_4 < +\infty \right\} \quad .$$

IV.2 Uniform $\epsilon-$regularity.

In this section we shall consider critical points of the following family of lagrangians where the parameter σ belongs to $[0, \sigma_0)$.

$$\begin{aligned} F^\sigma(\vec{\Phi}) &:= \left(\log \frac{1}{\sigma}\right)^{-1} \tilde{F}^{K_{g_0}}_{f_\sigma}(\vec{\Phi}) \\ &:= \left(\log \frac{1}{\sigma}\right)^{-1} \int_\Sigma \left[f_\sigma(H_{g_{\vec{\Phi}}}) + 2^{-1}\, |d\alpha|^2_{g_{\vec{\Phi}}} + K_{g_0}\ \alpha\ e^{-2\alpha} \right] \ dvol_{g_{\vec{\Phi}}} \\ &\quad - 2^{-1}\, K_{g_0} \ \left(\log \frac{1}{\sigma}\right)^{-1} \log\left(\int_\Sigma e^{2\alpha}\ dvol_{g_0} \right) \quad , \end{aligned} \tag{IV.1}$$

where $f_\sigma(t) = \log \sigma^{-1}[t^2 + \sigma^2(1+t^2)^2]$. Recall that we denote for any domain U of S^2

$$A_{\vec{\Phi}}(U) := \mathrm{Area}(\vec{\Phi}(U)) \quad .$$

The goal of this subsection is to establish the following result

Lemma IV.1. [uniform $\varepsilon-$regularity] *For any $C_1 > 0$, there exists $\varepsilon > 0$ and $\sigma_0 > 0$ such that for any $\sigma \in (0, \sigma_0)$ the following holds. Let $\vec{\Phi}$ be a critical point of F^σ satisfying*

$$F^\sigma(\vec{\Phi}) \le C_1 \quad . \tag{IV.2}$$

We keep denoting $\vec{\Phi}$ the expression of this immersion in a conformal chart from D^2 into $\mathbb{R}^3$. Denote respectively

$$g_{\vec{\Phi}} = e^{2\lambda}\ [dx_1^2 + dx_2^2] \quad \textit{and} \quad g_0 = e^{2\mu}\ [dx_1^2 + dx_2^2]$$

and denote moreover $l_\sigma := \left(\log \frac{1}{\sigma}\right)^{-1}$. Assume that α, μ, $\vec{n}$ and H satisfy

$$\inf_{D^2} \alpha \geq \log \sigma - C_1 \quad , \quad \|\nabla\mu\|_{L^\infty(D^2)} \leq C_1 \quad , \tag{IV.3}$$

and

$$l_\sigma \,\|\alpha\ e^{2\,\mu}\|_{L^\infty(D^2)} + \int_{D^2} |\nabla\vec{n}|^2 + \sigma^2\,[1+H^4]\ e^{2\lambda}\ dx^2 < \varepsilon \quad , \tag{IV.4}$$

then for any $j \in \mathbb{N}$ one has

$$\begin{aligned}
&|\nabla^{j+1}\vec{n}|^2(0) + |e^\lambda\,\nabla^j(\vec{H}\,(1+2\,\sigma^2\,(1+H^2))|^2(0) + \sigma^4\,H^2\,(1+H^2)^2\,e^{2\lambda}(0)\\
&\leq \tilde{C}_j \int_{D^2} |\nabla\vec{n}|^2\ dx^2 + \tilde{C}_j\ \left[\sigma^2 \int_{D^2} H^4\,e^{2\lambda}\ dx^2\right]^2 + \tilde{C}_j\ \left[l_\sigma \int_{D^2} |\nabla\alpha|^2\ dx^2\right]^2\\
&\quad +\tilde{C}_j\,[l_\sigma^2\,|\overline{\alpha}|^2 + l_\sigma^2\,\|e^{4\mu}\|_{L^\infty(D^2)}]\,\|e^{4\mu}\|_{L^\infty(D^2)} + \tilde{C}_j\,l_\sigma^2\ \left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^2 \quad ,
\end{aligned} \tag{IV.5}$$

where $\overline{\alpha} = |D^2_{1/2}|^{-1}\int_{D^2_{1/2}} \alpha$ and

$$\begin{aligned}
&l_\sigma|\nabla^{j+1}\alpha|^2(0) \leq\ \tilde{C}_j\,l_\sigma \int_{D^2} |\nabla\alpha|^2\ dx^2 + \tilde{C}_j\ \left[\int_{D^2} |\nabla\vec{n}|^2\ dx^2\right]^2\\
&\quad +\tilde{C}_j\ \left[\sigma^2 \int_{D^2} H^4\,e^{2\lambda}\ dx^2\right]^4 + \tilde{C}_j\,l_\sigma\,\|e^{4\mu}\|_{L^\infty(D^2)} + \tilde{C}_j\,l_\sigma^4\ \left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^4 \quad ,
\end{aligned} \tag{IV.6}$$

where $\tilde{C}_j$ only depends on C and j. □

Proof of lemma IV.1. In the first part of the proof, following the original ideas of [41], we derive from equation (III.53) more conservation laws. We do it first formally, not worrying of the regularity. In the second part of the proof we will revisit each step with estimates in relevant Banach spaces.

Step 1. Equations. Let $l_\sigma := \left(\log \frac{1}{\sigma}\right)^{-1}$

$$\begin{aligned}
\nabla\vec{L} &:= l_\sigma \nabla^\perp(f'_\sigma(H)\,\vec{n}) - 2\,l_\sigma\,f'_\sigma(H)\,\nabla^\perp\vec{n} - 2\ e^{-2\lambda}\,l_\sigma\,\nabla\vec{\Phi}\cdot\nabla\alpha\ \nabla^\perp\alpha\\
&\quad +l_\sigma\ \left[-2\,f_\sigma(H) + e^{-2\lambda}\,|\nabla\alpha|^2 - K_{g_0}\,\alpha\ e^{-2\alpha} + K_{g_0}\,A_{\vec{\Phi}}(\Sigma)^{-1}\right]\ \nabla^\perp\vec{\Phi} + 2\,l_\sigma\ e^{-2\lambda}\,\vec{\mathbb{I}}\llcorner\nabla^\perp\alpha \quad .
\end{aligned} \tag{IV.7}$$

Equation (III.53) gives

$$\nabla\vec{\Phi}\cdot\nabla^\perp\vec{L} = 2\,l_\sigma\ e^{2\lambda}\ \left(2\,f_\sigma(H) -\ H\,f'_\sigma(H) + K_{g_0}\,\alpha\,e^{-2\alpha} - K_{g_0}\,A_{\vec{\Phi}}(\Sigma)^{-1}\right) \quad . \tag{IV.8}$$

Let Y be the solution of

$$\left\{\begin{array}{l}
-\Delta Y = 2\,l_\sigma\,e^{2\lambda}\ \left(2\,f_\sigma(H) -\ H\,f'_\sigma(H) + K_{g_0}\,\alpha\,e^{-2\alpha} - K_{g_0}\,A_{\vec{\Phi}}(\Sigma)^{-1}\right) \quad \text{in } D^2\\[2mm]
Y = 0 \qquad \text{on } \partial D^2 \quad .
\end{array}\right. \tag{IV.9}$$

Observe that $2\, f_\sigma(H) -\, H\, f'_\sigma(H) = 2\, l_\sigma^{-1}\, \sigma^2\, (1 - H^4)$. So Y satisfies

$$\left\{\begin{array}{l} -\Delta Y = 4\, e^{2\lambda}\, \sigma^2\, (1-H^4) + 2\, l_\sigma\, K_{g_0}\, \alpha\, e^{2\mu} - 2\, K_{g_0}\, l_\sigma\, e^{2\lambda}\, A_{\vec{\Phi}}(\Sigma)^{-1} \quad \text{ in } D^2 \\ \\ Y = 0 \qquad \text{on } \partial D^2 \quad . \end{array}\right. \tag{IV.10}$$

Using Poincaré Lemma we deduce the existence of a function S such that

$$\nabla S = \vec{L} \cdot \nabla \vec{\Phi} + \nabla^\perp Y \quad . \tag{IV.11}$$

The equation (III.63) in conformal coordinates gives

$$\nabla \vec{\Phi} \times \nabla^\perp \vec{L} = -\, l_\sigma\, \nabla^\perp \vec{\Phi} \cdot \nabla f'_\sigma(H) + 2\, \nabla\alpha \cdot \nabla^\perp \vec{D} \quad , \tag{IV.12}$$

where

$$\nabla \vec{D} = \left(l_\sigma\ e^{-2\lambda} \sum_{i=1}^2 \mathbb{I}_{1i}\, \partial_{x_i} \vec{\Phi}\ ,\ l_\sigma\ e^{-2\lambda} \sum_{i=1}^2 \mathbb{I}_{2i}\, \partial_{x_i} \vec{\Phi} \right) \quad . \tag{IV.13}$$

Using again Poincaré Lemma on D^2 we obtain the existence of $\vec{V}$ such that

$$\nabla \vec{V} := \vec{L} \times \nabla \vec{\Phi} +\ l_\sigma\ f'_\sigma(H)\, \nabla \vec{\Phi} - 2\, (\alpha - \overline{\alpha})\, \nabla \vec{D} \quad . \tag{IV.14}$$

Using the explicit expression of $\nabla \vec{D}$ given by (III.64) we obtain

$$\vec{n} \cdot \nabla \vec{V} = \vec{n} \cdot (\vec{L} \times \nabla \vec{\Phi}) = \vec{L} \cdot \nabla^\perp \vec{\Phi} = \nabla^\perp S + \nabla Y \quad . \tag{IV.15}$$

We have also

$$\vec{n} \times \nabla \vec{V} = -\, (\vec{L} \cdot \vec{n})\, \nabla \vec{\Phi} -\ l_\sigma\ f'_\sigma(H)\, \nabla^\perp \vec{\Phi} - 2\, (\alpha - \overline{\alpha})\, \vec{n} \times \nabla \vec{D} \quad . \tag{IV.16}$$

Denote $\pi_T(\nabla^\perp \vec{V})$ the tangential projection of $\nabla^\perp \vec{V}$, we have

$$\pi_T(\nabla^\perp \vec{V}) = (\vec{L} \cdot \vec{n})\, \nabla \vec{\Phi} + l_\sigma\ f'_\sigma(H)\, \nabla^\perp \vec{\Phi} - 2\, (\alpha - \overline{\alpha})\, \nabla^\perp \vec{D} \quad . \tag{IV.17}$$

Hence

$$\vec{n} \times \nabla \vec{V} = -\nabla^\perp \vec{V} -\ 2\, (\alpha - \overline{\alpha})\ \left(\nabla^\perp \vec{D} + \vec{n} \times \nabla \vec{D} \right) - \vec{n}\, (\nabla S - \nabla^\perp Y) \quad . \tag{IV.18}$$

Let $\vec{v}$ be the unique solution to

$$\left\{\begin{array}{l} \Delta \vec{v} = \nabla^\perp Y \cdot \nabla \vec{n} \qquad \text{in } D^2 \\ \\ \vec{v} = 0 \qquad \text{on } \partial D^2 \quad . \end{array}\right. \tag{IV.19}$$

Using one more time Poincaré lemma we obtain the existence of $\vec{u}$ such that

$$\vec{n}\, \nabla^\perp Y = \nabla \vec{v} + \nabla^\perp \vec{u} \quad . \tag{IV.20}$$

Finally, let $\vec{R} := \vec{V} - \vec{u}$. We have

$$\vec{n} \times \nabla \vec{V} = \vec{n} \times \nabla \vec{R} + \vec{n} \times \nabla \vec{u} = \vec{n} \times \nabla \vec{R} + \vec{n} \times \nabla^\perp \vec{v} \quad . \tag{IV.21}$$

Hence (IV.18) becomes

$$\vec{n} \times \nabla \vec{R} + \vec{n} \times \nabla^\perp \vec{v} = -\nabla^\perp \vec{R} + \nabla \vec{v} - \vec{n}\, \nabla S -\ 2\, (\alpha - \overline{\alpha})\ \left(\nabla^\perp \vec{D} + \vec{n} \times \nabla \vec{D} \right) \quad , \tag{IV.22}$$

which gives

$$\left\{\begin{array}{l}\Delta S=-\nabla\vec{n}\cdot\nabla^\perp\vec{R}+\mathrm{div}\left(\vec{n}\cdot\nabla\vec{v}-\,2\,(\alpha-\overline{\alpha})\;\vec{n}\cdot\nabla^\perp\vec{D}\right)\\[5mm] \Delta\vec{R}=\nabla\vec{n}\times\nabla^\perp\vec{R}+\nabla\vec{n}\cdot\nabla^\perp S+\mathrm{div}\left(-\vec{n}\times\nabla\vec{v}+2\,(\alpha-\overline{\alpha})\;(-\nabla\vec{D}+\vec{n}\times\nabla^\perp\vec{D})\right)\quad .\end{array}\right. \tag{IV.23}$$

Taking the vectorial product between (IV.14) and $\nabla^\perp\vec{\Phi}$ we obtain

$$\begin{array}{l}\nabla\vec{V}\times\nabla^\perp\vec{\Phi}=(\vec{L}\cdot\nabla^\perp\vec{\Phi})\cdot\nabla\vec{\Phi}-2\,l_\sigma\,f'_\sigma(H)\,e^{2\lambda}\,\vec{n}-2\,(\alpha-\overline{\alpha})\,\nabla\vec{D}\times\nabla^\perp\vec{\Phi}\\[5mm] \quad=\nabla^\perp S\cdot\nabla\vec{\Phi}+\nabla Y\cdot\nabla\vec{\Phi}-2\,l_\sigma\,f'_\sigma(H)\,e^{2\lambda}\,\vec{n}-2\,(\alpha-\overline{\alpha})\,\nabla\vec{D}\times\nabla^\perp\vec{\Phi}\quad .\end{array} \tag{IV.24}$$

We have also

$$\begin{array}{l}\nabla\vec{V}\times\nabla^\perp\vec{\Phi}=\nabla\vec{R}\times\nabla^\perp\vec{\Phi}+\nabla\vec{u}\times\nabla^\perp\vec{\Phi}\\[5mm] \quad=\nabla\vec{R}\times\nabla^\perp\vec{\Phi}+\nabla^\perp\vec{v}\times\nabla^\perp\vec{\Phi}+\nabla Y\cdot(\vec{n}\times\nabla^\perp\vec{\Phi})\\[5mm] \quad=\nabla\vec{R}\times\nabla^\perp\vec{\Phi}+\nabla\vec{v}\times\nabla\vec{\Phi}+\nabla Y\cdot\nabla\vec{\Phi}\quad .\end{array} \tag{IV.25}$$

Combining (IV.24) and (IV.25) gives

$$2\,l_\sigma\,f'_\sigma(H)\,e^{2\lambda}\,\vec{n}=\nabla^\perp S\cdot\nabla\vec{\Phi}-2\,(\alpha-\overline{\alpha})\,\nabla\vec{D}\times\nabla^\perp\vec{\Phi}-\nabla\vec{R}\times\nabla^\perp\vec{\Phi}-\nabla\vec{v}\times\nabla\vec{\Phi}\quad . \tag{IV.26}$$

We have explicitly $l_\sigma\,f'_\sigma(H)=2\,H\,(1+2\,\sigma^2(1+H^2))$ moreover, a straightforward computation gives

$$-\,\nabla\vec{D}\times\nabla^\perp\vec{\Phi}=\,2\,l_\sigma\,\vec{H}\,e^{2\lambda}=l_\sigma\,\Delta\vec{\Phi}\quad . \tag{IV.27}$$

Inserting (IV.27) in (IV.26) gives then

$$2\,(1+2\,\sigma^2\,(1+H^2)-l_\sigma\,(\alpha-\overline{\alpha}))\;\Delta\vec{\Phi}=\nabla^\perp S\cdot\nabla\vec{\Phi}-\nabla\vec{R}\times\nabla^\perp\vec{\Phi}-\nabla\vec{v}\times\nabla\vec{\Phi}\quad . \tag{IV.28}$$

step 2 : We now prove that, for ε small enough, ∇S and $\nabla\vec{R}$ are in L^2.

Since $\int_{D^2}|\nabla\vec{n}|^2\;dx^2\,<\,\varepsilon$, for ε small enough, following Hélein's construction of energy controlled moving frames (see [21]), we get the existence of $\vec{e}_i$ such that

$$\vec{e}_1\times\vec{e}_2=\vec{n}\quad\text{ and }\quad\int_{D^2}|\nabla\vec{e}_i|^2\;dx^2\le C\;\int_{D^2}|\nabla\vec{n}|^2\;dx^2\quad . \tag{IV.29}$$

Using the assumptions (IV.2) we have

$$\begin{array}{l}2\,C_1\ge W(\vec{\Phi})+\sigma^2\;\displaystyle\int_{S^2}(1+H^2)^2\\[5mm] \quad+l_\sigma\;\left[\displaystyle\int_{S^2}2^{-1}\,|d\alpha|^2_{g_{\vec{\Phi}}}\;dvol_{g_{\vec{\Phi}}}+K_{g_0}\;\alpha\;e^{-2\alpha}-2^{-1}\,K_{g_0}\;\log A_{\vec{\Phi}}(\Sigma)\right]\quad .\end{array} \tag{IV.30}$$

Using Onofri inequality (III.42) we deduce that

$$W(\vec{\Phi})+\sigma^2\;\int_{S^2}(1+H^2)^2\le 2\,C_1 \tag{IV.31}$$

Since $W(\vec{\Phi})\le 2\,C_1$ using (III.78), we obtain the existence of a constant depending only on C_1 such that

$$\|\nabla\lambda\|_{L^{2,\infty}(D^2)}\le C \quad . \tag{IV.32}$$

Recall (see for instance [40]) that for $\vec{e}_i:=e^{-\lambda}\,\partial_{x_i}\vec{\Phi}$ the Liouville equation giving the expression of the Gauss curvature in conformal coordinates is equivalent to

$$-\Delta\lambda=(\nabla\vec{e}_1;\nabla^\perp\vec{e}_2) \quad . \tag{IV.33}$$

Let ν be the solution of

$$\left\{\begin{array}{l} -\Delta\nu=(\nabla\vec{e}_1;\nabla^\perp\vec{e}_2) \qquad \text{in } D^2\\[3mm] \nu=0 \qquad\quad \text{on } \partial D^2 \quad . \end{array}\right. \tag{IV.34}$$

We get, using Wente inequality

$$\|\nabla\nu\|_{L^2(D^2)}+\|\nu\|_{L^\infty(D^2)}\le C\sum_{i=1}^2\int_{D^2}|\nabla\vec{e}_i|^2\,dx^2\le C\int_{D^2}|\nabla\vec{n}|^2\,dx^2\le\varepsilon \quad . \tag{IV.35}$$

Since $\lambda-\nu$ is harmonic and since

$$\|\nabla(\lambda-\nu)\|_{L^{2,\infty}(D^2)}\le C \quad , \tag{IV.36}$$

we have

$$\|\nabla(\lambda-\nu)\|_{L^\infty(D^2_{1/2})}\le C \quad . \tag{IV.37}$$

Hence, using (IV.35), there exists $\overline{\lambda}=(\lambda-\nu)(0)\in\mathbb{R}$ such that

$$\|\lambda-\overline{\lambda}\|_{L^\infty(D^2_{1/2})}\le C \quad . \tag{IV.38}$$

Since $\|\nabla\mu\|_\infty\le\, C_1$, we have the existence of $\overline{\mu}\in\mathbb{R}$ such that

$$\|\mu-\overline{\mu}\|_{L^\infty(D^2)}< C \quad . \tag{IV.39}$$

Hence we deduce the existence of $\overline{\alpha}\in\mathbb{R}$ such that

$$\|\alpha-\overline{\alpha}\|_{L^\infty(D^2_{1/2})}\le C \quad . \tag{IV.40}$$

We rescale the domain so that $D^2_{1/2}$ becomes D^2. We now proceed to the introduction of $\vec{L}$, Y, $\vec{V}$, $\vec{v}$, $\vec{u}$,$\vec{R}$, S as in step 1. First of all using classical elliptic estimates we deduce from (IV.10), using the hypothesis (IV.4) and for σ small enough

$$\|\nabla Y\|_{L^{2,\infty}(D^2)}\le C\int_{D^2}\sigma^2\,[1+H^4]\,e^{2\lambda}\,dx^2+C\,l_\sigma\,|\overline{\alpha}|\int_{D^2}e^{2\mu}\,dx^2+C\,l_\sigma\,\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\le C\,\varepsilon \quad , \tag{IV.41}$$

and using (IV.19) we deduce by the mean of Wente estimates

$$\|\nabla\vec{v}\|_{L^2(D^2)}\le C\,\|\nabla\vec{n}\|_{L^2(D^2)}\,\left[\int_{D^2}\sigma^2\,[1+H^4]\,e^{2\lambda}\,dx^2+\,l_\sigma\,|\overline{\alpha}|\int_{D^2}e^{2\mu}\,dx^2+l_\sigma\right] \quad . \tag{IV.42}$$

Recall that for any $\vec{X}\in L^1(\mathbb{R}^2,\mathbb{R}^2)$ there exists a unique pair $a,b\in L^{2,\infty}$ such that

$$\vec{X}=\nabla a+\nabla^\perp b \qquad \text{and} \qquad \|a\|_{L^{2,\infty}(\mathbb{R}^2)}+\|b\|_{L^{2,\infty}(\mathbb{R}^2)}\le C\,\|\vec{X}\|_{L^1(\mathbb{R}^2)} \quad .$$

This pair is explicitly given by

$$a := -\frac{1}{2\pi}\frac{1}{r}\frac{\partial}{\partial r}\dot{\star}\vec{X} \quad \text{and} \quad b := \frac{1}{2\pi}\frac{1}{r}\frac{\partial}{\partial r}\dot{\star}\vec{X}^{\perp} \quad .$$

We apply this decomposition to each coordinates of the restriction to D^2 of

$$-2\,l_\sigma\, e^{\overline{\lambda}}\, f'_\sigma(H)\,\nabla^\perp\vec{n} - 2\; e^{-2\lambda+\overline{\lambda}}\, l_\sigma\,\nabla\vec{\Phi}\cdot\nabla\alpha\;\nabla^\perp\alpha + 2\, l_\sigma\; e^{-2\lambda+\overline{\lambda}}\,\mathbb{I}\llcorner\nabla^\perp\alpha$$

$$+l_\sigma\; e^{\overline{\lambda}}\,\left[-2\, f_\sigma(H) + e^{-2\lambda}\,|\nabla\alpha|^2 - K_{g_0}\,\alpha\; e^{-2\alpha} + K_{g_0}\, A_{\vec{\Phi}}(\Sigma)^{-1}\right]\;\nabla^\perp\vec{\Phi} \quad ,$$

and we get the existence of $\vec{a}$ and $\vec{b}$ in $L^{2,\infty}(D^2,\mathbb{R}^3)$ such that

$$\nabla(e^{\overline{\lambda}}\,\vec{L}) - \nabla^\perp(l_\sigma\, e^{\overline{\lambda}}\, f'_\sigma(H)\;\vec{n}) = \nabla\vec{a} + \nabla^\perp\vec{b} \quad \text{in } D^2 \quad ,$$

and

$$\begin{aligned} &\|\vec{a}\|_{L^{2,\infty}(D^2)} + \|\vec{b}\|_{L^{2,\infty}(D^2)} \le C\;\|\nabla\vec{n}\|_{L^2(D^2)} + C\, l_\sigma\,\|\nabla\alpha\|^2_{L^2(D^2)} \\ &+C\, l_\sigma\,|\overline{\alpha}|\int_{D^2_2} e^{2\mu}\; dx^2 + C\;\left[\int_{D^2}\sigma^2|\nabla\vec{n}|^4 e^{-2\lambda}\; dx^2\right] + C\; l_\sigma\;\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)} \quad . \end{aligned} \tag{IV.43}$$

We have that both $e^{\overline{\lambda}}\,\vec{L} + \vec{a}$ and $l_\sigma\, e^{\overline{\lambda}}\, f'_\sigma(H)\;\vec{n} + \vec{b}$ are harmonic conjugate to each other. Let $\mathcal{H}$ be the operator which sends an harmonic function on D^2 of average 0 to it's harmonic conjugate[13] of average 0. We have

$$e^{\overline{\lambda}}\,\vec{L} + \vec{a} = \mathcal{H}(l_\sigma\, e^{\overline{\lambda}}\, f'_\sigma(H)\;\vec{n} + \vec{b}) \quad .$$

Calderon Zygmund theory gives us that $\mathcal{H}$ maps continuously $L^{2,\infty}(D^2)$ into itself and $L^{4/3}(D^2)$ into itself : there exists C such that, for any harmonic function f in D^2

$$\|\mathcal{H}(f)\|_{L^{2,\infty}(D^2)} \le\; C\;\|f\|_{L^{2,\infty}(D^2)} \quad \text{and} \quad \|\mathcal{H}(f)\|_{L^{4/3}(D^2)} \le\; C\;\|f\|_{L^{4/3}(D^2)} \quad .$$

Let f be an harmonic function in $L^{2,\infty}(D^2)+\sigma^{-1/2}L^{4/3}(D^2)$. To any decomposition $f = f_1+f_2$ we assign $b(f_1)$ and $b(f_2)$ to be respectively the *Bergman projections* of f_1 and f_2. We have of course $b(f) = f$ and the L^p boundedness of the *Bergman projection* gives

$$\|b(f_1)\|_{2,\infty} \le C\;\|f_1\|_{2,\infty} \qquad \text{and} \qquad \|b(f_2)\|_{4/3} \le C\;\|f_2\|_{4/3} \quad .$$

Combining all the previous gives

$$\begin{aligned} &\|\mathcal{H}(f)\|_{L^{2,\infty}(D^2)+\sigma^{-1/2}L^{4/3}(D^2)} = \|\mathcal{H}(b(f))\|_{L^{2,\infty}(D^2)+\sigma^{-1/2}L^{4/3}(D^2)} \\ &\le \|\mathcal{H}(b(f_1))\|_{L^{2,\infty}(D^2)} + \sigma^{-1/2}\;\|\mathcal{H}(b(f_2))\|_{L^{4/3}(D^2)} \\ &\le C\;\left[\|b(f_1)\|_{L^{2,\infty}(D^2)} + \sigma^{-1/2}\;\|b(f_2)\|_{L^{4/3}(D^2)}\right] \le C\;\left[\|f_1\|_{L^{2,\infty}(D^2)} + \sigma^{-1/2}\;\|f_2\|_{L^{4/3}(D^2)}\right] \quad . \end{aligned}$$

Since this holds for any decomposition $f = f_1 + f_2$ we deduce that $\mathcal{H}$ is sending continuously $L^{2,\infty}(D^2)+\sigma^{-1/2}L^{4/3}(D^2)$ into itself with a constant independent of σ. Hence we have

$$\|e^{\overline{\lambda}}\,\vec{L}+\vec{a}\|_{L^{2,\infty}(D^2)+\sigma^{-1/2}L^{4/3}(D^2)} \le C\;\left[\|\vec{b}\|_{L^{2,\infty}(D^2)} + \|e^{\overline{\lambda}}\vec{H}\|_{L^{2,\infty}(D^2)} + \sigma^{-1/2}\|\, e^{\overline{\lambda}}\,\sigma^2\,\vec{H}\;(1+H^2)\|_{L^{4/3}(D^2)}\right] \quad .$$

[13] $\mathcal{H}$ is also the operator which to the real part of an holomorphic function on D^2 assigns it's imaginary part

Combining this with (IV.43) gives then

$$\begin{aligned}\|e^{\overline{\lambda}}\,\vec{L}\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)} &\le C\ \|\nabla\vec{n}\|_{L^2(D^2)} + C\,l_\sigma\ \|\nabla\alpha\|^2_{L^2(D^2)} + C\ \int_{D^2}\sigma^2|\nabla\vec{n}|^4 e^{-2\lambda}\ dx^2\\ &+C\,l_\sigma\,|\overline{\alpha}|\int_{D^2_2} e^{2\mu}\ dx^2 + C\ \left[\int_{D^2}\sigma^2|\nabla\vec{n}|^4e^{-2\lambda}\ dx^2\right]^{3/4} + C\ l_\sigma\,\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}\quad .\end{aligned}\tag{IV.44}$$

Combining lemma VII.3 with (IV.3) gives

$$\sigma^2\int_{D^2}|\nabla\vec{n}|^4\ e^{-2\lambda}\,dx^2 \le C\ \sigma^2\int_{D^2_2} H^4\,e^{2\lambda}\ dx^2 + C\left[\int_{D^2_2}|\nabla\vec{n}|^2\ dx^2\right]^2 < C\,\varepsilon\quad .\tag{IV.45}$$

Hence, using the explicit expressions (IV.11), (IV.14) and (IV.20) we obtain that $\nabla\vec{R}$ and $\nabla\vec{S}$ are uniformly bounded in $L^{2,\infty}+\sigma^{-1/2}L^{4/3}$ and we have respectively

$$\begin{aligned}\|\nabla\vec{R}\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)} + \|\nabla S\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)} &\le C\,\|\nabla\vec{n}\|_{L^2(D^2)} + C\,l_\sigma\,\|\nabla\alpha\|^2_{L^2(D^2)}\\ +C\,l_\sigma\,|\overline{\alpha}|\int_{D^2_2}e^{2\mu}\ dx^2 + \left[\int_{D^2}\sigma^2|\nabla\vec{n}|^4e^{-2\lambda}\ dx^2\right]^{3/4} &+ C\ l_\sigma\,\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}\quad .\end{aligned}\tag{IV.46}$$

Let φ and $\vec{\Psi}$ the unique solutions in $W^{1,2}_0(D^2)$ of the linear system

$$\left\{\begin{array}{l}\Delta\varphi = -\nabla\vec{n}\cdot\nabla^\perp\vec{R} + \operatorname{div}\Big(\vec{n}\cdot\nabla\vec{v} - 2\,(\alpha-\overline{\alpha})\ \vec{n}\cdot\nabla^\perp\vec{D}\Big)\\[2mm] \Delta\vec{\Psi} = \nabla\vec{n}\times\nabla^\perp\vec{R} + \nabla\vec{n}\cdot\nabla^\perp S + \operatorname{div}\Big(-\vec{n}\times\nabla\vec{v} + 2\,(\alpha-\overline{\alpha})\ (-\nabla\vec{D}+\vec{n}\times\nabla^\perp\vec{D}\Big)\quad .\end{array}\right.\tag{IV.47}$$

Using lemma VII.1 and in particular (VII.2), together with the fact that both $\nabla\vec{v}$ and $(\alpha-\overline{\alpha})\,\nabla\vec{D}$ are in L^2 we deduce using $l_\sigma\ \|\alpha-\overline{\alpha}\|_{L^\infty(D^2)}<\varepsilon$

$$\begin{aligned}\|\nabla\varphi\|_{L^{2,\infty}(D^2)} + \|\nabla\vec{\Psi}\|_{L^{2,\infty}(D^2)} &\le \varepsilon^{1/2}\ \|\nabla\vec{n}\|_{L^2(D^2)} + \ C\,\varepsilon^{1/2}\,l_\sigma\,\|\nabla\alpha\|^2_{L^2(D^2)}\\ +\,C\,\varepsilon^{1/2}\,l_\sigma\,|\overline{\alpha}|\int_{D^2_2}e^{2\mu}\ dx^2 + \varepsilon^{1/2}\ &\left[\int_{D^2}\sigma^2\,[e^{2\lambda}+|\nabla\vec{n}|^4e^{-2\lambda}]\ dx^2\right]^{3/4} + C\,l_\sigma\,\varepsilon^{1/2}\,\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}\quad .\end{aligned}\tag{IV.48}$$

Recall that for any harmonic function v the following holds

$$\left[\int_{D^2_{1/2}}|v|^2(x)\ dx^2\right]^{1/2} \le C\ \left[\int_{D^2}|v|^{4/3}\ dx^2\right]^{3/4}\quad .$$

Recall also the Hölder inequality in Lorentz spaces (see [20])

$$\forall f\in L^{2,\infty}(D^2)\qquad \|f\|_{L^{4/3}(D^2)}\le C\ \|f\|_{L^{2,\infty}(D^2)}\quad .$$

Observe that the triangular inequality gives

$$\begin{aligned}\|f\|_{L^{4/3}(D^2)} &\le \inf_{f=f_1+f_2}\|f_1\|_{L^{4/3}(D^2)} + \|f_2\|_{L^{4/3}(D^2)} \le \inf_{f=f_1+f_2}\|f_1\|_{L^{4/3}(D^2)} + \sigma^{-1/2}\|f_2\|_{L^{4/3}(D^2)}\\ &\le C\inf_{f=f_1+f_2}\|f_1\|_{L^{2,\infty}(D^2)} + \sigma^{-1/2}\|f_2\|_{L^{4/3}(D^2)} = C\,\|f\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}\quad .\end{aligned}$$

1 Since $S-\varphi$ and $\vec{R}-\vec{\Psi}$ are harmonic, we have using the two previous inequalities

$$\begin{array}{l}
\|\nabla(S-\varphi)\|_{L^{2,\infty}(D^2_{1/2})}+\|\nabla(\vec{R}-\vec{\Psi})\|_{L^{2,\infty}(D^2_{1/2})}\le C\,\|\nabla(S-\varphi)\|_{L^{4/3}(D^2)}+\|\nabla(\vec{R}-\vec{\Psi})\|_{L^{4/3}(D^2)}\\[3mm]
\quad\le C\,\|\nabla S\|_{L^{4/3}(D^2)}+C\,\|\nabla\vec{R}\|_{L^{4/3}(D^2)}+C\,\|\nabla\varphi\|_{L^{2,\infty}(D^2)}+C\,\|\nabla\vec{\Psi}\|_{L^{2,\infty}(D^2)}\\[3mm]
\quad\le C\,\|\nabla S\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}+C\,\|\nabla\vec{R}\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}+C\,\|\nabla\varphi\|_{L^{2,\infty}(D^2)}+C\,\|\nabla\vec{\Psi}\|_{L^{2,\infty}(D^2)}
\end{array}\tag{IV.49}$$

2 Hence

$$\begin{array}{l}
\|\nabla S\|_{L^{2,\infty}(D^2_{1/2})}+\|\nabla\vec{R}\|_{L^{2,\infty}(D^2_{1/2})}\le\|\nabla(S-\varphi)\|_{L^{2,\infty}(D^2_{1/2})}+\|\nabla(\vec{R}-\vec{\Psi})\|_{L^{2,\infty}(D^2_{1/2})}\\[3mm]
\quad\quad+\|\nabla\varphi\|_{L^{2,\infty}(D^2_{1/2})}+2\,\|\nabla\vec{\Psi}\|_{L^{2,\infty}(D^2_{1/2})}\\[3mm]
\le C\,\|\nabla S\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}+C\,\|\nabla\vec{R}\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}+C\,\|\nabla\varphi\|_{L^{2,\infty}(D^2)}+C\,\|\nabla\vec{\Psi}\|_{L^{2,\infty}(D^2)}\\[3mm]
\quad\quad\le C\,\|\nabla\vec{n}\|_{L^2(D^2)}+C\,l_\sigma\,\|\nabla\alpha\|^2_{L^2(D^2)}\\[3mm]
\quad\quad+C\,l_\sigma\,\,|\overline{\alpha}|\displaystyle\int_{D^2_2}e^{2\mu}\,\,dx^2+\left[\int_{D^2}\sigma^2\,[e^{2\lambda}+|\nabla\vec{n}|^4e^{-2\lambda}]\,\,dx^2\right]^{3/4}+C\,l_\sigma\,\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}\quad.
\end{array}\tag{IV.50}$$

3 Let φ_1 and $\vec{\Psi}_1$ the unique solutions in $W^{1,2}_0(D^2_{1/2})$ of the linear system

$$\left\{\begin{array}{l}
\Delta\varphi_1=-\nabla\vec{n}\cdot\nabla^\perp\vec{R}+\mathrm{div}\left(\vec{n}\cdot\nabla\vec{v}-\,2\,(\alpha-\overline{\alpha})\,\,\vec{n}\cdot\nabla^\perp\vec{D}\right)\\[3mm]
\Delta\vec{\Psi}_1=\nabla\vec{n}\times\nabla^\perp\vec{R}+\nabla\vec{n}\cdot\nabla^\perp S+\mathrm{div}\left(-\,\vec{n}\times\nabla\vec{v}+2\,(\alpha-\overline{\alpha})\,\,(-\nabla\vec{D}+\vec{n}\times\nabla^\perp\vec{D})\right)\quad.
\end{array}\right.\tag{IV.51}$$

4 Wente estimates, combined with (IV.42) and the pointwize bound $|\nabla\vec{D}|^2\le C\,l_\sigma^2\,|\nabla\vec{n}|^2$ give

$$\begin{array}{l}
\|\nabla\varphi_1\|_{L^2(D^2_{1/2})}+\|\nabla\vec{\Psi}_1\|_{L^2(D^2_{1/2})}\le\,\,C\,\|\nabla\vec{n}\|_{L^2(D^2_{1/2})}\,\,\left[\|\nabla S\|_{L^{2,\infty}(D^2_{1/2})}+\|\nabla\vec{R}\|_{L^{2,\infty}(D^2_{1/2})}\right]\\[3mm]
+\,C\,\|\nabla\vec{n}\|_{L^2(D^2)}\,\,\left[\displaystyle\int_{D^2}[|\nabla\vec{n}|^2+\sigma^2[e^{2\lambda}+e^{-2\lambda}|\nabla\vec{n}|^4]\,\,dx^2+C\,l_\sigma\,|\overline{\alpha}|\int_{D^2}e^{2\mu}\,\,dx^2+C\,l_\sigma\,\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}\right]\\[3mm]
+\,C\,l_\sigma\,\,\|\alpha-\overline{\alpha}\|_{L^\infty(D^2)}\,\,\|\nabla\vec{n}\|^2_{L^2(D^2)}\quad.
\end{array}\tag{IV.52}$$

5 Since $S-\varphi_1$ and $\vec{R}-\vec{\Psi}_1$ are harmonic we obtain finally

$$\begin{array}{l}
\|\nabla S\|_{L^2(D^2_{1/4})}+\|\nabla\vec{R}\|_{L^2(D^2_{1/4})}\le\|\nabla S\|_{L^{2,\infty}(D^2_{1/2})}+\|\nabla\vec{R}\|_{L^{2,\infty}(D^2_{1/2})}\\[3mm]
\quad\quad+2\,\|\nabla\varphi_1\|_{L^2(D^2_{1/2})}+2\,\|\nabla\vec{\Psi}_1\|_{L^2(D^2_{1/2})}\\[3mm]
\le C\,\|\nabla\vec{n}\|_{L^2(D^2)}+C\,l_\sigma\,\|\nabla\alpha\|^2_{L^2(D^2)}+C\,l_\sigma\,|\overline{\alpha}|\displaystyle\int_{D^2_2}e^{2\mu}\,\,dx^2\\[3mm]
\quad+\,C\,\left[\displaystyle\int_{D^2}\sigma^2\,[e^{2\lambda}+|\nabla\vec{n}|^4e^{-2\lambda}]\,\,dx^2\right]^{3/4}+C\,l_\sigma\,\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}\quad.
\end{array}\tag{IV.53}$$

Combining (IV.53) and (IV.45) gives then by changing 1 into 1/2

$$
\begin{aligned}
&\|\nabla S\|_{L^2(D^2_{1/8})}+\|\nabla\vec{R}\|_{L^2(D^2_{1/8})}\le C\,\|\nabla\vec{n}\|_{L^2(D^2)}+C\,l_\sigma\,\|\nabla\alpha\|^2_{L^2(D^2)}\\
&+C\,l_\sigma\,|\overline{\alpha}|\int_{D^2_2}e^{2\mu}\ dx^2+C\ \sigma^2\int_{D^2}[1+H^4]\,e^{2\lambda}\ dx^2+C\,l_\sigma\,\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}\quad .
\end{aligned}
\tag{IV.54}
$$

Hence ∇S and $\nabla\vec{R}$ are in $L^2(D^2_{1/8})$ and under the assumptions of the lemma, using also lemma III.11, we have that $\|\nabla S\|_{L^2(D^2_{1/8})}+\|\nabla\vec{R}\|_{L^2(D^2_{1/8})}$ are bounded by a constant depending only on C_1. We rescale the domain in such a way that ∇S and $\nabla\vec{R}$ are in $L^2(D^2)$.

Step 3. Uniform Morrey decrease of the Willmore energy. Precisely we are going to prove the existence of $\gamma>0$ independent of σ and independent of the solution such that

$$
\sup_{x_0\in D^2_{1/2}\ ;\ r<1/4}\quad r^{-\gamma}\int_{B^2_r(x_0)}H^2\,(1+\sigma^2(1+H^2))^2\ e^{2\lambda}\ dx^2\le C\quad . \tag{IV.55}
$$

Following step 1 of the proof of the theorem the map $\vec{U}:=(e^{-\overline{\lambda}}\vec{\Phi},\vec{v},S,\vec{R})$ satisfies the following system on $B_r(x_0)$

$$
\left\{
\begin{aligned}
&\Delta(e^{-\overline{\lambda}}\,\vec{\Phi})=h_\sigma\ e^{-\overline{\lambda}}\ \left[\nabla^\perp S\cdot\nabla\vec{\Phi}-\nabla\vec{R}\times\nabla^\perp\vec{\Phi}-\nabla\vec{v}\times\nabla\vec{\Phi}\right]\\
&\Delta\vec{v}=\nabla^\perp Y\cdot\nabla\vec{n}\\
&\Delta S=-\nabla\vec{n}\cdot\nabla^\perp\vec{R}+\mathrm{div}\left(\vec{n}\cdot\nabla\vec{v}-\ 2\,(\alpha-\overline{\alpha})\ \vec{n}\cdot\nabla^\perp\vec{D}\right)\\
&\Delta\vec{R}=\nabla\vec{n}\times\nabla^\perp\vec{R}+\nabla\vec{n}\cdot\nabla^\perp S+\mathrm{div}\left(-\vec{n}\times\nabla\vec{v}+2\,(\alpha-\overline{\alpha})\ (-\nabla\vec{D}+\vec{n}\times\nabla^\perp\vec{D})\right)\quad ,
\end{aligned}
\right.
\tag{IV.56}
$$

where $0\le h_\sigma:=2\,(1+2\,\sigma^2\,(1+H^2)-l_\sigma\,(\alpha-\overline{\alpha}))^{-1}\le 1$ and where we shall use the fact that

$$
\partial_{x_i}\vec{n}=-\,e^{-2(\lambda-\overline{\lambda})}\sum_{j=1}^2\vec{n}\cdot\partial^2_{x_ix_j}\left(e^{-\overline{\lambda}}\,\vec{\Phi}\right)\ e^{-\overline{\lambda}}\partial_{x_j}\vec{\Phi}\quad , \tag{IV.57}
$$

and that

$$
\nabla\vec{D}=l_\sigma\ e^{-2(\lambda-\overline{\lambda})}\left(\sum_{i=1}^2\vec{n}\cdot\partial^2_{x_1x_i}\left(e^{-\overline{\lambda}}\vec{\Phi}\right)\,e^{-\overline{\lambda}}\partial_{x_i}\vec{\Phi}\ ,\ \vec{n}\cdot\partial^2_{x_2x_i}\left(e^{-\overline{\lambda}}\vec{\Phi}\right)\,e^{-\overline{\lambda}}\partial_{x_i}\vec{\Phi}\right)\quad . \tag{IV.58}
$$

Let $\vec{w}$ in $W^{1,2}_0(B_r(x_0))$ the solutions of

$$
\Delta\vec{w}=\nabla^\perp Y\cdot\nabla\vec{n}\quad . \tag{IV.59}
$$

Using Wente estimates we obtain

$$
\int_{B_r(x_0)}|\nabla\vec{w}|^2\ dx^2\le C\ \|\nabla Y\|^2_{L^{2,\infty}(D^2)}\ \int_{B_r(x_0)}|\nabla\vec{n}|^2\ dx^2\quad . \tag{IV.60}
$$

Since $\vec{v}-\vec{w}$ is harmonic for any $t\in(0,1)$ the monotonicity formula for harmonic functions gives

$$
\int_{B_{t\,r}(x_0)}|\nabla(\vec{v}-\vec{w})|^2\ dx^2\le t^2\ \int_{B_r(x_0)}|\nabla(\vec{v}-\vec{w})|^2\ dx^2\quad . \tag{IV.61}
$$

We deduce from (IV.41), (IV.60) and (IV.61)

$$\int_{B_{t\,r}(x_0)} |\nabla\vec{v}|^2\ dx^2 \le t^2\ \int_{B_r(x_0)} |\nabla\vec{v}|^2\ dx^2 + C\ \sqrt{\varepsilon}\ \frac{\|\nabla Y\|^2_{L^{2,\infty}(D^2)}}{\sqrt{\varepsilon}} \int_{B_r(x_0)} |\nabla\vec{n}|^2\ dx^2 \quad . \tag{IV.62}$$

Let T and $\vec{Q}$ in $W^{1,2}_0(B_{t\,r}(x_0))$ solving

$$\left\{\begin{array}{l} \Delta T = -\nabla\vec{n}\cdot\nabla^\perp\vec{R} + \mathrm{div}\left(\vec{n}\cdot\nabla\vec{v} -\ 2\,(\alpha-\overline{\alpha})\ \vec{n}\cdot\nabla^\perp\vec{D}\right) \\[3mm] \Delta\vec{Q} = \nabla\vec{n}\times\nabla^\perp\vec{R} + \nabla\vec{n}\cdot\nabla^\perp S + \mathrm{div}\left(-\,\vec{n}\times\nabla\vec{v} + 2\,(\alpha-\overline{\alpha})\ (-\nabla\vec{D} + \vec{n}\times\nabla^\perp\vec{D})\right) \quad . \end{array}\right. \tag{IV.63}$$

Wente inequalities combined with classical elliptic estimates and the following facts : respectively

$$\int_{D^2} |\nabla\vec{n}|^2\ dx^2 + l_\sigma^2\ \|\alpha-\overline{\alpha}\|^2_{L^\infty(D^2)} < \varepsilon \quad \text{and} \quad |\nabla\vec{D}|^2 \le C\, l_\sigma^2\, |\nabla\vec{n}|^2$$

gives

$$\begin{aligned} \int_{B_{t\,r}(x_0)} [|\nabla T|^2 + |\nabla\vec{Q}|^2]\ dx^2 \le C\ \sqrt{\varepsilon}\ &\left[\int_{B_{t\,r}(x_0)} [|\nabla S|^2 + |\nabla\vec{R}|^2 + \delta^2\,|\nabla\vec{n}|^2\right]\ dx^2 \\ &+ C\int_{B_{t\,r}(x_0)} |\nabla\vec{v}|^2\ dx^2 \quad , \end{aligned} \tag{IV.64}$$

where

$$\delta^2 := \frac{l_\sigma^2\ \|\alpha-\overline{\alpha}\|^2_{L^\infty(D^2)} + \|\nabla Y\|^2_{L^{2,\infty}(D^2)}}{\sqrt{\varepsilon}} \quad .$$

Since $S-T$ and $\vec{R}-\vec{Q}$ are harmonic, the monotonicity formula gives

$$\begin{aligned} &\int_{B_{t^2\,r}(x_0)} \left[|\nabla(S-T)|^2 + |\nabla(\vec{R}-\vec{Q})|^2\right]\ dx^2 \\ &\quad \le t^2\ \int_{B_{t\,r}(x_0)} \left[|\nabla(S-T)|^2 + |\nabla(\vec{R}-\vec{Q})|^2\right]\ dx^2 \quad . \end{aligned} \tag{IV.65}$$

Hence combining (IV.62), (IV.64) and (IV.65) we obtain

$$\begin{aligned} \int_{B_{t^2\,r}(x_0)} \left[|\nabla S|^2 + |\nabla\vec{R}|^2 + |\nabla\vec{v}|^2\right]\ dx^2 \le\ & t^2 \int_{B_r(x_0)} \left[|\nabla S|^2 + |\nabla\vec{R}|^2 + |\nabla\vec{v}|^2\right] \\ &+ C\,\sqrt{\varepsilon}\ \int_{B_r(x_0)} \left[|\nabla S|^2 + |\nabla\vec{R}|^2 + \delta^2\,|\nabla\vec{n}|^2\right]\ dx^2 \quad . \end{aligned} \tag{IV.66}$$

We recall the structural equation (see [40])

$$\nabla\vec{n} = \nabla^\perp\vec{n}\times\vec{n} - 2\,H\,\nabla\vec{\Phi} \quad . \tag{IV.67}$$

Taking the divergence gives then

$$\Delta\vec{n} = \nabla^\perp\vec{n}\times\nabla\vec{n} - 2\,\mathrm{div}\left[2\,H\,\nabla\vec{\Phi}\right] \quad . \tag{IV.68}$$

We introduce $\vec{\xi}$ to be the solution of

$$\left\{\begin{array}{l}\Delta\vec{\xi}=\nabla^\perp\vec{n}\times\nabla\vec{n}-2\,\text{div}\left[2\,H\,\nabla\vec{\Phi}\right]\quad\text{in } B_r(x_0)\\[3mm] \vec{\xi}=0\qquad\qquad\text{on }\partial B_r(x_0)\quad.\end{array}\right. \tag{IV.69}$$

Classical elliptic estimates combined with the first equation of (IV.56) and the fact that $\|\nabla\vec{n}\|_2^2<\varepsilon$ gives

$$\int_{B_r(x_0)}|\nabla\vec{\xi}|^2\ dx^2\le C\int_{B_r(x_0)}\left[|\nabla S|^2+|\nabla\vec{R}|^2+|\nabla\vec{v}|^2\right]\ dx^2+C\,\varepsilon\int_{B_r(x_0)}|\nabla\vec{n}|^2\ dx^2\quad. \tag{IV.70}$$

Since $\vec{n}-\vec{\xi}$ is harmonic on $B_r(x_0)$ we have

$$\int_{B_{t\,r}(x_0)}|\nabla(\vec{n}-\vec{\xi})|^2\ dx^2\le t^2\int_{B_r(x_0)}|\nabla(\vec{n}-\vec{\xi})|^2\ dx^2\quad. \tag{IV.71}$$

Hence we have

$$\begin{array}{l}\displaystyle\int_{B_{t\,r}(x_0)}|\nabla\vec{n}|^2\ dx^2\le[t^2+C\,\varepsilon]\int_{B_r(x_0)}|\nabla\vec{n}|^2\ dx^2\\[3mm] \displaystyle\qquad+C\int_{B_r(x_0)}\left[|\nabla S|^2+|\nabla\vec{R}|^2+|\nabla\vec{v}|^2\right]\ dx^2\quad.\end{array} \tag{IV.72}$$

Inserting (IV.66) in (IV.72) we finally obtain, taking σ small enough in such a way that $\delta<1$

$$\begin{array}{l}\displaystyle\int_{B_{t^2\,r}(x_0)}\left[|\nabla S|^2+|\nabla\vec{R}|^2+|\nabla\vec{v}|^2+\delta^2\,|\nabla\vec{n}|^2\right]\ dx^2\\[3mm] \displaystyle\quad\le\ C\ [t^2+C\,\sqrt{\varepsilon}+\delta^2]\int_{B_r(x_0)}\left[|\nabla S|^2+|\nabla\vec{R}|^2+|\nabla\vec{v}|^2+\delta^2\,|\nabla\vec{n}|^2\right]\ dx^2\quad.\end{array} \tag{IV.73}$$

We fix $t>0$ and $\varepsilon>0$ independent of r, x_0, $\sigma<\sigma_0$ and the solution such that $C\ [t^2+C\,\sqrt{\varepsilon}]\le 1/2$. By classical iteration argument we deduce the existence of $\gamma>0$ such that

$$\begin{array}{l}\displaystyle\sup_{x_0\in D^2_{1/8}\ ;\ r<1/16}\quad r^{-\gamma}\int_{B^2_r(x_0)}\left[|\nabla S|^2+|\nabla\vec{R}|^2+|\nabla\vec{v}|^2+\delta^2\,|\nabla^2(e^{-\lambda}\vec{\Phi})|^2\right]\ dx^2\\[3mm] \displaystyle\quad\le\int_{D^2_{1/4}}\left[|\nabla S|^2+|\nabla\vec{R}|^2+|\nabla\vec{v}|^2+\delta^2\,|\nabla\vec{n}|^2\right]\ dx^2\quad.\end{array} \tag{IV.74}$$

Combining (IV.42), (IV.54) and (IV.74) we obtain in particular

$$\begin{array}{l}\displaystyle\sup_{x_0\in D^2_{1/8}\ ;\ r<1/16}\quad r^{-\gamma/2}\left[\int_{B^2_r(x_0)}\left[|\nabla S|^2+|\nabla\vec{R}|^2+|\nabla\vec{v}|^2\right]\ dx^2\right]^{1/2}\\[3mm] \displaystyle\quad\le C\,\|\nabla\vec{n}\|_{L^2(D^2)}+C\,l_\sigma\,\|\nabla\alpha\|^2_{L^2(D^2)}\\[3mm] \displaystyle\quad+C\,l_\sigma\,|\overline{\alpha}|\int_{D^2}e^{2\mu}\ dx^2+C\ \sigma^2\int_{D^2}[1+H^4]\,e^{2\lambda}\ dx^2+C\,l_\sigma\,\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\quad.\end{array} \tag{IV.75}$$

Combining (IV.75) and (IV.28) gives

$$
\begin{aligned}
&\sup_{x_0\in D^2_{1/8}\ ;\ r<1/16} r^{-\gamma}\int_{B^2_r(x_0)} H^2\left[1+\sigma^2\,(1+H^2)\right]^2\, dx^2]\ \ e^{2\lambda}\, dx^2\\
&\le C\int_{D^2}|\nabla\vec{n}|^2\, dx^2+C\left[l_\sigma\int_{D^2}|\nabla\alpha|^2\, dx^2\right]^2\\
&+C\,l_\sigma^2\,|\overline{\alpha}|^2\left[\int_{D^2}e^{2\mu}\, dx^2\right]^2+C\,\left[\sigma^2\int_{D^2}[1+H^4]\,e^{2\lambda}\, dx^2\right]^2+C\,l_\sigma^2\,\left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^2 \quad .
\end{aligned}
\tag{IV.76}
$$

Step 4. Bootstraping (IV.76). Lemma VII.4 applied to (IV.68) implies that

$$
\begin{aligned}
&\sup_{x_0\in D^2_{1/16}\ ;\ r<1/32} r^{-\gamma}\int_{B^2_r(x_0)}|\nabla\vec{n}|^2\, dx^2\\
&\le C\int_{D^2}|\nabla\vec{n}|^2\, dx^2+C\left[l_\sigma\int_{D^2}|\nabla\alpha|^2\, dx^2\right]^2\\
&+C\,l_\sigma^2\,|\overline{\alpha}|^2\left[\int_{D^2}e^{2\mu}\, dx^2\right]^2+C\,\left[\sigma^2\int_{D^2}[1+H^4]\,e^{2\lambda}\, dx^2\right]^2+C\,l_\sigma^2\,\left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^2 \quad .
\end{aligned}
\tag{IV.77}
$$

The Liouville equation reads as follows

$$
-\Delta\alpha=e^{2\lambda}\ K\ +\Delta\mu=e^{2\lambda}\ K-e^{2\mu}\ K_{g_0}\quad . \tag{IV.78}
$$

Thus

$$
\int_{B^2_r(x_0)}|\Delta\alpha|\le\int_{B^2_r(x_0)}|\nabla\vec{n}|^2\, dx^2+4\,\pi^2\,r^2\,\|e^{2\mu}\|_{L^\infty(D^2)}\quad . \tag{IV.79}
$$

Combining (IV.77), (IV.79) and Adams-Morrey embedding gives that

$$
\begin{aligned}
&\forall\ p<\tfrac{2-\gamma}{1-\gamma}\qquad\|\nabla\alpha\|_{L^p(D^2_{1/32})}\\
&\le\ C_p\sup_{x_0\in D^2_{1/16}\ ;\ r<1/32} r^{-\gamma}\int_{B^2_r(x_0)}|\nabla\vec{n}|^2+C\,\|e^{2\mu}\|_{L^\infty(D^2)}+\|\nabla\alpha\|_{L^2(D^2)}\quad .
\end{aligned}
\tag{IV.80}
$$

This gives in particular

$$
\begin{aligned}
&\left[\int_{D^2_{1/32}}[l_\sigma\ |\nabla\alpha|^2]^{p/2}\right]^{2/p}\le\ C\,l_\sigma\left[\sup_{x_0\in D^2_{1/16}\ ;\ r<1/32} r^{-\gamma}\int_{B^2_r(x_0)}|\nabla\vec{n}|^2\, dx^2\right]^2\\
&\quad+C\,l_\sigma\,\|e^{2\mu}\|^2_{L^\infty(D^2)}+\ C\,l_\sigma\int_{D^2}|\nabla\alpha|^2\,dx^2\quad .
\end{aligned}
\tag{IV.81}
$$

The equation (IV.7) gives

$$
\begin{aligned}
&\Delta\left(2\,\vec{H}(1+2\,\sigma^2\,(1+H^2))\right)=\mathrm{div}\Big(2\,l_\sigma f'_\sigma(H)\,\nabla\vec{n}+2\ e^{-2\lambda}\,l_\sigma\,\nabla\vec{\Phi}\cdot\nabla\alpha\ \nabla\alpha\\
&\quad-l_\sigma\ [-2\,f_\sigma(H)+e^{-2\lambda}\,|\nabla\alpha|^2-K_{g_0}\,\alpha\ e^{-2\alpha}+K_{g_0}\,A_{\vec{\Phi}}(\Sigma)^{-1}]\ \nabla\vec{\Phi}-2\,l_\sigma\ e^{-2\lambda}\,(\vec{\mathbb{I}}\llcorner\nabla^\perp\alpha)^\perp\Big)\quad .
\end{aligned}
\tag{IV.82}
$$

This gives that $\vec{V} := e^{\overline{\lambda}}\, 2\, \vec{H}(1 + 2\,\sigma^2\,(1 + H^2))$ satisfies an equation of the form

$$\Delta \vec{V} = \mathrm{div}(\vec{I} + \vec{J}) \quad , \tag{IV.83}$$

where

$$\vec{I} := 2\; l_\sigma\; e^{\overline{\lambda}}\, f'_\sigma(H)\, \nabla\vec{n} + 2\, l_\sigma\; f_\sigma(H)\; \nabla\vec{\Phi} + l_\sigma\, K_{g_0}\, \alpha\; e^{\overline{\lambda}-2\alpha}\; \nabla\vec{\Phi} \quad .$$

Thanks to (IV.76) we have in one hand

$$\begin{array}{l}\displaystyle\sup_{x_0\in D^2_{1/32}\;;\;r<1/4} \quad r^{-\gamma}\int_{B^2_r(x_0)} |\vec{I}|\; dx^2 \le C\int_{D^2}|\nabla\vec{n}|^2\; dx^2 + C\,\left[l_\sigma\int_{D^2}|\nabla\alpha|^2\; dx^2\right]^2\\[4mm] \displaystyle\quad +C\, l_\sigma^2\,|\overline{\alpha}|^2\;\|e^{4\mu}\|_{L^\infty(D^2)} + C\;\left[\sigma^2\int_{D^2}[1+H^4]\, e^{2\lambda}\; dx^2\right]^2 + C\, l^2_\sigma\;\left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^2\end{array} \tag{IV.84}$$

where γ and C are independent of the solution and of σ but depend only on the constant C_1 in the statement of the lemma.

In the other hand, using the fact that $2-\gamma/(1-\gamma) > 2$ and using (IV.81) we deduce the existence of $q > 1$ such that

$$\begin{array}{l}\displaystyle\left[\int_{D^2_{1/32}}|\vec{J}|^q\; dx^2\right]^{1/q} \le\; C\, l_\sigma\;\left[\sup_{x_0\in D^2_{1/16}\;;\;r<1/32}\quad r^{-\gamma}\int_{B^2_r(x_0)}|\nabla\vec{n}|^2\; dx^2\right]^2\\[4mm] \displaystyle\quad +C\,[l_\sigma\,\|e^{2\mu}\|_{L^\infty(D^2)} + l_\sigma\,|\overline{\alpha}|]\;\|e^{2\mu}\|_{L^\infty(D^2)} +\; C\, l_\sigma\int_{D^2}|\nabla\alpha|^2\, dx^2\end{array} \tag{IV.85}$$

Using (III.72) and classical Adams-Sobolev inequalities (see [40]) give then the existence of $p > 2$ such that

$$\begin{array}{l}\displaystyle 2\,\|e^{\overline{\lambda}}\,\vec{H}(1+2\,\sigma^2\,(1+H^2))\|^2_{L^p(D_{1/64})} = \|V\|^2_{L^p(D_{1/64})}\\[4mm] \displaystyle\le\; C\int_{D^2}|\nabla\vec{n}|^2\; dx^2 + C\;\left[l_\sigma\int_{D^2}|\nabla\alpha|^2\; dx^2\right]^2 + C\;\left[\sigma^2\int_{D^2}H^4\, e^{2\lambda}\; dx^2\right]^2\\[4mm] \displaystyle\quad +C\,[l^2_\sigma\,|\overline{\alpha}|^2\; + l^2_\sigma\,\|e^{4\mu}\|_{L^\infty(D^2)}]\;\|e^{4\mu}\|_{L^\infty(D^2)} + C\, l^2_\sigma\;\left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^2\end{array} \tag{IV.86}$$

where we have used (III.74), (III.75) .

Bootstraping this information respectively in the three elliptic systems (IV.68), (IV.78) and (IV.82) (which are now becoming sub-critical for $\vec{V} \in L^p$ with $p > 2$) one obtains (IV.5), (IV.6) and lemma IV.1 is proved. □

V The Palais-Smale Condition for the Frame Energies

V.1 Sequential weak compactness of weak Immersions in $\mathcal{E}_{\Sigma,2}$ with uniformly bounded Frame Energies.

In this section we are working with the lagrangian F^σ defined in the previous section but the parameter σ will be fixed all along the section. So, in order to simplify the presentation, we will simply work with the following lagrangian

$$F(\vec{\Phi}) := \int_\Sigma \left[H^2 + \left[(1+H^2)^2 + 2^{-1}\,|d\alpha|^2_{g_{\vec{\Phi}}} + K_{g_0}\,\alpha\, e^{-2\alpha}\right]\right]\; dvol_{g_{\vec{\Phi}}} - 2^{-1}\, K_{g_0}\;\log(\mathrm{Area}(\vec{\Phi}(\Sigma))) \quad , \tag{V.1}$$

where, as before g_0 is a constant Gauss curvature metric of volume 1 on Σ, $g_{\vec{\Phi}} = e^{2\alpha}\, g_0$. The following lemma was proved in [35] when Σ is the torus. We extend it now to general surfaces following the same line of ideas as in [35].

Lemma V.1. *Let Σ be a closed surface and $\vec{\Phi}^k$ be a sequence of weak immersions in $\mathcal{E}_{\Sigma,2}$ satisfying*

$$\limsup_{k\to+\infty} F(\vec{\Phi}^k) < +\infty \quad , \tag{V.2}$$

then the conformal class of the associated sequence g_0^k of constant scalar curvature metric of volume 1 such that $g_{\vec{\Phi}^k} = e^{2\alpha^k}\, g_0^k$ is pre-compact in the moduli space $\mathcal{M}(\Sigma)$ moreover, there exists a sequence of diffeomorphisms Ψ^k of Σ such that $(\Psi^k)^* g_0^k$ is converging strongly in any C^l topology to a limiting constant curvature metric h, $\vec{\Phi}^k \circ \Psi^k$ is conformal from $(\Sigma, (\Psi^k)^* g_0^k)$ and is sequentially weakly pre-compact in $W^{2,4}$ and for any weakly converging sub-sequence the limit $\vec{\xi}^\infty$ is still a weak immersion of $\mathcal{E}_{\Sigma,2}$ and

$$\log |d(\vec{\Phi}^k \circ \Psi^k)|^2_{(\Psi^k)^* g_0^k} \rightharpoonup \log |d\vec{\xi}^\infty|^2_h \qquad \textit{weakly in } W^{1,4}_h \quad . \tag{V.3}$$

□

Proof of lemma V.1. We are working with a Aubin Gauge in the case $K_{g_0} > 0$. Using respectively lemma III.12 for $K_{g_0} < 0$ or lemma III.11 for $K_{g_0} > 0$ we have in all cases

$$\limsup_{k\to+\infty} \int_\Sigma (1 + H^2_{\vec{\Phi}^k})^2 \; dvol_{g_{\vec{\Phi}^k}} < +\infty \quad . \tag{V.4}$$

Hence using again (III.81) for genus$(\Sigma) > 1$ or lemma III.11 for $K_{g_0} > 0$ gives in all cases

$$\limsup_{k\to+\infty} \int_\Sigma |d\alpha^k|^2_{g_0^k} \; dvol_{g_0^k} < +\infty \quad . \tag{V.5}$$

Moreover since Σ is a closed surface we have also using Willmore-Li-Yau universal Lower Bound of the Willmore energy (see [40])

$$16\pi^2 \le \left[\int_\Sigma e^{2\alpha^k} \; dvol_{g_0^k}\right] \; \left[\int_\Sigma H^4_{\vec{\Phi}^k} \; dvol_{g_{\vec{\Phi}^k}}\right] \quad . \tag{V.6}$$

Hence, combining (V.4) and (V.6) we then have proved in all cases

$$\limsup_{k\to+\infty} \left|\log\left(\text{Area}(\vec{\Phi}^k(\Sigma))\right)\right| < +\infty \quad . \tag{V.7}$$

These preliminary estimates having been established we now prove the pre-compactness of the conformal class in the non zero genus case. The case when Σ is a torus has already been considered in [35]. So we can restrict to the case where genus$(\Sigma) > 1$. Assume the conformal class associated to $g_{\vec{\Phi}^k}$ and hence g_0^k would degenerate we have a rather precise description of this degeneration (see [23]). It requires the formation of at least a *collar* which is a subdomain of Σ diffeomorphic to an annulus that we identify to a cylinder of the form

$$\mathcal{C} := \left\{(x_1, x_2) \; ; \; \frac{2\pi}{l^k}\varphi^k < x_2 < \frac{2\pi}{l^k}(\pi - \varphi^k) \; ; \; 0 \le x_1 \le 2\pi\right\}$$

where the vertical lines $x_1 = 0$ and $x_1 = 2\pi$ are identified, l^k is the length of a closed geodesic for the hyperbolic metric g_0^k

$$l^k \longrightarrow 0$$

and $\varphi^k := \arctan(\sinh(l^k/2))$. The closed geodesic of length l^k is given by $x_2 = \pi^2/l^k$. On this cylinder the hyperbolic metric g_0^k has the following explicit expression

$$g_0^k = \left(\frac{l^k}{2\pi\ \sin\left(\frac{l^k x_2}{2\pi}\right)} \right)^2\ [dx_1^2 + dx_2^2]$$

Denote in these coordinates

$$g_{\vec{\Phi}^k} := e^{2\,\lambda^k}\ [dx_1^2 + dx_2^2] \quad \text{ and } \quad {\vec{e}_i}^{\,k} := e^{-\lambda^k}\ \partial_{x_i}\vec{\Phi}^k$$

The unit vector field ${\vec{e}_1}^{\,k}$ is tangent to a foliation of the image in $\mathbb{R}^3$ by $\vec{\Phi}^k$ of the collar region by circles. We apply Fenchel theorem to each of these circles. Precisely for each $t \in (\frac{2\pi}{l^k}\varphi^k, \frac{2\pi}{l^k}(\pi - \varphi^k))$ we have

$$2\pi \le \int_{\{x_2=t\}\cap\mathcal{C}} \left| \frac{\partial {\vec{e}_1}^{\,k}}{\partial x_1} \right|\ dx_1 \tag{V.8}$$

Integrating this inequality for x_2 between $\frac{2\pi}{l^k}\varphi^k$ and $\frac{2\pi}{l^k}(\pi-\varphi^k)$ and using Cauchy-Schwartz gives

$$\frac{(2\pi)^2}{l^k}\ (\pi - 2\,\varphi^k) \le \frac{2\pi}{\sqrt{l^k}}\ \sqrt{\pi - 2\,\varphi^k}\ \left[\int_{\mathcal{C}} |\nabla {\vec{e}_1}^{\,k}|^2\ dx^2\right]^{1/2} \tag{V.9}$$

We have $|\nabla {\vec{e}_1}^{\,k}|^2 \le |\nabla \vec{n}_{\vec{\Phi}^k}|^2 + |\nabla\lambda^k|^2$ moreover $\lambda^k = \alpha^k + \mu^k$ where

$$\mu^k = 2\ \log\left(\frac{l^k}{2\pi}\right) - 2\ \log\left(\sin\left(\frac{l^k\ x_2}{2\pi}\right)\right)$$

Thus

$$|\nabla\mu^k|^2 = \frac{(l^k)^2}{\pi^2}\ \frac{\cos^2\left(\frac{l^k\ x_2}{2\pi}\right)}{\sin^2\left(\frac{l^k\ x_2}{2\pi}\right)}$$

Hence

$$\int_{\mathcal{C}} |\nabla\mu^k|^2\ dx^2 \le \frac{(l^k)^2}{\pi^2} \int_{\frac{2\pi\varphi^k}{l^k}}^{\frac{2\pi\,(\pi-\varphi^k)}{l^k}} \frac{dx_2}{\sin^2\left(\frac{l^k\ x_2}{2\pi}\right)} \le\ C\ (l^k)^2 \int_{\frac{2\pi\varphi^k}{l^k}}^{\infty} \frac{dx_2}{(l^k)^2\,x_2^2} \le C \tag{V.10}$$

and since from (V.5) we have that

$$\limsup_{k\rightarrow+\infty} \int_{\mathcal{C}} |\nabla\alpha^k|^2\ dx^2 = \limsup_{k\rightarrow+\infty} \int_{\mathcal{C}} |d\alpha^k|^2_{g_{\vec{\Phi}^k}}\ dvol_{g_{\vec{\Phi}^k}} < +\infty$$

we deduce that

$$\limsup_{k\rightarrow+\infty} \int_{\mathcal{C}} |\nabla {\vec{e}_1}^{\,k}|^2\ dx^2 < +\infty \tag{V.11}$$

Combining (V.9) and (V.11) gives that l^k is bounded from below by a positive number which contradicts the formation of a collar and the degeneracy of the conformal class of $[g_0^k]$. Modulo the composition with isometries g_0^k is strongly converging in every Banach space $C^l(\Sigma)$. In order to simplify the presentation we assume that g_0^k is fixed. We cover the riemannian surface (Σ, g_0) by finitely many conformal charts

$\phi_j\ :\ D^2\ \to\ \phi_j(D^2)$ for $j\in J$ such that $\Sigma\subset\cup_j\phi_j(D^2_{1/2})$. Denote again $\phi_j^*g_{\vec{\Phi}^k}=e^{2\lambda_j^k}\,[dx_1^2+dx_2^2]$ the expression of $g_{\vec{\Phi}^k}$ in the chart ϕ_j. We have

$$\limsup_{k\to+\infty}\int_{D^2}|\nabla\vec{n}_{\vec{\Phi}^k\circ\phi_j}|^2+|\nabla\lambda_j^k|^2\ dx^2<+\infty\quad . \tag{V.12}$$

For $i=1,2$ we denote $\vec{e}_{j,i}^{\,k}:=e^{-\lambda_j^k}\partial_{x_i}(\vec{\Phi}^k\circ\phi_j)$ and Liouville equation gives

$$-\Delta\lambda_j^k=(\nabla\vec{e}_{j,1}^{\,k}\cdot\nabla^\perp\vec{e}_{j,2}^{\,k})\qquad\text{in } D^2\quad .$$

Inequality (V.12) implies

$$\limsup_{k\to+\infty}\int_{D^2}|\nabla\vec{e}_{j,i}^{\,k}|^2\ dx^2<+\infty\quad . \tag{V.13}$$

Combining (V.12) and (V.13) together with Wente estimates we obtain the existence of $\overline{\lambda_j}^k\in\mathbb{R}$ such that

$$\limsup_{k\to+\infty}\|\lambda_j^k-\overline{\lambda_j}^k\|_{L^\infty(D^2_{3/4})}<+\infty\quad . \tag{V.14}$$

Due to the connectedness of Σ we deduce that

$$\sup_{j\neq l}\limsup_{k\to+\infty}\|\overline{\lambda_l}^k-\overline{\lambda_j}^k\|_{L^\infty(D^2_{3/4})}<+\infty\quad . \tag{V.15}$$

We have moreover

$$\limsup_{k\to+\infty}\int_{D^2}e^{2\,\lambda_j^k}\ dx^2<+\infty\quad . \tag{V.16}$$

We have

$$\begin{aligned}4\pi&\le\sum_{j\in J}\int_{D^2_{1/2}}H^2_{\vec{\Phi}^k\circ\phi_j}\ e^{2\,\lambda_j^k}\ dx^2\\&\le\sum_{j\in J}\left(\int_{D^2_{1/2}}e^{2\,\lambda_j^k}\ dx^2\right)^{1/2}\ \left(\int_{D^2_{1/2}}H^4_{\vec{\Phi}^k\circ\phi_j}\ e^{2\,\lambda_j^k}\ dx^2\right)^{1/2}\quad .\end{aligned} \tag{V.17}$$

Since

$$\limsup_{k\to+\infty}\int_{D^2_{1/2}}H^4_{\vec{\Phi}^k\circ\phi_j}\ e^{2\,\lambda_j^k}\ dx^2<+\infty\quad , \tag{V.18}$$

we deduce from (V.15), (V.16) and (V.17) that

$$\max_{j\in J}\limsup_{k\to+\infty}\|\lambda_j^k\|_{L^\infty(D^2_{3/4})}<+\infty\quad . \tag{V.19}$$

Using

$$\Delta(\vec{\Phi}^k\circ\phi_j)=e^{2\,\lambda_j^k}\ H_{\vec{\Phi}^k\circ\phi_j}$$

together with (V.18) we deduce that $\vec{\Phi}^k\circ\phi_j$ is sequentially weakly pre-compact in $W^{2,4}(D^2_{1/2})$. Bootstrapping this information with (V.19) gives (V.3) and lemma V.1 is proved. □

V.2 The Finsler Structure on the Space of $W^{2,4}-$immersions

While aiming to apply *Palais Deformation Theory* we are going to equip $\mathcal{E}_{\Sigma,2}$, the space of $W^{2,4}-$immersions of a *Finsler structure* $\|\cdot\|_{\vec{\Phi}}$ given in [44] for which the metric space given by the *Palais distance* is complete.

V.3 The Palais-Smale Condition.

The aim of the present subsection is to establish the following lemma which was proved for the exact Willmore functional in [6].

Lemma V.2. *Let Σ be a closed surface and $\vec{\Phi}^k$ be a sequence of weak immersions in $\mathcal{E}_{\Sigma,2}$ satisfying*

$$\limsup_{k\rightarrow+\infty} F(\vec{\Phi}^k)<+\infty \quad , \tag{V.20}$$

where F is given by (IV.1) and such that we have

$$\lim_{k\rightarrow+\infty}\sup_{\|\vec{w}\|_{\vec{\Phi}^k}\le 1}\partial F(\vec{\Phi}^k)\cdot\vec{w}=0 \tag{V.21}$$

where for any $\vec{\Phi}\in\mathcal{E}_{\Sigma,2}$ we denote

$$\|\vec{w}\|^4_{\vec{\Phi}}:=\int_\Sigma\left[|\nabla^{g_{\vec{\Phi}}}d\vec{w}|^4_{g_{\vec{\Phi}}}+|d\vec{w}|^4_{g_{\vec{\Phi}}}+|w|^4\right]\,dvol_{g_{\vec{\Phi}}} \quad .$$

Then, modulo extraction of a subsequence, there exists a sequence of parametrization Ψ^k such that $\vec{\Phi}^k\circ\Psi^k$ strongly converges in $W^{2,4}$ towards a critical point of F in $\mathcal{E}_{\Sigma,2}$ □

Proof of lemma V.2. We are taking an Aubin Gauge for the S^2 case. We denote $f(t):=t^2+(1+t^2)^2$. We consider the charts ϕ^j given by the previous lemma and we omit to mention the index j. We also skip writing the index k when it is not absolutely needed. In each of this chart the information given by (V.21) is saying that

$$\begin{array}{l}\vec{G}:=\mathrm{div}\Big[\nabla(f'(H)\,\vec{n})-2\,f'(H)\,\nabla\vec{n}-2\ e^{-2\lambda}\,\nabla\vec{\Phi}\cdot\nabla\alpha\ \nabla\alpha\\[2mm] \quad+\ \ [-2\,f(H)+e^{-2\lambda}\,|\nabla\alpha|^2-K_{g_0}\,\alpha\ e^{-2\alpha}+K_{g_0}\,A_{\vec{\Phi}}(\Sigma)^{-1}]\ \nabla\vec{\Phi}-2\ e^{-2\lambda}\ \left(\vec{\mathbb{I}}\llcorner\nabla^\perp\alpha\right)^\perp\Big]\\[2mm] \quad\longrightarrow 0 \qquad \text{strongly in } W^{-2,4/3}(D^2)=(W^{2,4}_0(D^2))^*\end{array} \tag{V.22}$$

Let

$$\begin{array}{l}\vec{\phi}_k:=-\dfrac{1}{2\pi}\dfrac{1}{r}\dfrac{\partial}{\partial r}\star\Big[-2\,f'(H)\,\nabla\vec{n}-2\ e^{-2\lambda}\,\nabla\vec{\Phi}\cdot\nabla\alpha\ \nabla\alpha\\[2mm] \quad+\ \ [-2\,f(H)+e^{-2\lambda}\,|\nabla\alpha|^2-K_{g_0}\,\alpha\ e^{-2\alpha}+K_{g_0}\,A_{\vec{\Phi}}(\Sigma)^{-1}]\ \nabla\vec{\Phi}-2\ e^{-2\lambda}\ \left(\vec{\mathbb{I}}\llcorner\nabla^\perp\alpha\right)^\perp\Big]\ \mathbf{1}_{D^2} \quad .\end{array}$$

This convolution is justified by observing that $\frac{1}{r}\frac{\partial}{\partial r}\in L^{2,\infty}(\mathbb{R}^2)$ and the other term in the convolution is uniformly bounded in $L^1(\mathbb{R}^2)$. Hence we have

$$\limsup_{k\rightarrow+\infty}\|\vec{\phi}_k\|_{2,\infty}<+\infty \quad .$$

We have obviously

$$\left\{\begin{array}{l}\vec{G}_k=\Delta(f'(H_k)\,\vec{n}_k+\vec{\phi}_k)\longrightarrow 0 \qquad \text{strongly in } W^{-2,4/3}(D^2)=(W^{2,4}_0(D^2))^*\\[3mm] \text{and}\quad \displaystyle\limsup_{k\rightarrow+\infty}\|f'(H_k)\,\vec{n}_k+\vec{\phi}_k\|_{L^{4/3}(D^2)}<+\infty \quad .\end{array}\right.$$

Let $\vec{h}$ be a weak limit (modulo extraction of a subsequence) of $f'(H_k)\,\vec{n}_k+\vec{\phi}_k$ in $L^{4/3}(D^2)$. It is obviously harmonic an we have using Rellich Kondrachov

$$f'(H_k)\,\vec{n}_k+\vec{\phi}_k \longrightarrow \vec{h} \qquad \text{strongly in } L^{4/3}_{loc}(D^2) \quad .$$

Let $\vec{M}_k := f'(H_k)\,\vec{n}_k+\vec{\phi}_k-\vec{h}$. We have

$$\vec{G}_k=\Delta\vec{M}_k \quad \text{ and } \quad \vec{M}_k \longrightarrow 0 \qquad \text{strongly in } L^{4/3}_{loc}(D^2) \quad . \tag{V.23}$$

Applying Poincaré Lemma we obtain the existence of $\vec{L}$ such that

$$\begin{aligned}\nabla\vec{L} &:= \nabla^\perp(f'(H)\,\vec{n})-2\,f'(H)\,\nabla^\perp\vec{n}-2\;e^{-2\lambda}\,\sigma^2\nabla\vec{\Phi}\cdot\nabla\alpha\;\nabla^\perp\alpha\\ &\quad+[-2\,f(H)+e^{-2\lambda}\,|\nabla\alpha|^2-K_{g_0}\,\alpha\;e^{-2\alpha}+K_{g_0}\,A_{\vec{\Phi}}(\Sigma)^{-1}]\;\nabla^\perp\vec{\Phi}\\ &\quad+2\;\;e^{-2\lambda}\,\vec{\mathbb{I}}\llcorner\nabla^\perp\alpha-\nabla^\perp\vec{M} \quad .\end{aligned} \tag{V.24}$$

Equation (III.53) gives

$$\nabla\vec{\Phi}\cdot\nabla^\perp\vec{L}-\text{div}\left[\nabla\vec{\Phi}\cdot\vec{M}\right]=2\;\;e^{2\lambda}\,(2\,f(H)-\;H\,f'(H)+K_{g_0}\,\alpha\,e^{-2\alpha}-\;K_{g_0}\,A_{\vec{\Phi}}(\Sigma)^{-1})-\vec{M}\cdot\Delta\vec{\Phi} \quad . \tag{V.25}$$

Let Y be the solution of

$$\left\{\begin{array}{l}\Delta Y=2\,e^{2\lambda}\,(2\,f(H)-\;H\,f'(H)+K_{g_0}\,\alpha\,e^{-2\alpha}-K_{g_0}\,A_{\vec{\Phi}}(\Sigma)^{-1})-\vec{M}\cdot\Delta\vec{\Phi} \quad \text{ in } D^2\\[3mm] Y=0 \qquad \text{on } \partial D^2 \quad .\end{array}\right. \tag{V.26}$$

Observe that $2\,f(H)-\;H\,f'(H)=2\,(1-H^4)$. So Y satisfies

$$\left\{\begin{array}{l}-\Delta Y=4\,e^{2\lambda}\,(1-H^4)+2\,K_{g_0}\,\alpha\,e^{2\mu}-2\,K_{g_0}\,A_{\vec{\Phi}}(\Sigma)^{-1}-\vec{M}\cdot\Delta\vec{\Phi} \quad \text{ in } D^2\\[3mm] Y=0 \qquad \text{on } \partial D^2 \quad .\end{array}\right. \tag{V.27}$$

Since $\Delta\vec{\Phi}$ is uniformly bounded in L^4 we have $\vec{M}\cdot\Delta\vec{\Phi}\rightarrow 0$ strongly in L^1. Hence

$$\limsup_{k\rightarrow+\infty}\|\nabla Y_k\|_{L^{2,\infty}(D^2)}<+\infty \quad . \tag{V.28}$$

Using Poincaré theorem we deduce the existence of a function S such that

$$\nabla S=\vec{L}\cdot\nabla\vec{\Phi}-\vec{M}\cdot\nabla^\perp\vec{\Phi}+\nabla^\perp Y \quad . \tag{V.29}$$

The equation (III.63) in conformal coordinates gives

$$\nabla\vec{\Phi}\times\left[\nabla^\perp\vec{L}-\nabla\vec{M}\right]=-\,\nabla^\perp\vec{\Phi}\cdot\nabla f'(H)+2\,\nabla\alpha\cdot\nabla^\perp\vec{D} \quad , \tag{V.30}$$

where

$$\nabla\vec{D}=\left(\;e^{-2\lambda}\sum_{i=1}^2\mathbb{I}_{1i}\,\partial_{x_i}\vec{\Phi}\;,\;\;e^{-2\lambda}\sum_{i=1}^2\mathbb{I}_{2i}\,\partial_{x_i}\vec{\Phi}\right) \quad . \tag{V.31}$$

Let $\vec{W}$ be the solution of

$$\begin{cases} \Delta\vec{W} = \vec{M} \times \Delta\vec{\Phi} & \text{in } D^2 \\ \\ \vec{W} = 0 & \text{on } \partial D^2 \quad . \end{cases} \tag{V.32}$$

Since $\Delta\vec{\Phi}$ is uniformly bounded in L^4 we have $\vec{M} \times \Delta\vec{\Phi} \to 0$ strongly in L^1. Hence

$$\lim_{k\to+\infty} \|\nabla\vec{W}_k\|_{L^{2,\infty}(D^2)} = 0 \quad . \tag{V.33}$$

Using again Poincaré Lemma on D^2 we obtain the existence of $\vec{V}$ such that

$$\nabla\vec{V} := \vec{L} \times \nabla\vec{\Phi} - \vec{M} \times \nabla^\perp\vec{\Phi} + f'(H)\,\nabla\vec{\Phi} - 2\,(\alpha - \overline{\alpha})\,\nabla\vec{D} + \nabla^\perp\vec{W} \quad . \tag{V.34}$$

Using the explicit expression of $\nabla\vec{D}$ given by (III.64) we obtain

$$\begin{aligned} \vec{n} \cdot (\nabla\vec{V} - \nabla^\perp\vec{W}) &= \vec{n} \cdot (\vec{L} \times \nabla\vec{\Phi} - \vec{M} \times \nabla^\perp\vec{\Phi}) \\ &= \vec{L} \cdot \nabla^\perp\vec{\Phi} + \vec{M} \cdot \nabla\vec{\Phi} = \nabla^\perp S + \nabla Y \quad . \end{aligned} \tag{V.35}$$

We have also

$$\begin{aligned} \vec{n} \times (\nabla\vec{V} - \nabla^\perp\vec{W}) = &- (\vec{L} \cdot \vec{n})\,\nabla\vec{\Phi} + (\vec{n} \cdot \vec{M})\nabla^\perp\vec{\Phi} \\ &- f'(H)\,\nabla^\perp\vec{\Phi} - 2\,(\alpha - \overline{\alpha})\,\vec{n} \times \nabla\vec{D} \quad . \end{aligned} \tag{V.36}$$

Denote $\pi_T(\nabla^\perp\vec{V} + \nabla\vec{W})$ the tangential projection of $\nabla^\perp\vec{V} + \nabla\vec{W}$, we have

$$\begin{aligned} \pi_T(\nabla^\perp\vec{V} + \nabla\vec{W}) = &(\vec{L} \cdot \vec{n})\,\nabla\vec{\Phi} - (\vec{n} \cdot \vec{M})\nabla^\perp\vec{\Phi} \\ &+ f'(H)\,\nabla^\perp\vec{\Phi} - 2\,(\alpha - \overline{\alpha})\,\nabla^\perp\vec{D} \quad . \end{aligned} \tag{V.37}$$

Hence

$$\vec{n} \times (\nabla\vec{V} - \nabla^\perp\vec{W}) = -\nabla^\perp\vec{V} - \nabla\vec{W} - \,2\,(\alpha - \overline{\alpha})\, \left(\nabla^\perp\vec{D} + \vec{n} \times \nabla\vec{D}\right) - \vec{n}\,(\nabla S - \nabla^\perp Y) \quad . \tag{V.38}$$

Let $\vec{v}$ be the unique solution to

$$\begin{cases} \Delta\vec{v} = \nabla^\perp Y \cdot \nabla\vec{n} & \text{in } D^2 \\ \\ \vec{v} = 0 & \text{on } \partial D^2 \quad . \end{cases} \tag{V.39}$$

Using one more time Poincaré lemma we obtain the existence of $\vec{u}$ such that

$$\vec{n}\,\nabla^\perp Y = \nabla\vec{v} + \nabla^\perp\vec{u} \quad . \tag{V.40}$$

Finally, let $\vec{R} := \vec{V} - \vec{u}$. We have

$$\vec{n} \times \nabla\vec{V} = \vec{n} \times \nabla\vec{R} + \vec{n} \times \nabla\vec{u} = \vec{n} \times \nabla\vec{R} + \vec{n} \times \nabla^\perp\vec{v} \quad . \tag{V.41}$$

Hence (V.38) becomes

$$\begin{aligned} \vec{n} \times (\nabla\vec{R} - \nabla^\perp\vec{W}) + \vec{n} \times \nabla^\perp\vec{v} = &-\nabla^\perp\vec{R} - \nabla\vec{W} + \nabla\vec{v} \\ &-\vec{n}\,\nabla S - \,2\,(\alpha - \overline{\alpha})\, \left(\nabla^\perp\vec{D} + \vec{n} \times \nabla\vec{D}\right) \quad , \end{aligned} \tag{V.42}$$

which gives

$$\left\{\begin{array}{l}\Delta S=-\nabla\vec{n}\cdot\nabla^{\perp}\vec{R}+\mathrm{div}\left(\vec{n}\cdot\nabla\vec{z}-\,2\,(\alpha-\overline{\alpha})\;\vec{n}\cdot\nabla^{\perp}\vec{D}\right)\\[5mm] \Delta\vec{R}=\nabla\vec{n}\times\nabla^{\perp}\vec{R}+\nabla\vec{n}\cdot\nabla^{\perp}S+\mathrm{div}\left(-\,\vec{n}\times\nabla\vec{z}+2\,(\alpha-\overline{\alpha})\;(-\nabla\vec{D}+\vec{n}\times\nabla^{\perp}\vec{D}\right)\quad ,\end{array}\right. \tag{V.43}$$

where $\vec{z}:=\vec{v}-\vec{W}$. Let $\vec{U}:=\vec{R}+2\,(\alpha-\overline{\alpha})\,\vec{D}$, with this notation (V.43) becomes

$$\left\{\begin{array}{l}\Delta S=-\nabla\vec{n}\cdot\nabla^{\perp}\vec{U}+\mathrm{div}\left(\vec{n}\cdot\nabla\vec{z}+\,2\,\vec{n}\cdot\vec{D}\;\nabla^{\perp}\alpha\right)\\[5mm] \Delta\vec{U}=\nabla\vec{n}\times\nabla^{\perp}\vec{U}+\nabla\vec{n}\cdot\nabla^{\perp}S-\mathrm{div}\left(\vec{n}\times\nabla\vec{z}-\,2\,\nabla\alpha\;\vec{D}+\vec{n}\times\vec{D}\;\nabla^{\perp}\alpha\right)\quad .\end{array}\right. \tag{V.44}$$

From lemma V.1 we know that $\nabla\vec{n}$ is weakly sequentially pre-compact in $L^4(D^2)$. The factor α satisfies the following Liouville type equation

$$e^{-2\mu}\Delta\alpha=e^{2\alpha}\;K_{g_{\vec{\Phi}}}-K_{g_0}$$

Since α is uniformly bounded in L^∞ as well as μ on D^2 we deduce that $\Delta\alpha$ is uniformly bounded in L^2 and hence, $\nabla\alpha$ is strongly pre-compact in the spaces $L^p_{loc}(D^2)$ for any $p<+\infty$. We know that $\nabla\vec{W}$ converges to zero strongly in $L^{2,\infty}$ and that $\nabla\vec{v}$ is strongly precompact in any space $L^q(D^2)$ for any $q<4$ hence, since also $\nabla\vec{D}$ is weakly pre-compact in L^4 and since we have chosen $\vec{D}$ to be of average zero on D^2, it is precompact in any $L^p_{loc}(D^2)$ for any $p<+\infty$. We can then apply lemma VII.2 and deduce that ∇S and $\nabla\vec{U}$ are strongly pre-compact in $L^{4/3}_{loc}(D^2)$.

Taking now the vectorial product between (V.34) and $\nabla^{\perp}\vec{\Phi}$ we obtain

$$\begin{array}{l}(\nabla\vec{V}-\nabla^{\perp}\vec{W})\times\nabla^{\perp}\vec{\Phi}\\[5mm] \quad=(\vec{L}\cdot\nabla^{\perp}\vec{\Phi}-\vec{M}\cdot\nabla\vec{\Phi})\cdot\nabla\vec{\Phi}-\,2\,f'(H)\,e^{2\lambda}\,\vec{n}-2\,(\alpha-\overline{\alpha})\,\nabla\vec{D}\times\nabla^{\perp}\vec{\Phi}+2\,\vec{M}\,e^{2\lambda}\\[5mm] \quad=\nabla^{\perp}S\cdot\nabla\vec{\Phi}+\nabla Y\cdot\nabla\vec{\Phi}-2\,(\vec{M}\cdot\nabla\vec{\Phi})\cdot\nabla\vec{\Phi}-2\,f'(H)\,e^{2\lambda}\,\vec{n}\\[5mm] \qquad-2\,(\alpha-\overline{\alpha})\,\nabla\vec{D}\times\nabla^{\perp}\vec{\Phi}+2\,\vec{M}\,e^{2\lambda}\quad .\end{array} \tag{V.45}$$

We have also

$$\begin{array}{l}\nabla\vec{V}\times\nabla^{\perp}\vec{\Phi}=\nabla\vec{R}\times\nabla^{\perp}\vec{\Phi}+\nabla\vec{u}\times\nabla^{\perp}\vec{\Phi}\\[5mm] \quad=\nabla\vec{R}\times\nabla^{\perp}\vec{\Phi}+\nabla^{\perp}\vec{v}\times\nabla^{\perp}\vec{\Phi}+\nabla Y\cdot(\vec{n}\times\nabla^{\perp}\vec{\Phi})\\[5mm] \quad=\nabla\vec{R}\times\nabla^{\perp}\vec{\Phi}+\nabla\vec{v}\times\nabla\vec{\Phi}+\nabla Y\cdot\nabla\vec{\Phi}\quad .\end{array} \tag{V.46}$$

Combining (V.45) and (V.46) gives

$$\begin{array}{l}2\,f'(H)\,e^{2\lambda}\,\vec{n}=\nabla^{\perp}S\cdot\nabla\vec{\Phi}-2\,(\alpha-\overline{\alpha})\,\nabla\vec{D}\times\nabla^{\perp}\vec{\Phi}-\nabla\vec{R}\times\nabla^{\perp}\vec{\Phi}-\nabla\vec{v}\times\nabla\vec{\Phi}\\[5mm] \qquad+2\,\vec{M}\;e^{2\lambda}-2\,(\vec{M}\cdot\nabla\vec{\Phi})\cdot\nabla\vec{\Phi}+\nabla\vec{W}\times\nabla\vec{\Phi}\quad .\end{array} \tag{V.47}$$

This implies

$$\begin{array}{l}2\,f'(H)\,e^{2\lambda}\,\vec{n}=\nabla^{\perp}S\cdot\nabla\vec{\Phi}-\nabla\vec{U}\times\nabla^{\perp}\vec{\Phi}-\nabla\vec{v}\times\nabla\vec{\Phi}+2\,\vec{D}\times\nabla^{\perp}\vec{\Phi}\cdot\nabla\alpha\\[5mm] \qquad+2\,\vec{M}\;e^{2\lambda}-2\,(\vec{M}\cdot\nabla\vec{\Phi})\cdot\nabla\vec{\Phi}+\nabla\vec{W}\times\nabla\vec{\Phi}\quad .\end{array} \tag{V.48}$$

From the results we established above we have then that $f'(H)$ is strongly pre-compact in $L^{4/3}_{loc}(D^2)$. We have explicitly $f'(H) = 2\,H\,(3 + H^2))$. Denote

$$\vec{J}^\infty := \lim_{k\to+\infty} \vec{H}^k\,(3 + |H^k|^2) \quad \text{strongly in } L^{4/3}_{loc}(D^2) \quad . \tag{V.49}$$

Since $\nabla\alpha^k$ is strongly pre-compact in any $L^p_{loc}(D^2)$ for $p < +\infty$, this is also the case for $\nabla\lambda^k$. Moreover $\Delta\vec{\Phi}^k$ is uniformly bounded in L^4 so, modulo extraction of a subsequence we have

$$\vec{H}^k \rightharpoonup \vec{H}^\infty := 2^{-1}\, e^{-2\lambda^\infty}\Delta\vec{\Phi}^\infty \text{ weakly in } L^4(D^2_{3/4}) \quad ,$$

and $\vec{\Phi}^\infty$ is a conformal immersion of the disc $D^2_{1/2}$ in $\mathcal{E}_{D^2_{1/2},2}$. Observe that

$$\vec{H}^k\cdot\vec{H}^k\,(3+|H^k|^2) \rightharpoonup \vec{H}^\infty\cdot\vec{J}^\infty \in L^1(D^2_{1/2}) \quad \text{weakly in } \mathcal{D}'(D^2_{1/2}) \quad .$$

Hence the sequence of non-negative measures $|H^k|^2\,(3+|H^k|^2)$ does not concentrate with respect to the Lebesgue measure :

$$\forall\varepsilon>0 \quad \exists\,\delta>0 \quad \text{s. t. } \forall\, E\subset D^2_{1/2} \quad \text{measur.} \quad |E|\le\delta \quad \Rightarrow \quad \limsup_{k\to+\infty}\int_E |H^k|^4\,dx^2 \le \varepsilon \quad . \tag{V.50}$$

From the strong convergence (V.49) we deduce an almost everywhere convergence

$$\vec{H}^k \longrightarrow \vec{I}^\infty \qquad \text{a. e. in } D^2$$

Using Egorov theorem, for any $\delta > 0$ there exists $E_\delta \subset D^2$ such that $|E_\delta| < \delta$ and

$$\vec{H}^k \longrightarrow \vec{I}^\infty \qquad \text{uniformly in } D^2\setminus E_\delta$$

Hence for any test function $\varphi \in C^\infty_0(D^2_{1/2})$ we have

$$\limsup_{k\to 0}\left|\int_{D^2}\varphi\,\vec{H}^k\;dx^2 - \int_{D^2\setminus E_\delta}\varphi\,\vec{I}^\infty\;dx^2\right| \le \|\varphi\|_\infty\;\limsup_{k\to+\infty}\int_{E_\delta}|H^k|dx^2 \quad . \tag{V.51}$$

Combining (V.50) and (V.51) we deduce that

$$\vec{I}^\infty = \vec{H}^\infty \quad , \tag{V.52}$$

and, using (V.49), that $\vec{H}^k$ converges strongly towards $\vec{H}^\infty$ in L^4. Hence $\vec{\Phi}^k$ is <u>strongly</u> pre-compact in $W^{2,4}(D^2_{1/2})$. Inserting this information in (V.22) gives that the limiting immersion satisfies the Euler Lagrange equation of F which concludes the proof of lemma V.2. □

V.4 Min-Max procedures for Frame Energies.

V.4.1 The free case.

Let $\mathcal{P}(\mathcal{E}_{S^2})$ the space of subsets of the space of weak immersions $\mathcal{E}_{S^2}$.

Definition V.3. *A non empty subset $\mathcal{A}$ of $\mathcal{P}(\mathcal{E}_{\Sigma,2})$ is called admissible if for any homeomorphism $\varphi \in$ Hom$(\mathcal{E}_{S^2,2})$ isotopic to the identity we have that*

$$\forall A\in\mathcal{A} \qquad \varphi(A)\in\mathcal{A} \quad .$$

Moreover, there exists a topological set X such that for any $A \in \mathcal{A}$ there exists $\vec{\Phi}^A$ in $C^0(X,\mathcal{E}_{\Sigma,2})$ such that

$$A = \vec{\Phi}^A(X) \quad .$$

□

Let now $\mathcal{A}$ be admissible. Because of Willmore universal lower bounds of W on the space of closed surfaces we have obviously

$$\beta(0) := \inf_{A\in\mathcal{A}} \max_{\vec{\Phi}\in A} W(\vec{\Phi}) \ge 4\pi > 0 \quad . \tag{V.53}$$

Since for any arbitrary fixed $\vec{\Phi}$ the map $\sigma \to F^\sigma(\vec{\Phi})$ is increasing we can use a beautiful argument initially introduced by Michael Struwe in [48] and follow word by word the proof of theorem 6.4 in [32] together with the Palais Smale property of F^σ established in lemma V.2 in order to deduce the following lemma which was the main goal of the present subsection.

Lemma V.3. *Let $\mathcal{A}$ be an admissible family. There exists a sequence $\sigma^k \to 0$ and a sequence of critical points $\vec{\Phi}^{\sigma^k}$ of F^{σ^k} such that*

$$\beta(\sigma^k) = F^{\sigma^k}(\vec{\Phi}^{\sigma^k}) \quad \textit{and} \quad \partial_\sigma F^{\sigma^k}(\vec{\Phi}^{\sigma^k}) = o\left(\frac{1}{\sigma^k\ \log(\sigma^k)^{-1}}\right) \tag{V.54}$$

so in particular we have that

$$\lim_{k\to+\infty} W(\vec{\Phi}^{\sigma^k}) = \beta(0) \quad . \tag{V.55}$$

□

V.4.2 The Area Constrained Case.

We are now adapting the previous case to the situation when we fix the area to be equal to 1. Precisely we define

$$\mathcal{E}^1_{\Sigma,2} := \left\{\vec{\Phi} \in \mathcal{E}_{\Sigma,2} \quad ; \quad \int_\Sigma dvol_{g_{\vec{\Phi}}} = 1\right\} \quad .$$

It is not difficult to check that it defines a Finsler Manifold structure based on the $W^{2,4}$ topology. The notion of admissible set is defined identically as above but replacing general homeomorphisms of $\mathcal{E}_{\Sigma,2}$ by homeomorphisms of $\mathcal{E}^1_{\Sigma,2}$. The construction of the pseudo-gradient restricted to $\mathcal{E}^1_{\Sigma,2}$ applies and we can follow again word by word the proof of theorem 6.4 in [32] and the Palais Smale property of F^σ established in lemma V.2 in order to deduce the same statement as lemma V.3 but under the area constraint $\text{Area}(\vec{\Phi}^{\sigma^k}) = 1$. We shall now establish the following lemma which is a consequence of the scaling invariance of the Willmore energy in $\mathbb{R}^m$

Lemma V.4. *Let $\vec{\Phi}^{\sigma^k}$ be a critical point of F^{σ^k} under the constraint $Area(\vec{\Phi}^{\sigma^k}) = 1$ and satisfying*

$$\partial_\sigma F^{\sigma^k}(\vec{\Phi}^{\sigma^k}) = o\left(\frac{1}{\sigma^k\ \log(\sigma^k)^{-1}}\right) \tag{V.56}$$

Then it satisfies the equation (for any choice of gauge α^k in the case $\Sigma = S^2$)

$$\begin{aligned} & d\Big[*_{g_{\vec{\Phi}^k}} d[l_{\sigma^k} f'_{\sigma^k}(H^k)\,\vec{n}_{\vec{\Phi}^k}] - 2\,l_{\sigma^k}\, f'_{\sigma^k}(H^k)\ *_{g_{\vec{\Phi}^k}} d\vec{n}_{\vec{\Phi}^k} \\ & \quad + l_{\sigma^k}\left[-2\,f_{\sigma^k}(H^k) + |d\alpha^k|^2_{g_{\vec{\Phi}^k}} - K_{g_0}\,\alpha^k\, e^{-2\alpha^k} + K_{g_0}\,[A_{\vec{\Phi}^k}(\Sigma)]^{-1}\right] *_{g_{\vec{\Phi}^k}} d\vec{\Phi}^k \\ & \quad - 2\,l_{\sigma^k}\left\langle d\vec{\Phi}^k, d\alpha^k\right\rangle_{g_{\vec{\Phi}^k}} *_{g_{\vec{\Phi}^k}} d\alpha^k + 2\,l_{\sigma^k}\,\vec{\mathbb{I}}^k \llcorner_{g_{\vec{\Phi}^k}}(*_{g_{\vec{\Phi}^k}} d\alpha^k)\Big] = C^k\ d\left[*_{g_{\vec{\Phi}^k}} d\vec{\Phi}^k\right] \quad , \end{aligned} \tag{V.57}$$

where

$$C^k = 2\,(\sigma^k)^2 \int_\Sigma (1 - |H^k|^4) + l_{\sigma^k} \int_\Sigma K_{g_0}\alpha\ dvol_{g_0} - l_{\sigma^k}\, K_{g_0} \quad . \tag{V.58}$$

Hence (for the choice of an Aubin gauge in the case $\Sigma = S^2$) we have

$$\lim_{k\to+\infty} |C^k| = 0 \quad . \tag{V.59}$$

□

Remark V.1. *Observe that a-priori C^k depends on the choice of gauge α^k.* □

Proof of Lemma V.4. We omit to write the index k. The fact that equation (V.57) is satisfied comes from (III.49) and classical lagrange multiplier theory bearing in mind that the first derivative of the fixed area constraint is proportional to $d\left[*_{g_{\vec{\Phi}^k}} d\vec{\Phi}^k\right]$ which cannot be zero since obviously there is no compact minimal immersion in $\mathbb{R}^m$ (hence the constraint is non degenerate). We take the scalar product between (V.57) and $\vec{\Phi}$ and we integrate the resulting 2-form over the closed surface Σ. This gives

$$C = l_\sigma \int_\Sigma 2\, f_\sigma(H) - H\, f_\sigma'(H) + K_{g_0} \int_\Sigma \alpha\, e^{-2\alpha}\, dvol_{g_{\vec{\Phi}}} + 4\pi\, l_\sigma\, (\text{genus}(\Sigma) - 1) \quad . \tag{V.60}$$

Since $2\, f_\sigma(H) - H\, f_\sigma'(H) = 2\,\sigma^2\, l_\sigma^{-1}(1 - H^4)$ we obtain (V.58). For the choice of an Aubin gauge we have due to (V.56) combined with theorem III.10 and (III.74)

$$l_\sigma \|\alpha\|_{L^\infty(S^2)} = o(1) \quad . \tag{V.61}$$

This implies (V.59) for $\Sigma = S^2$. For $\Sigma \neq S^2$, lemma III.12 combined with the assumption (V.56) implies also

$$l_\sigma \|\alpha\|_{L^\infty(\Sigma)} = o(1) \quad , \tag{V.62}$$

and lemma V.4 is proved in any case. □

For area constrained critical point we then have the following $\epsilon-$regularity lemma whose proof is following step by step the proof of lemma IV.1 since (V.59) holds.

Lemma V.5. [uniform $\varepsilon-$regularity under area constraint.] *For any $C_0 > 0$, there exists $\varepsilon > 0$ and $\sigma_0 > 0$ such that for any $\sigma \in (0, \sigma_0)$ and any critical point $\vec{\Phi}$ of F^σ under the constraint $Area(\vec{\Phi}) = 1$ satisfying*

$$F^\sigma(\vec{\Phi}) \le C_1 \quad \text{and} \quad \partial_\sigma F^\sigma(\vec{\Phi}) \le \frac{\varepsilon}{\sigma \log(\sigma)^{-1}} \quad , \tag{V.63}$$

assume moreover

$$\int_{D^2} |\nabla \vec{n}|^2 < \varepsilon \quad , \tag{V.64}$$

then for any $j \in \mathbb{N}$ the estimates (IV.5) and (IV.6) hold in any case for $\Sigma \neq S^2$ and for any Coulomb gauge in case $\Sigma = S^2$. □

VI The passage to the limit $\sigma \to 0$.

We shall give two results regarding the passage to the limit in the equation. A subsection will be devoted to each of the two results.

VI.1 The limiting immersions.

In the sequel we shall denote by $\mathcal{M}^+(S^2)$ the non-compact Möbius group of positive conformal diffeomorphism of the 2-sphere S^2.

Lemma VI.1. *Let $\sigma^k \to 0$ and a sequence of weak immersions $\vec{\Phi}^k \in \mathcal{E}_{S^2}$ which are critical points of F^{σ_k} under area constraint equal to 1 and such that*

$$\limsup_{k\to+\infty} F^{\sigma^k}(\vec{\Phi}^k) < +\infty \quad \text{and} \quad \partial_\sigma F^{\sigma^k}(\vec{\Phi}^k) = o\left(\frac{1}{\sigma^k\ \log(\sigma^k)^{-1}}\right) \quad . \tag{VI.1}$$

Then modulo translation there exists a subsequence that we still denote $\vec{\Phi}^k$, there exists a family of bilipschitz homeomorphism Ψ^k, there exists a finite family of sequences $(f_i^k)_{i=1\cdots N}$ of elements in $\mathcal{M}^+(S^2)$, there exists a finite family of natural integers $(N_i)_{i=1\cdots N}$ and for each $i \in \{1\cdots N\}$ there exists finitely many points of S^2, $b_{i,1}\cdots b_{i,N_i}$ such that

$$\vec{\Phi}^k \circ \Psi^k \longrightarrow \vec{f}^\infty \quad \text{strongly in } C^0(S^2,\mathbb{R}^m) \quad , \tag{VI.2}$$

where $\vec{f}^\infty \in W^{1,\infty}(S^2,\mathbb{R}^m)$, moreover

$$\vec{\Phi}^k \circ f_i^k \rightharpoonup \vec{\xi}_i^\infty \quad \text{strongly in } C^l_{loc}(S^2 \setminus \{b_{i,1}\cdots b_{i,N_i}\}) \quad , \tag{VI.3}$$

for any $l \in \mathbb{N}$ where $\vec{\xi}_j^\infty$ is a Willmore conformal possibly branched immersion of S^2. In addition we have

$$\vec{f}^\infty(S^2) = \bigcup_{i=1}^N \vec{\xi}_i^\infty(S^2) \quad , \tag{VI.4}$$

moreover

$$A(\vec{\Phi}_k) = \int_{S^2} 1\ dvol_{g_{\vec{\Phi}_k}} \longrightarrow A(\vec{f}^\infty) = \sum_{i=1}^N A(\vec{\xi}_i^\infty) \quad , \tag{VI.5}$$

and finally

$$(\vec{f}^\infty)_*[S^2] = \sum_{i=1}^N (\vec{\xi}_i^\infty)_*[S^2] \quad , \tag{VI.6}$$

where for any Lipschitz mapping $\vec{a}$ from S^2 into $\mathbb{R}^m$, $(\vec{a})_*[S^2]$ denotes the current given by the push-forward by $\vec{a}$ of the current of integration over S^2 : for any smooth two-form ω on $\mathbb{R}^m$

$$\left\langle (\vec{a})_*[S^2], \omega \right\rangle := \int_{S^2} (\vec{a})^*\omega \quad .$$

$\Box$

Remark VI.1. *Lemma VI.1 is "detecting" the Willmore spheres "visible" at the limit but is ignoring the possible "asymptotic Willmore spheres" which are shrinking and disappearing at the limit. The detection of these asymptotic Willmore spheres is the purpose of the next subsection. Finally the detection of the possible loss of energy in the so called "neck regions" and the energy quantization effect is going to be investigated in section VII of the paper.*

Proof of lemma VI.1. We are working modulo extraction of subsequences. Consider the various diffeomorphisms of S^2 f_i^k given by theorem I.2 of [34]. We choose the gauges given by f_i^k that is the pairs (f_i^k, α_i^k) satisfying

$$g_{\vec{\Phi}^k \circ f_i^k} = e^{2\alpha_i^k}\ g_{S^2}$$

From the analysis in [34] we have the existence of N_i points of S^2, $b_{i,1}\cdots b_{i,N_i}$ such that

$$\limsup_{k\rightarrow+\infty}\|\alpha_i^k\|_{L^\infty_{loc}(S^2\setminus\{b_{i,1}\cdots b_{i,N_i}\})}+\|\nabla\alpha_i^k\|_{L^2_{loc}(S^2\setminus\{b_{i,1}\cdots b_{i,N_i}\})}<+\infty\quad . \tag{VI.7}$$

Hence we have in particular obviously

$$\lim_{k\rightarrow+\infty} l_{\sigma^k}\|\alpha_i^k\|_{L^\infty_{loc}(S^2\setminus\{b_{i,1}\cdots b_{i,N_i}\})}+\sqrt{l_{\sigma^k}}\,\|\nabla\alpha_i^k\|_{L^2_{loc}(S^2\setminus\{b_{i,1}\cdots b_{i,N_i}\})}=0\quad . \tag{VI.8}$$

The assumptions (VI.1) implies moreover

$$(\sigma^k)^2\int_{S^2}(1+H^2_{\vec{\Phi}^k})^2\ dvol_{\vec{\Phi}^k}=o(l_{\sigma^k})\quad . \tag{VI.9}$$

Again from the analysis of [34] we have that the density of energy

$$|d\vec{n}_{\vec{\Phi}^k\circ f_i^k}|^2_{g_{\vec{\Phi}^k\circ f_i^k}}\ dvol_{g_{\vec{\Phi}^k\circ f_i^k}}$$

remains uniformly absolutely continuous with respect to Lebesgue in $S^2\setminus\cup_{j=1}^{N_i}B_\delta(b_{i,j})$. Hence all the assumptions which permit to apply the uniform $\epsilon-$regularity lemma IV.1 are fulfilled and we deduce the strong convergence (VI.3) towards a Willmore sphere that can be possibly branched at the $b_{i,j}$.

We claim now that these Willmore spheres are true spheres Willmore in the sense that the Willmore residues are zero :

$$\int_{\partial B_\delta(b_{i,j})}\partial_\nu\vec{H}_{\vec{\xi}_i^\infty}-\,2\,H_{\vec{\xi}_i^\infty}\,\partial_\nu\vec{n}_{\vec{\xi}_i^\infty}-2\,H^2_{\vec{\xi}_i^\infty}\,\partial_\nu\vec{\xi}_i^\infty\ dl=0$$

Indeed, because of the strong convergence (in any C^l-norm) away from the $b_{i,j}$ we have for any $\delta>0$

$$\int_{\partial B_\delta(b_{i,j})}\partial_\nu\vec{H}_{\vec{\xi}_i^k}-2\,H_{\vec{\xi}_i^k}\,\partial_\nu\vec{n}_{\vec{\xi}_i^k}-2\,H^2_{\vec{\xi}_i^k}\,\partial_\nu\vec{\xi}_i^k\ dl\longrightarrow\int_{\partial B_\delta(b_{i,j})}\partial_\nu\vec{H}_{\vec{\xi}_i^\infty}-2\,H_{\vec{\xi}_i^\infty}\,\partial_\nu\vec{n}_{\vec{\xi}_i^\infty}-2\,H^2_{\vec{\xi}_i^\infty}\,\partial_\nu\vec{\xi}_i^\infty\ dl=R$$

where $\vec{\xi}_i^k=\vec{\Phi}^k\circ f_i^k$. Since $\vec{\xi}_i^\infty$ is obviously Willmore we have that R is independent of $\delta<\inf_{j\neq l}\{|b_{i,j}-b_{i,l}|\}$. For any (i,j) we chose Let $\chi_{i,j}\in C^\infty_0(0,\delta_i))$ such that $0<\delta_i<\inf_{j\neq l}\{|b_{i,j}-b_{i,l}|\}$ and satisfying $\int_{\mathbb{R}_+}\chi_{i,j}(s)\ ds=1$. Hence we have

$$R=\int_{S^2}\chi_{i,j}(|x-b_{i,j}|)\ d|x-b_{i,j}|\wedge\left[*d\vec{H}_{\vec{\xi}_i^\infty}-\,2\,H_{\vec{\xi}_i^\infty}\ *d\vec{n}_{\vec{\xi}_i^\infty}-2\,H^2_{\vec{\xi}_i^\infty}\ *d\vec{\xi}_i^\infty\right]\quad .$$

Because of the strong C^l convergence we have

$$R=\lim_{k\rightarrow+\infty}\int_{S^2}\chi_{i,j}(|x-b_{i,j}|)\ d|x-b_{i,j}|\wedge\left[*d\vec{H}_{\vec{\xi}_i^k}-\,2\,H_{\vec{\xi}_i^k}\ *d\vec{n}_{\vec{\xi}_i^k}-2\,H^2_{\vec{\xi}_i^k}\ *d\vec{\xi}_i^k\right]\quad .$$

Since we are on a sphere, and because of the Euler Lagrange equation (IV.7) we have the existence of $\vec{L}_k$ on $B_{\delta_i}(b_{i,j})$ such that

$$\begin{array}{l}
d\vec{L}^k=*d\vec{H}_{\vec{\xi}_i^k}-\,2\,H_{\vec{\xi}_i^k}\ *d\vec{n}_{\vec{\xi}_i^k}-2\,H^2_{\vec{\xi}_i^k}\ *d\vec{\xi}_i^k\\[2mm]
\quad+4\,\sigma^2\ *d(\vec{H}_{\vec{\xi}_i^k}\,(1+H^2_{\vec{\xi}_i^k}))-8\,\sigma^2\,\vec{H}_{\vec{\xi}_i^k}\,(1+H^2_{\vec{\xi}_i^k})\ *d\vec{n}_{\vec{\xi}_i^k}-2\,\sigma^2\ (1+H^2_{\vec{\xi}_i^k})^2\ *d\vec{\xi}_i^k\\[2mm]
\quad-2\ l_\sigma\ <d\vec{\Phi}\cdot d\alpha^k>_{g_{\vec{\xi}_i^k}}\ *d\alpha^k+l_\sigma\left[|d\alpha^k|^2_{g_{\vec{\xi}_i^k}}-K_{g_0}\,\alpha^k\ e^{-2\alpha^k}+K_{g_0}\,A_{\vec{\xi}_i^k}(\Sigma)^{-1}\right]\ *d\vec{\xi}_i^k+2\,l_\sigma\ \vec{\mathbb{I}}_{\vec{\xi}_i^k}\llcorner *d\alpha^k\quad .
\end{array}$$

Since clearly

$$\int_{S^2}\chi_{i,j}(|x-b_{i,j}|)\ d|x-b_{i,j}|\wedge d\vec{L}^k=0\quad ,$$

we have

$$R=-4\,\sigma^2\lim_{k\to+\infty}\int_{S^2}\chi_{i,j}(|x-b_{i,j}|)\ d|x-b_{i,j}|\wedge *d(\vec{H}_{\vec{\xi}_i^k}\,(1+H^2_{\vec{\xi}_i^k}))-2\,\vec{H}_{\vec{\xi}_i^k}\,(1+H^2_{\vec{\xi}_i^k})\,*\,d\vec{n}_{\vec{\xi}_i^k}$$

$$+2\,\sigma^2\lim_{k\to+\infty}\int_{S^2}\chi_{i,j}(|x-b_{i,j}|)\ d|x-b_{i,j}|\wedge\ (1+H^2_{\vec{\xi}_i^k})^2\ *d\vec{\xi}_i^k$$

$$-2\,l_\sigma\lim_{k\to+\infty}\int_{S^2}\chi_{i,j}(|x-b_{i,j}|)\ d|x-b_{i,j}|\wedge\left[<d\vec{\Phi}\cdot d\alpha^k>_{g_{\vec{\xi}_i^k}}\ *d\alpha^k-\vec{\mathbb{I}}_{\vec{\xi}_i^k}\llcorner *d\alpha^k\right]$$

$$+l_\sigma\lim_{k\to+\infty}\int_{S^2}\chi_{i,j}(|x-b_{i,j}|)\ \left[|d\alpha^k|^2_{g_{\vec{\xi}_i^k}}-K_{g_0}\,\alpha^k\ e^{-2\alpha^k}+K_{g_0}\,A_{\vec{\xi}_i^k}(\Sigma)^{-1}\right]\ d|x-b_{i,j}|\wedge *d\vec{\xi}_i^k\quad .$$

Using the strong precompactness of $\vec{\xi}_i^k$ in any C^l topology in the domain where $\chi_{i,j}(|x-b_{i,j}|)\neq 0$ we conclude that $R=0$ and this finishes the proof of lemma VI.1. □

VI.2 Bubble Detection.

Lemma VI.2. [Bubble Detection Lemma.] *Let Σ be a closed surface, let $C_0>0$ and $\varepsilon(C_0)>0$ given by lemma V.5 . Let $\sigma^k\to 0$ and $\vec{\Phi}^{\,k}$ be a sequence of critical points of F^{σ^k} under the constraint Area$(\vec{\Phi}^k)=1$ and satisfying*

$$F^{\sigma^k}(\vec{\Phi}^k)\le C_0\quad\text{ and }\quad\partial_\sigma F^\sigma(\vec{\Phi})=o\left(\frac{1}{\sigma\log(\sigma)^{-1}}\right)\tag{VI.10}$$

assume when $\Sigma\neq S^2$ that the sequence of constant Gauss curvature metric g_0^k such that $g_{\vec{\Phi}^k}=e^{2\alpha^k}\ g_0^k$ is pre-compact in any $C^l(\Sigma)$ topology. Let $B_{\rho^k}(p^k)$ be a geodesic ball for the metric g_0^k such that

$$\int_{B_{\rho^k}(p^k)}|d\vec{n}|^2_{g_{\vec{\Phi}^k}}\ dvol_{g_{\vec{\Phi}^k}}<\varepsilon\tag{VI.11}$$

Compose $\vec{\Phi}^{\,k}$ in a sequence of conformal charts (x_1^k,x_2^k) from D^2 into $(B_{\rho^k}(x^k),g_0^k)$ such that there exists $\overline{\mu}_k\in\mathbb{R}$ satisfying $g_0^k=e^{2\mu^k}\ [dx_1^2+dx_2^2]$ and $\mu^k-\overline{\mu}^k$ is weakly precompact for the $(L^\infty)^*$ topology. Let $\overline{\alpha}^k$ be the average of α^k on $D^2_{1/2}$. Then

$$e^{-\overline{\mu}^k-\overline{\alpha}^k}\ \left[\vec{\Phi}^{\,k}(x)-\vec{\Phi}^{\,k}(0)\right]$$

is strongly converging in $D^2_{1/2}$ in any C^l norm towards a Willmore disc. □

Remark VI.2. *We are mostly interested with the balls $B_{\rho^k}(p^k)$ for which*

$$\limsup_{k\to 0}\int_{B_{\rho^k}(p^k)}|d\vec{n}|^2_{g_{\vec{\Phi}^k}}\ dvol_{g_{\vec{\Phi}^k}}>0\tag{VI.12}$$

where there is indeed a non flat Willmore bubble forming which is swallowing part of the energy. The lemma however applies also when (VI.12) does not hold. □

Proof of lemma VI.2. We keep denoting $\vec{\Phi}^{\,k}$ the composition of $\vec{\Phi}^{\,k}$ with the given chart and we work on D^2. Because of lemma V.4 $\vec{\Phi}$ satisfies the equation (where we omit to write explicitly the upper index k)

$$\begin{array}{l}
e^{\overline{\lambda}}\ \mathrm{div}\left[\nabla\vec{H}_{\vec{\Phi}}-2\,H_{\vec{\Phi}}\,\nabla\vec{n}_{\vec{\Phi}}-2\,H^2_{\vec{\Phi}}\,\nabla\vec{\Phi}\right]\\[2mm]
\quad=\mathrm{div}\left[-2\,e^{\overline{\lambda}}\,\sigma^2\,\nabla\left[\vec{H}_{\vec{\Phi}}\,(1+H^2_{\vec{\Phi}})\right]+4\,\sigma^2\ e^{\overline{\lambda}}\,H_{\vec{\Phi}}\,(1+H^2_{\vec{\Phi}})\ \nabla\vec{n}_{\vec{\Phi}}\right]\\[2mm]
\quad+\mathrm{div}\left[\sigma^2\,(1+H^2_{\vec{\Phi}})^2\,e^{\overline{\lambda}}\ \nabla\vec{\Phi}+\,l_\sigma\ e^{\overline{\lambda}-2\lambda}\ \left[(\mathbb{I}_{\vec{\Phi}}\llcorner\nabla^\perp\alpha)^\perp+\nabla\vec{\Phi}\cdot\nabla\alpha\ \nabla\alpha\right]\right]\\[2mm]
\quad-\mathrm{div}\left[2^{-1}\,l_\sigma\ \left[e^{-2\lambda}\,|\nabla\alpha|^2-\ K_{g_0}\,\alpha\ e^{-2\alpha}+K_{g_0}\right]\ e^{\overline{\lambda}}\right]+\ 2^{-1}\,C\,e^{\overline{\lambda}}\,\vec{H}_{\vec{\Phi}}
\end{array}\qquad\text{(VI.13)}$$

where $\overline{\lambda}$ is the average of λ on $D^2_{1/2}$ and is equal to $\overline{\mu}+\overline{\alpha}$. The uniform $\epsilon-$regularity under area constraint, lemma V.5, combined with our assumption (VI.10) and lemma V.4 imply that the right hand side of (VI.13) is converging towards 0 in any norm. Denote $\vec{\xi}:=e^{-\overline{\lambda}}(\vec{\Phi}(x)-\vec{\Phi}(0))$ we have

$$\vec{n}_{\vec{\xi}}=\vec{n}_{\vec{\Phi}}\quad,\quad\vec{H}_{\vec{\xi}}=e^{\overline{\lambda}}\,\vec{H}_{\vec{\Phi}}\quad\text{ and }\quad\nabla\vec{\xi}=e^{-\overline{\lambda}}\,\nabla\vec{\Phi}\quad.$$

Hence we have

$$\mathrm{div}\left[\nabla\vec{H}_{\vec{\xi}^k}-2\,H_{\vec{\xi}^k}\,\nabla\vec{n}_{\vec{\xi}^k}-2\,H^2_{\vec{\xi}^k}\,\nabla\vec{\xi}^k\right]\longrightarrow 0\quad\text{ in }C^l(D^2_{1/2})\quad\forall\,l\in\mathbb{N}\quad,$$

and

$$\limsup_{k\rightarrow+\infty}\int_{D^2_{1/2}}|\nabla\vec{n}_{\vec{\xi}^k}|^2<+\infty\quad\text{ and }\quad\limsup_{k\rightarrow+\infty}\log|\nabla\vec{\xi}^k|^2<+\infty\quad.$$

Hence, since it is passing also through the origin, adapting the arguments in [41] (see also precisely theorem 7.11 [43]) to this perturbed case gives the strong convergence of $\vec{\xi}^k$ to a limiting Willmore immersion in $C^l(D^2_{1/2})$ $\forall\,l\in\mathbb{N}$ and the lemma VI.2 is proved. □

VI.3 The energy quantization

With lemma VI.2 at hand, in order to prove our main result theorem I.1, following the scheme of the paper [8] it remains only to establish the vanishing of the energy in the so called *neck region.* We are now restricting to the sphere case exclusively. Precisely we are going to prove the following lemma.

Lemma VI.1. *Let $\sigma^k\rightarrow 0$ and a sequence of weak immersions $\vec{\Phi}^k\in\mathcal{E}_{S^2}$ which are critical points of F^{σ_k} under area constraint and such that*

$$\limsup_{k\rightarrow+\infty}F^{\sigma^k}(\vec{\Phi}^k)<+\infty\quad\textit{and}\quad\partial_\sigma F^{\sigma^k}(\vec{\Phi}^k)=o\left(\frac{1}{\sigma^k\ \log(\sigma^k)^{-1}}\right)\qquad\text{(VI.1)}$$

Then there exists $\vec{\xi}_1\cdots\vec{\xi}_n$, finitely Willmore immersions of S^2 minus finitely many points, such that

$$\lim_{k\rightarrow+\infty}W(\vec{\Phi}^k)=\sum_{i=1}^n W(\vec{\xi}_i)-\ 4\pi\,N\quad,\qquad\text{(VI.2)}$$

where $N\in\mathbb{N}$. □

Proof of lemma VI.1. We shall work with an Aubin gauge (Ψ^k,α^k) satisfying

$$g_{\vec{\Phi}^k\circ\Psi^k}=e^{2\,\alpha^k}\ \frac{g_{S^2}}{4\pi}\quad\text{ and }\quad\forall i=1,2,3\quad\int_{S^2}x_i\ e^{2\,\alpha^k}\ dvol_{S^2}=0\quad,$$

where g_{S^2} and $dvol_{S^2}$ are respectively the standard metric and standard associated volume form on S^2. In order to simplify the notation we omit to write Ψ_k that we assume to be the identity. The assumption (VI.1) reads as follows

$$\begin{aligned} o\left(\frac{1}{\sigma^k\,\log(\sigma^k)^{-1}}\right) &= 2\,\sigma^k \int_{S^2}(1+H^2_{\vec{\Phi}^k})^2\ e^{2\alpha_k}\ dvol_{g_{S^2}} \\ &+\frac{1}{\sigma^k\,(\log\sigma^k)^2}\left[\int_{S^2}2^{-1}|d\alpha^k|_{g_{S^2}}\ dvol_{g_{S^2}}+\int_{S^2}\alpha^k\ dvol_{g_{S^2}}-2\pi\ \log\mathrm{Area}(\vec{\Phi}_k)\right]\quad . \end{aligned} \tag{VI.3}$$

The non negativity of both the first term and the second term in the r.h.s. of (VI.3) gives respectively

$$(\sigma^k)^2\int_{S^2}(1+H^2_{\vec{\Phi}^k})^2\ e^{2\alpha_k}\ dvol_{g_{S^2}}=o\left(\frac{1}{\log(\sigma^k)^{-1}}\right)\quad , \tag{VI.4}$$

and

$$\frac{1}{\log(\sigma^k)^{-1}}\left[\int_{S^2}2^{-1}|d\alpha^k|_{g_{S^2}}\ dvol_{g_{S^2}}+\int_{S^2}\alpha^k\ dvol_{g_{S^2}}-2\pi\ \log\mathrm{Area}(\vec{\Phi}_k)\right]=o(1)\quad . \tag{VI.5}$$

We keep using the notation from previous sections that is $l_{\sigma^k}:=1/\log((\sigma^k)^{-1})$. Using lemma III.10 and lemma III.11, we get for this Aubin gauge

$$l_{\sigma^k}\int_{S^2}|d\alpha^k|^2_{g_{S^2}}\ dvol_{g_{S^2}}=o(1) \tag{VI.6}$$

and

$$l_{\sigma^k}\,\|\alpha^k\|_{L^\infty(S^2)}=o(1) \tag{VI.7}$$

Hence in order to apply the uniform ϵ-regularity lemma V.5 on any geodesic ball $B_r(x_0)$ for the S^2 metric it suffices to assume that σ is small enough and

$$\int_{B_r(x_0)}|d\vec{n}|^2_{g_{S^2}}\ dvol_{g_{S^2}}<\varepsilon$$

then in particular

$$\begin{aligned} &r^4\,\|e^\lambda\,\nabla(\vec{H}_{\vec{\Phi}^k}\,(1+2\sigma^2(1+H^2_{\vec{\Phi}^k})))\|^2_{L^\infty(B_{r/2}(x_0))}+r^2\,\sigma^4\,\|e^\lambda\,H_{\vec{\Phi}^k}(1+H^2_{\vec{\Phi}^k})\|^2_{L^\infty(B_{r/2}(x_0))} \\ &+r^2\,\|\nabla\vec{n}_{\vec{\Phi}^k}\|^2_{L^\infty(B_{r/2}(x_0))}\le C\int_{B^2_r(x_0)}|\nabla\vec{n}|^2\ dx^2+C\ \left[\int_{B_r(x_0)}[l_{\sigma^k}\ e^{2\alpha^k}+l_{\sigma^k}\,\|\alpha^k\|_{L^\infty(S^2)}]\ dvol_{g_{S^2}}\right]^2 \\ &+C\left[\int_{B^2_r(x_0)}(\sigma^k)^2H^4\ e^{2\lambda}\ dx^2\right]^2+\ C\ \left[\int_{B^2_r(x_0)}l_{\sigma^k}\ |\nabla\alpha^k|^2\ dx^2\right]^2+C\ l_{\sigma^k}\|e^{4\mu_k}\|_{L^\infty(B_r(x_0))}\quad , \end{aligned} \tag{VI.8}$$

and

$$\begin{aligned} &r^2\ l_{\sigma^k}\,\|\nabla\alpha^k\|^2_{L^\infty(B_{r/2}(x_0))}\le\ C\,l_{\sigma^k}\int_{B^2_r(x_0)}|\nabla\alpha^k|^2\ dx^2+\ C\ \left[\int_{B^2_r(x_0)}|\nabla\vec{n}_{\vec{\Phi}^k}|^2\ dx^2\right]^2 \\ &+C\left[\int_{B^2_r(x_0)}(\sigma^k)^2H^4_{\vec{\Phi}^k}\ e^{2\lambda}\ dx^2\right]^4+C\ \left[\int_{B_r(x_0)}(\sqrt{l_{\sigma^k}}+l_{\sigma^k}\ e^{2\alpha^k})\ dvol_{g_{S^2}}\right]^2+C\ l_{\sigma^k}\|e^{4\mu_k}\|_{L^\infty(B_r(x_0))}\quad . \end{aligned} \tag{VI.9}$$

Recall that the fact that $\vec{\Phi}^k$ is a critical point of F^{σ^k} is equivalent to the existence of $\vec{L}_{\vec{\Phi}^k}$ such that

$$\begin{aligned}\nabla\vec{L}_{\vec{\Phi}^k} &:= 2\,\nabla^\perp\left(\vec{H}_{\vec{\Phi}^k}\,(1+2\,\sigma^2\,(1+H^2_{\vec{\Phi}^k})\right) - 4\,H_{\vec{\Phi}^k}\,(1+2\,\sigma^2\,(1+H^2_{\vec{\Phi}^k}))\,\nabla^\perp\vec{n}_{\vec{\Phi}^k}\\ &-2\,e^{-2\lambda^k}\,l_{\sigma^k}\,\nabla\vec{\Phi}^k\cdot\nabla\alpha^k\,\nabla^\perp\alpha^k + 2\,l_\sigma\,e^{-2\lambda^k}\,\vec{\mathbb{I}}_{\vec{\Phi}^k}\llcorner\nabla^\perp\alpha^k - 2\,(H^2_{\vec{\Phi}^k}+\sigma^2(1+H^2_{\vec{\Phi}^k})^2)\,\nabla^\perp\vec{\Phi}^k \\ &+l_{\sigma^k}\,\left[e^{-2\lambda^k}\,|\nabla\alpha^k|^2 - K_{g_0}\,\alpha\,e^{-2\alpha^k} + 4\pi\,A_{\vec{\Phi}^k}(S^2)^{-1}\right]\,\nabla^\perp\vec{\Phi}^k\quad .\end{aligned}\tag{VI.10}$$

Since $2\,(\sigma^k)^2\,(1+H^2_{\vec{\Phi}^k})|H_{\vec{\Phi}^k}|\,|\nabla\vec{n}_{\vec{\Phi}^k}|\,e^\lambda \le |\nabla\vec{n}_{\vec{\Phi}^k}|^2 + (\sigma^k)^4\,H^2_{\vec{\Phi}^k}(1+H^2_{\vec{\Phi}^k})^2\,e^{2\lambda^k}$, the previous estimates imply

$$\begin{aligned}r^2\|e^{\lambda^k}\,\nabla\vec{L}_{\vec{\Phi}^k}\|_{L^\infty(B_{r/2}(x_0))} &\le C\,\left[\int_{B^2_r(x_0)}|\nabla\vec{n}_{\vec{\Phi}^k}|^2\,dx^2\right]^{1/2} + C\,l_{\sigma^k}\int_{B^2_r(x_0)}|\nabla\alpha^k|^2\,dx^2\\ &+\int_{B^2_r(x_0)}(\sigma^k)^2H^4_{\vec{\Phi}^k}\,e^{2\lambda^k}\,dx^2 + l_{\sigma^k}\int_{B_r(x_0)}dvol_{g_{\vec{\Phi}^k}} + l_{\sigma^k}\,\|\alpha^k\|_\infty\int_{B_r(x_0)}dvol_{g_{S^2}}\quad .\end{aligned}\tag{VI.11}$$

We now follow step by step the arguments from section VI of [8] and check how each estimate is slightly modified by the viscous terms. We consider a neck region that is an annulus (for the S^2 metric) of the form $B^2_{R^k}(0)\setminus B^2_{r^k}(0)$ where

$$\lim_{k\to+\infty} R^k = 0 \qquad \text{and} \qquad \lim_{k\to+\infty}\frac{r^k}{R^k} = 0\quad ,$$

and such that

$$\lim_{k\to+\infty}\ \sup_{r^k<s<R^k/4}\int_{B^2_{4s}(0)\setminus B^2_s(0)}|\nabla\vec{n}_{\vec{\Phi}^k}|^2\,dx^2 = 0\quad .\tag{VI.12}$$

We shall omit very often when it is not necessary to write explicitly the superscript k. Denote for $s<R^k/4$

$$\begin{aligned}s\,\delta^k(s) &:= \left[\int_{B^2_{2s}(0)\setminus B^2_{s/2}(0)}|\nabla\vec{n}|^2\,dx^2\right]^{1/2} + C\,\left[l_{\sigma^k}\int_{B^2_{2s}(0)\setminus B^2_{s/2}(0)}|\nabla\alpha^k|^2\,dx^2\right]^{1/2}\\ &+\left[\int_{B^2_{2s}(0)\setminus B^2_{s/2}(0)}(\sigma^k)^2(1+H^2)^2\,e^{2\lambda}\,dx^2\right]^{1/2} + \left[l_\sigma\int_{B^2_{2s}(0)\setminus B^2_{s/2}(0)}dvol_{g_{\vec{\Phi}}}\right]^{1/2}\\ &+\left[l_\sigma\,\|\alpha\|_\infty\int_{B^2_{2s}(0)\setminus B^2_{s/2}(0)}dvol_{g_{S^2}}\right]^{1/2}\quad .\end{aligned}\tag{VI.13}$$

We have

$$\lim_{k\to+\infty}\ \sup_{s\in(r^k,R^k)} s\,\delta^k(s) = 0\quad .$$

We shall omit very often when it is not necessary to write explicitly the superscript k. From (VI.11), we have in the neck region

$$|x|^2|\nabla\vec{L}|(x)\le|x|\,\delta(|x|)\,e^{-\lambda(x)}\quad ,\tag{VI.14}$$

and such that

$$\limsup_{k\to+\infty}\int_r^{R/4}\delta^2(s)\,s\,ds < +\infty\tag{VI.15}$$

We keep up with the notations in [8] and introduce

$$\vec{L}_t := \frac{1}{|\partial B_t(0)|}\int_{\partial B_t(0)} \vec{L}\ dl_{\partial B_t(0)} \quad \text{ and } \quad \lambda(t) := \frac{1}{|\partial B_t(0)|}\int_{\partial B_t(0)} \lambda\ dl_{\partial B_t(0)} \quad .$$

Using (VI.11) we have

$$\begin{array}{l}
\displaystyle\frac{d\vec{L}_t}{dt} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \vec{L}}{\partial t}(t,\theta)\ d\theta = -\frac{2}{\pi}\int_0^{2\pi} H\,(1+2\,\sigma^2\,(1+H^2))\,\frac{1}{t}\frac{\partial \vec{n}}{\partial \theta}\ d\theta\\[2mm]
\displaystyle -\frac{1}{\pi}\int_0^{2\pi} e^{-2\lambda}\, l_\sigma\, \nabla\vec{\Phi}\cdot\nabla\alpha\ \frac{1}{t}\frac{\partial\alpha}{\partial\theta} + \frac{1}{\pi}\int_0^{2\pi} e^{-2\lambda}\, l_\sigma\ \left(\mathbb{I}\llcorner\nabla^\perp\alpha\right)\cdot\frac{\partial}{\partial r}\ d\theta\\[2mm]
\displaystyle -\frac{1}{\pi}\int_0^{2\pi} (H^2+\sigma^2\,(1+H^2)^2)\,\frac{1}{t}\frac{\partial\vec{\Phi}}{\partial\theta}\ d\theta\\[2mm]
\displaystyle +\frac{1}{2\pi}\int_0^{2\pi} l_\sigma\ \left[e^{-2\lambda}\,|\nabla\alpha|^2 - K_{g_0}\,\alpha\, e^{-2\alpha} + 4\pi\, A_{\vec{\Phi}}(S^2)^{-1}\right]\ \frac{1}{t}\frac{\partial\vec{\Phi}}{\partial\theta}\ d\theta \quad .
\end{array} \tag{VI.16}$$

This gives using again $\sigma^2(1+H^2)^2 \le 2\,\sigma^4(1+H^2)^2H^2 + 3\,(1+H^2)$

$$\begin{array}{l}
\displaystyle e^{\lambda(t)}\left|\frac{d\vec{L}}{dt}\right|(t) \le C\ \sigma^4\|e^\lambda H\,(1+H^2)\|^2_{L^\infty(\partial B_t(0))} + C\,\|\nabla\vec{n}\|^2_{L^\infty(\partial B_t(0))}\\[2mm]
\displaystyle +C\, l_\sigma\,\|\nabla\alpha\|^2_{L^\infty(\partial B_t(0))} + l_\sigma\, e^{2\,\lambda(t)} + l_\sigma\,\|\alpha\|_\infty \quad .
\end{array} \tag{VI.17}$$

Combining the previous inequality with (VI.8), (VI.9) and (VI.13) we finally obtain

$$e^{\lambda(t)}\left|\frac{d\vec{L}}{dt}\right|(t) \le C\,\delta^2(t) \quad . \tag{VI.18}$$

Following the arguments in [8] (proof of lemma VI.1) we can choose a normalization in such a way that

$$e^{\lambda(x)}|\vec{L}|(x) \le C\ |x|^{-1} \quad \text{on } B_{R/4}\setminus B_{2r} \quad , \tag{VI.19}$$

where C is independent of k. We adopt the notations of the proof of lemma IV.1. Let Y satisfying

$$\left\{\begin{array}{l}
-\Delta Y = 4\,e^{2\lambda}\,\sigma^2\,(1-H^4) + 2\,l_\sigma\,K_{g_0}\,\alpha\,e^{2\mu} - 8\pi\,l_\sigma\,e^{2\lambda}\,A_{\vec{\Phi}}(S^2)^{-1} \quad \text{ in } B_R(0)\\[3mm]
Y = 0 \qquad \text{ on } \partial B_R(0) \quad .
\end{array}\right. \tag{VI.20}$$

Inequality gives (IV.41)

$$\begin{array}{l}
\displaystyle \|\nabla Y\|_{L^{2,\infty}(B_R(0))} \le C\ \int_{B_R(0)} \sigma^2\,[1+H^4]\ e^{2\lambda}\ dx^2 +\\[2mm]
\displaystyle \qquad +C\,l_\sigma\,\|\alpha\|_\infty\int_{B_R(0)} e^{2\mu}\ dx^2 + C\,l_\sigma\,\frac{A_{\vec{\Phi}}(B_R(0))}{A_{\vec{\Phi}}(S^2)} = o(1) \quad .
\end{array} \tag{VI.21}$$

On $B_{2t}\setminus B_{t/2}$ we have

$$\|\Delta Y\|_{L^\infty(B_{2t}\setminus B_{t/2})} \le C\,\delta^2(t) \quad . \tag{VI.22}$$

Hence we deduce using standard interpolation theory[14]

$$t^2\,\|\nabla Y\|^2_{L^\infty(\partial B_t(0))}\le C\,\|\nabla Y\|_{L^{2,\infty}(B_R(0))}\,t^2\,\|\Delta Y\|_{L^\infty(B_{2t}\setminus B_{t/2})}+\|\nabla Y\|^2_{L^{2,\infty}(B_R(0))}\quad . \tag{VI.23}$$

Combining (VI.21), (VI.22) and (VI.23) give

$$\|\nabla Y\|_{L^\infty(\partial B_t(0))}\le o(1)\ [\delta(t)+t^{-1}]=o(1)\,t^{-1}\quad . \tag{VI.24}$$

Using Poincaré theorem we deduce the existence of a function S such that

$$\nabla S=\vec{L}\cdot\nabla\vec{\Phi}+\nabla^\perp Y\qquad\text{on } B_R(0)\quad . \tag{VI.25}$$

Combining (VI.19) and (VI.24) gives

$$\|\nabla S\|_{L^\infty(\partial B_t(0))}\le\ C\,t^{-1}\qquad\text{for } t\in[2r,R/2]\quad . \tag{VI.26}$$

Let $\vec{v}$ satisfying

$$\left\{\begin{array}{l}\Delta\vec{v}=\nabla^\perp Y\cdot\nabla\vec{n}\qquad\text{in } B_R(0)\\[5mm] \vec{v}=0\qquad\text{on } \partial B_R(0)\quad .\end{array}\right. \tag{VI.27}$$

This gives using Wente estimates (see [40])

$$\int_{B_R(0)}|\nabla\vec{v}|^2\ dx^2\le C\,\|\nabla Y\|^2_{2,\infty}\ \int_{B_R(0)}|\nabla\vec{n}|^2\ dx^2=o(1)\quad , \tag{VI.28}$$

and

$$t^2\,\|\nabla\vec{v}\|^2_{L^\infty(\partial B_t(0))}\le C\,\|\nabla\vec{v}\|_{L^2}\ t^2\ \|\Delta\vec{v}\|_{L^\infty(B_{2t}\setminus B_{t/2})}+\int_{B_{2t}\setminus B_{t/2}}|\nabla\vec{v}|^2\ dx^2\quad . \tag{VI.29}$$

Let

$$\eta_1(t):=t^{-1}\ \left[\int_{B_{2t}\setminus B_{t/2}}|\nabla\vec{v}|^2\ dx^2\right]^{1/2}=o(1)\,t^{-1}\quad .$$

We have

$$\int_r^R\eta_1^2(t)\,t\ dt=o(1)\quad . \tag{VI.30}$$

Hence using estimates (VI.29) we have

$$\|\nabla\vec{v}\|_{L^\infty(\partial B_t(0))}\le o(1)\ \delta(t)+\eta_1(t)=o(1)\,t^{-1}\quad . \tag{VI.31}$$

Let now $\vec{u}$ such that

$$\vec{n}\,\nabla^\perp Y=\nabla\vec{v}+\nabla^\perp\vec{u}\quad . \tag{VI.32}$$

It satisfies

$$\|\nabla\vec{u}\|_{L^\infty(\partial B_t(0))}\le o(1)\,t^{-1}\quad . \tag{VI.33}$$

Let

$$\nabla\vec{D}=\left(l_\sigma\ e^{-2\lambda}\sum_{i=1}^2\mathbb{I}_{1i}\,\partial_{x_i}\vec{\Phi}\ ,\ l_\sigma\ e^{-2\lambda}\sum_{i=1}^2\mathbb{I}_{2i}\,\partial_{x_i}\vec{\Phi}\right)\quad . \tag{VI.34}$$

Using again Poincaré Lemma on $B_R(0)$ we obtain the existence of $\vec{V}$ such that

$$\nabla\vec{V}:=\vec{L}\times\nabla\vec{\Phi}+2\,H(1+2\,\sigma^2\,(1+H^2))\,\nabla\vec{\Phi}-2\,\alpha\,\nabla\vec{D}\quad . \tag{VI.35}$$

[14] See for instance the proof of lemma A.1 in [10] and replace $\|u\|_\infty$ by $\|\nabla u\|_{2,\infty}$ which has the same scaling in 2 dimension.

Using again $2\,|H|\,\sigma^2\,(1+H^2)) \le 2\,\sqrt{H^2\,e^{2\lambda}\,\sigma^4\,(1+H^2)^2}$ together with (VI.8) and using also (VI.19) we obtain

$$\|\nabla\vec{V}\|_{L^\infty(\partial B_t(0))} \le C\,[t^{-1}+\delta(t)] \le 2\,C\,t^{-1} \quad . \tag{VI.36}$$

We denote $\vec{R} := \vec{V} - \vec{u}$ and (IV.22) implies

$$\vec{n}\times\nabla\vec{R} + \vec{n}\times\nabla^\perp\vec{v} = -\nabla^\perp\vec{R} + \nabla\vec{v} - \vec{n}\,\nabla S - 2\,\alpha\,\left(\nabla^\perp\vec{D} + \vec{n}\times\nabla\vec{D}\right) \quad . \tag{VI.37}$$

Combining (VI.33) and (VI.36) we have

$$\|\nabla\vec{R}\|_{L^\infty(\partial B_t(0))} \le C\,t^{-1} \quad . \tag{VI.38}$$

Let $\vec{E}$ be the solution of

$$\left\{\begin{array}{l}\Delta\vec{E} = 2\,\nabla\alpha\cdot\nabla^\perp\vec{D} \quad \text{on } B_R(0)\\[3mm] \vec{E} = 0 \qquad \text{on } \partial\vec{D} \quad .\end{array}\right. \tag{VI.39}$$

We have using Wente estimates (see [40])

$$\|\nabla\vec{E}\|_{L^2(B_R(0))} \le \|\nabla\vec{E}\|_{L^{2,1}(B_R(0))} \le C\,l_\sigma\,\|\nabla\alpha\|_{L^2(B_R(0))}\,\|\nabla\vec{n}\|_{L^2(B_R(0))} = o(1) \quad . \tag{VI.40}$$

Denote

$$\eta_2(t) := t^{-1}\,\left[\int_{B_{2t}\setminus B_{t/2}} |\nabla\vec{E}|^2\,dx^2\right]^{1/2} = o(1)\,t^{-1} \quad .$$

From (VI.40) we obtain

$$\int_{2r}^{R/2} \eta_2^2(t)\;t\,dt = o(1) \quad . \tag{VI.41}$$

Interpolation inequalities give again

$$\begin{aligned} t^2\,\|\nabla\vec{E}\|^2_{L^\infty(\partial B_t(0))} &\le C\,\|\nabla\vec{E}\|_{L^2(B_R(0))}\;t^2\;\|\Delta\vec{E}\|_{L^\infty(B_{2t}\setminus B_{t/2})} + C\int_{B_{2t}\setminus B_{t/2}} |\nabla\vec{E}|^2\,dx^2\\ &\le C\,t^2\;o(1)\,l_\sigma\|\nabla\alpha\|_{L^\infty(B_{2t}\setminus B_{t/2})}\;\|\nabla\vec{n}\|_{L^\infty(B_{2t}\setminus B_{t/2})} + t^2\;\eta_2^2(t)\\ &\le C\,t^2\;[o(1)\;\delta^2(t) + \eta_2^2(t)] = o(1) \quad . \end{aligned} \tag{VI.42}$$

Hence we deduce

$$\|\nabla\vec{E}\|_{L^\infty(\partial B_t(0))} \le C\;[o(1)\,\delta(t) + \eta_2(t)] = o(1)\,t^{-1} \quad . \tag{VI.43}$$

Let $\vec{F}$ such that

$$2\,\alpha\,\nabla^\perp\vec{D} = \nabla^\perp\vec{F} + \nabla\vec{E} \quad .$$

Observe that we have in one hand

$$\|\nabla\vec{F}\|_{L^2(B_R(0))} \le 2\,l_\sigma\,\|\alpha\|_\infty\;\|\nabla\vec{n}\|_{L^2(B_R(0))} + \|\nabla\vec{E}\|_{L^2(B_R(0))} = o(1) \quad , \tag{VI.44}$$

and in the other hand for $t\in[4r, R/4]$

$$\begin{aligned} \|\nabla\vec{F}\|_{L^\infty(\partial B_t(0))} &\le 2\,l_\sigma\,\|\alpha\|_\infty\;\|\nabla\vec{n}\|_{L^\infty(\partial B_t(0))} + \|\nabla\vec{E}\|_{L^\infty(\partial B_t(0))}\\ &= C\;[o(1)\,\delta(t) + \eta_2(t)] = o(1)\,t^{-1} \quad . \end{aligned} \tag{VI.45}$$

Let $\vec{X} := \vec{R} + \vec{F}$ we have using (VI.38)

$$\|\nabla\vec{X}\|_{L^\infty(\partial B_t(0))} \le C\, t^{-1} \tag{VI.46}$$

and it satisfies

$$\vec{n}\times\nabla\vec{X} + \vec{n}\times\nabla^\perp\vec{v} = -\nabla^\perp\vec{X} + \nabla\vec{v} - \vec{n}\,\nabla S - \nabla\vec{E} + \vec{n}\times\nabla^\perp\vec{E} \quad . \tag{VI.47}$$

Let $\vec{w}$ be the solution of

$$\left\{\begin{array}{l} \Delta\vec{w} = \nabla\vec{n}\cdot\nabla^\perp(\vec{v}-\vec{E}) \quad \text{on } B_R(0) \\[5mm] \vec{w} = 0 \qquad \text{on } \partial B_R(0) \quad . \end{array}\right. \tag{VI.48}$$

Using Wente estimates we obtain

$$\|\nabla\vec{w}\|_{L^2(B_R(0))} \le C\,\|\nabla(\vec{v}-\vec{E})\|_{L^2(B_R(0))}\,\|\nabla\vec{n}\|_{L^2(B_R(0))} = o(1) \quad . \tag{VI.49}$$

Denote

$$\eta_3(t) := t^{-1}\ \left[\int_{B_{2t}\setminus B_{t/2}} |\nabla\vec{w}|^2\ dx^2\right]^{1/2} = o(1)\, t^{-1} \quad .$$

From (VI.49) we obtain

$$\int_{2r}^{R/2} \eta_3^2(t)\ t\, dt = o(1) \quad . \tag{VI.50}$$

Interpolation inequalities give again using (VI.31) we obtain

$$\|\nabla\vec{w}\|_{L^\infty(\partial B_t(0))} \le C\ [o(1)\,\delta(t) + \eta_3(t)] = \ o(1)\ t^{-1} \quad . \tag{VI.51}$$

Let $\vec{Z}$ such that

$$\vec{n}\times\nabla^\perp(\vec{v}-\vec{E}) = \nabla^\perp\vec{Z} + \nabla\vec{w} \quad .$$

Clearly from the above we have also

$$\begin{array}{l} \|\nabla\vec{Z}\|_{L^2(B_R(0))} = o(1) \\[5mm] \text{and} \quad \|\nabla\vec{Z}\|_{L^\infty(\partial B_t(0))} \le C\ \left[o(1)\ \delta(t) + \displaystyle\sum_{i=1}^{3}\eta_i(t)\right] = o(1)\ t^{-1} \quad . \end{array} \tag{VI.52}$$

Let $\vec{T} := \vec{X} + \vec{Z}$. We have

$$\nabla\vec{T} = -\nabla^\perp\vec{v} + \nabla^\perp\vec{E} + \vec{n}\ \nabla^\perp S + \vec{n}\times\nabla^\perp\vec{X} \quad , \tag{VI.53}$$

which implies in particular

$$\nabla^\perp S + \vec{n}\cdot\nabla^\perp(\vec{E}-\vec{v}) = \vec{n}\cdot\nabla\vec{T} \quad . \tag{VI.54}$$

Regarding the estimates we have from (VI.46) and (VI.52)

$$\|\nabla\vec{T}\|_{L^\infty(\partial B_t(0))} \le C\ t^{-1} \qquad \Longrightarrow \qquad \|\nabla\vec{T}\|_{L^{2,\infty}(B_R\setminus B_r)} \le C \quad , \tag{VI.55}$$

where the constant C (as for all constants C above) is independent of k. Let B be the solution of

$$\left\{\begin{array}{l} \Delta B = \nabla\vec{n}\cdot\nabla^\perp(\vec{E}-\vec{v}) \quad \text{in } B_R(0) \\[5mm] B = 0 \quad . \end{array}\right. \tag{VI.56}$$

Similarly as above we obtain

$$\|\nabla B\|_{L^{2,1}(B_R(0))} = o(1) \quad \text{and} \quad \|\nabla B\|_{L^\infty(\partial B_t(0))} \le o(1)\ t^{-1} \quad . \tag{VI.57}$$

Denote

$$\eta_4(t) := t^{-1}\ \left[\int_{B_{2t}\setminus B_{t/2}} |\nabla B|^2\ dx^2\right]^{1/2} = o(1)\, t^{-1} \quad .$$

From (VI.57) we obtain

$$\int_{2r}^{R/2} \eta_4^2(t)\ t\, dt = o(1) \quad . \tag{VI.58}$$

Interpolation inequalities give again

$$\|\nabla B\|_{L^\infty(\partial B_t(0))} \le C\ [o(1)\,\delta(t) + \eta_4(t)] \le o(1)\ t^{-1} \quad . \tag{VI.59}$$

Let D such that

$$\vec{n}\cdot\nabla^\perp(\vec{E}-\vec{v}) = \nabla B + \nabla^\perp D \quad . \tag{VI.60}$$

Similarly as above we obtain

$$\begin{aligned} &\|\nabla D\|_{L^2(B_R(0))} = o(1) \\ &\text{and} \quad \|\nabla D\|_{L^\infty(\partial B_t(0))} \le C\ [o(1)\,\delta(t) + \eta_1(t) + \eta_2(t) + \eta_4(t)] = o(1)\ t^{-1} \quad . \end{aligned} \tag{VI.61}$$

Denoting $U := S + D$ we have form (VI.26) and (VI.61)

$$\|\nabla U\|_{L^\infty(\partial B_t(0))} \le C\ t^{-1} \qquad \Longrightarrow \qquad \|\nabla U\|_{L^{2,\infty}(B_R\setminus B_r)} \le C \quad . \tag{VI.62}$$

The pair $(U,\vec{T})$ satisfies the following system

$$\left\{\begin{array}{l} \nabla U = \nabla^\perp C - \vec{n}\cdot\nabla^\perp\vec{T} \\[2mm] \nabla\vec{T} = -\nabla^\perp\vec{v} + \nabla^\perp\vec{E} + \vec{n}\ \nabla^\perp S + \vec{n}\times\nabla^\perp\vec{X} \quad . \end{array}\right. \tag{VI.63}$$

Let $U_t := |\partial B_t|^{-1}\ \int_{\partial B_t} U$ and $\vec{T}_t := |\partial B_t|^{-1}\ \int_{\partial B_t} \vec{T}$. The system (VI.63) implies

$$\left\{\begin{array}{l} \displaystyle\frac{dU_t}{dt} = -\frac{1}{2\pi}\int_0^{2\pi} [\vec{n}-\vec{n}_t]\cdot\frac{1}{t}\frac{\partial\vec{T}}{\partial\theta}\, d\theta \\[4mm] \displaystyle\frac{d\vec{T}_t}{dt} = \frac{1}{2\pi}\int_0^{2\pi} [\vec{n}-\vec{n}_t]\ \frac{1}{t}\frac{\partial S}{\partial\theta}\, d\theta + \frac{1}{2\pi}\int_0^{2\pi} [\vec{n}-\vec{n}_t]\times\frac{1}{t}\frac{\partial\vec{X}}{\partial\theta}\, d\theta \quad , \end{array}\right. \tag{VI.64}$$

where $\vec{n}_t := |\partial B_t|^{-1}\ \int_{\partial B_t}\vec{n}$. We have

$$\|\vec{n}-\vec{n}_t\|_{L^\infty(\partial B_t)} \le C\ t\ \delta(t) \quad .$$

Hence using the estimates above we obtain

$$\left|\frac{dU_t}{dt}\right| + \left|\frac{d\vec{T}_t}{dt}\right| \le C\,\delta(t) \quad , \tag{VI.65}$$

which implies

$$\int_r^R \left|\frac{dU_t}{dt}\right|^2+\left|\frac{d\vec{T}_t}{dt}\right|^2\ t\ dt\le C\quad , \tag{VI.66}$$

where C is again independent of k. The system (VI.63) implies

$$\left\{\begin{array}{l}\Delta U=-\nabla\vec{n}\cdot\nabla^\perp\vec{T}\\[5mm] \Delta\vec{T}=\nabla\vec{n}\cdot\nabla^\perp S+\nabla\vec{n}\times\nabla^\perp\vec{X}\quad .\end{array}\right. \tag{VI.67}$$

We can then make use of lemma 10 of [28] we obtain that ∇U and $\nabla\vec{T}$ are uniformly bounded in L^2

$$\int_{B_R\setminus B_r}|\nabla U|^2+|\nabla\vec{T}|^2\ dx^2\le C\quad , \tag{VI.68}$$

which itself implies using (VI.61) and (VI.52)

$$\int_{B_R\setminus B_r}|\nabla S|^2+|\nabla\vec{X}|^2\ dx^2\le C\quad , \tag{VI.69}$$

where C is independent of k. We bootstrap this information in (VI.64) as follows

$$\begin{array}{l}\displaystyle\int_r^R\left|\frac{dU_t}{dt}\right|+\left|\frac{d\vec{T}_t}{dt}\right|\ dt\le\int_r^R\ \delta(t)\ \int_{\partial B_t}|\nabla\vec{T}|+|\nabla\vec{X}|+|\nabla S|\\[5mm] \displaystyle\le\left[\int_r^R\ \delta^2(t)\ t\ dt\right]^{1/2}\ \left[\int_r^R\ t^{-1}\left[\int_{\partial B_t}|\nabla\vec{T}|+|\nabla\vec{X}|+|\nabla S|\right]^2\ dt\right]^{1/2}\\[5mm] \displaystyle\le\left[\int_r^R\ \delta^2(t)\ t\ dt\right]^{1/2}\ \left[\int_{B_R\setminus B_r}|\nabla S|^2+|\nabla\vec{T}|^2+|\nabla\vec{X}|^2\ dx^2\right]^{1/2}\le C\quad .\end{array} \tag{VI.70}$$

We can chose S and $\vec{R}$ which were fixed modulo the addition of an arbitrary constant in such a way that

$$0=U_r=|\partial B_r|^{-1}\int_{\partial B_r}U\qquad\text{and}\qquad 0=\vec{T}_r=|\partial B_r|^{-1}\int_{\partial B_r}\vec{T}\quad . \tag{VI.71}$$

Combining this choice with (VI.70) we obtain

$$|U_t|_{L^\infty([r,R])}+|\vec{T}_t|_{L^\infty([r,R])}\le C\quad . \tag{VI.72}$$

We can then make use of lemma 8 of [28] and deduce

$$\|\nabla U\|_{L^{2,1}(B_R\setminus B_r)}+\|\nabla\vec{T}\|_{L^{2,1}(B_R\setminus B_r)}\le C\quad . \tag{VI.73}$$

We have from (IV.28)

$$\begin{array}{l}2\,(1+2\,\sigma^2\,(1+H^2)-l_\sigma\,\alpha)\ e^{2\,\lambda}\vec{H}=\nabla^\perp S\cdot\nabla\vec{\Phi}-\nabla\vec{R}\times\nabla^\perp\vec{\Phi}-\nabla\vec{v}\times\nabla\vec{\Phi}\\[5mm] =\nabla^\perp U\cdot\nabla\vec{\Phi}-\nabla\vec{T}\times\nabla^\perp\vec{\Phi}-\nabla\vec{v}\times\nabla\vec{\Phi}-\nabla^\perp D\cdot\nabla\vec{\Phi}+\nabla(\vec{F}+\vec{Z})\times\nabla^\perp\vec{\Phi}\quad .\end{array} \tag{VI.74}$$

Hence (VI.73) implies

$$\|h_\sigma^{-1}\ e^\lambda\vec{H}+[\nabla\vec{v}+\nabla^\perp(\vec{F}+\vec{Z})]\times\nabla\vec{\Phi}\,e^{-\lambda}+\nabla^\perp D\cdot\nabla\vec{\Phi}\,e^{-\lambda}\|_{L^{2,1}(B_R\setminus B_r)}\le C\quad , \tag{VI.75}$$

where $h_\sigma^{-1} := 2\,(1+2\,\sigma^2\,(1+H^2)-l_\sigma\,\alpha)$. For any $\varepsilon>0$ we choose r^k and R^k such that

$$\|s\,\delta^k(s)\|_{L^\infty([r^k,R^k])}\le\varepsilon\quad ,$$

which implies in particular using (VI.8) that

$$\|(h^k_\sigma)^{-1}\ e^{\lambda^k}\,\vec{H}^k\|_{L^{2,\infty}(B_{R^k}\setminus B_{r^k})}\le C\,\varepsilon\quad . \tag{VI.76}$$

Using (VI.28), (VI.44), (VI.52) and (VI.61) , for k large enough we have

$$\|[\nabla\vec{v}^{\,k}+\nabla^\perp(\vec{F}^{\,k}+\vec{Z}^{\,k})]\times\nabla\vec{\Phi}^{\,k}\,e^{-\lambda^k}+\nabla^\perp D^k\cdot\nabla\vec{\Phi}^{\,k}\,e^{-\lambda^k}\|_{L^2(B_{R^k}\setminus B_{r^k})}\le\varepsilon\quad . \tag{VI.77}$$

Combining (VI.76) and (VI.77) we obtain in particular that

$$\|(h^k_\sigma)^{-1}\ e^{\lambda^k}\,\vec{H}^k+[\nabla\vec{v}^{\,k}+\nabla^\perp(\vec{F}^{\,k}+\vec{Z}^{\,k})]\times\nabla\vec{\Phi}^{\,k}\,e^{-\lambda^k}+\nabla^\perp D^k\cdot\nabla\vec{\Phi}^{\,k}\,e^{-\lambda^k}\|_{L^{2,\infty}(B_{R^k}\setminus B_{r^k})}\le C\,\varepsilon\quad . \tag{VI.78}$$

Combining (VI.75) and (VI.78) we obtain

$$\|(h^k_\sigma)^{-1}\ e^{\lambda^k}\,\vec{H}^k+[\nabla\vec{v}^{\,k}+\nabla^\perp(\vec{F}^{\,k}+\vec{Z}^{\,k})]\times\nabla\vec{\Phi}^{\,k}\,e^{-\lambda^k}+\nabla^\perp D^k\cdot\nabla\vec{\Phi}^{\,k}\,e^{-\lambda^k}\|_{L^{2}(B_{R^k}\setminus B_{r^k})}\le C\,\sqrt{\varepsilon}\quad . \tag{VI.79}$$

Combining (VI.77) and (VI.79) we then obtain

$$\|(h^k_\sigma)^{-1}\ e^{\lambda^k}\,\vec{H}^k\|_{L^2(B_{R^k}\setminus B_{r^k})}\le C\,[\sqrt{\varepsilon}+\varepsilon]\quad , \tag{VI.80}$$

which implies that the Willmore energy is as small as we want in any neck region for k large enough. We deduce lemma VI.1 from this fact and the final arguments of [8]

VII Appendix

Lemma VII.1. *There exists $C>0$ such that for any $\sigma\in(0,1)$ and any $\nabla a\in L^4(D^2)$ and $\nabla b\in L^{4/3}(D^2)$, denoting φ the $W^{1,1}$ solution to the following equation*

$$\left\{\begin{array}{l}-\Delta\varphi=\partial_{x_1}a\,\partial_{x_2}b-\partial_{x_2}a\,\partial_{x_1}b\quad \text{in } D^2\\[2mm] \varphi=0\qquad \text{on }\partial D^2\quad ,\end{array}\right. \tag{VII.1}$$

then the following inequality holds

$$\|\nabla\varphi\|_{L^{2,\infty}}\le C\,\|\nabla a\|_{L^{2,\infty}\cap\sigma^{1/2}L^4(D^2)}\ \|\nabla b\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}\quad , \tag{VII.2}$$

and

$$\|\nabla\varphi\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}\le C\,\|\nabla a\|_{L^{2,\infty}}\ \|\nabla b\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}\quad , \tag{VII.3}$$

where

$$\|f\|_{L^{2,\infty}\cap\sigma^{1/2}L^4(D^2)}:=\|f\|_{L^{2,\infty}(D^2)}+\sigma^{1/2}\,\|f\|_{L^4(D^2)}\quad ,$$

and

$$\|f\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}:=\inf\left\{\|f_1\|_{L^{2,\infty}(D^2)}+\sigma^{-1/2}\|f_2\|_{L^{4/3}(D^2)}\quad ;\quad f=f_1+f_2\right\}\quad .$$

Proof of lemma VII.1. Let X_1 and X_2 such that

$$\nabla b=X_1+X_2\quad \|X_1\|_{L^{2,\infty}(D^2)}+\sigma^{-1/2}\|X_2\|_{L^{4/3}(D^2)}\le 2\,\|\nabla b\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)} \tag{VII.4}$$

Let c_i for $i=1,2$ satisfying

$$\left\{\begin{array}{ll} -\Delta c_i = \operatorname{div}(X_i^\perp) & \text{in } D^2\\[5mm] c_i=0 & \text{on } \partial D^2\end{array}\right. \tag{VII.5}$$

There exists a constant C independent of σ such that

$$\|\nabla c_1\|_{L^{2,\infty}(D^2)}\le C\,\|X_1\|_{L^{2,\infty}(D^2)}\quad\text{and}\quad \|\nabla c_2\|_{L^{4/3}(D^2)}\le C\,\|X_2\|_{L^{4/3}(D^2)}\quad . \tag{VII.6}$$

Applying Poincaré Lemma we obtain the existence of b_i such that $\int_{D^2} b_i=0$ and

$$X_i^\perp+\nabla c_i=\nabla^\perp b_i\quad\Longleftrightarrow\quad X^i=\nabla b_i+\nabla^\perp c_i\quad , \tag{VII.7}$$

and we have

$$\|\nabla b_1\|_{L^{2,\infty}(D^2)}\le (C+1)\,\|X_1\|_{L^{2,\infty}(D^2)}\quad\text{and}\quad \|\nabla b_2\|_{L^{4/3}(D^2)}\le (C+1)\,\|X_2\|_{L^{4/3}(D^2)}\quad . \tag{VII.8}$$

Observe that

$$\nabla b=\nabla b_1+\nabla b_2+\nabla^\perp c_1+\nabla^\perp c_2\qquad \Delta(c_1+c_2)=0\quad . \tag{VII.9}$$

Since $c_1+c_2=0$ on ∂D^2 we have $c_1+c_2\equiv 0$ on D^2 and

$$\nabla b=\nabla b_1+\nabla b_2 \tag{VII.10}$$

Let

$$\left\{\begin{array}{ll} -\Delta\varphi_i=\partial_{x_1}a\,\partial_{x_2}b_i-\partial_{x_2}a\,\partial_{x_1}b_i & \text{in } D^2\\[5mm] \varphi_i=0 & \text{on } \partial D^2\quad .\end{array}\right. \tag{VII.11}$$

Using Wente estimates (see for instance [40]) we obtain respectively

$$\|\nabla\varphi_1\|_{L^{2,\infty}(D^2)}\le C\;\|\nabla a\|_{L^{2,\infty}(D^2)}\,\|\nabla b_1\|_{L^{2,\infty}(D^2)}\quad , \tag{VII.12}$$

and

$$\|\nabla\varphi_2\|_{L^{2}(D^2)}\le C\;\|\nabla a\|_{L^{4}(D^2)}\,\|\nabla b_2\|_{L^{4/3}(D^2)}\quad . \tag{VII.13}$$

Combining (VII.12) and (VII.13) we obtain (VII.2). We now write

$$\left\{\begin{array}{ll} -\Delta\varphi_2=\operatorname{div}(b_2\,\nabla^\perp a) & \text{in } D^2\\[5mm] \varphi=0 & \text{on } \partial D^2\quad .\end{array}\right. \tag{VII.14}$$

Sobolev-Lorentz embedding theorem gives, since $\int_{D^2} b_2=0$,

$$\|b_2\|_{L^{4,4/3}(D^2)}\le C\,\|\nabla b_2\|_{L^{4/3}(D^2)}\quad . \tag{VII.15}$$

Hence we have using fundamental properties of Lorentz spaces (see [20])

$$\|b_2\,\nabla^\perp a\|_{L^{4/3}(D^2)}\le C\;\|\nabla a\|_{L^{2,\infty}(D^2)}\;\|b_2\|_{L^{4,4/3}(D^2)}\le C\,\|\nabla a\|_{L^{2,\infty}(D^2)}\;\|\nabla b_2\|_{L^{4/3}(D^2)}\quad . \tag{VII.16}$$

Hence using classical elliptic estimates we have

$$\|\nabla\varphi_2\|_{L^{4/3}(D^2)}\le C\;\|\nabla a\|_{L^{2,\infty}(D^2)}\;\|\nabla b_2\|_{L^{4/3}(D^2)}\quad . \tag{VII.17}$$

Combining (VII.12) and (VII.17) we obtain using (VII.4)

$$
\begin{aligned}
&\|\nabla\varphi_1\|_{L^{2,\infty}(D^2)}+\sigma^{-1/2}\|\nabla\varphi_2\|_{L^{4/3}(D^2)}\\
&\quad\le C\,\|\nabla a\|_{L^{2,\infty}(D^2)}\ \left[\|\nabla b_1\|_{L^{2,\infty}(D^2)}+\sigma^{-1/2}\|\nabla b_2\|_{L^{4/3}(D^2)}\right]\\
&\quad\le C\,\|\nabla a\|_{L^{2,\infty}(D^2)}\ \left[\|X_1\|_{L^{2,\infty}(D^2)}+\sigma^{-1/2}\|X_2\|_{L^{4/3}(D^2)}\right]\\
&\quad\le C\,\|\nabla a\|_{L^{2,\infty}(D^2)}\ \|\nabla b\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}\quad .
\end{aligned}
\tag{VII.18}
$$

This implies (VII.3) and lemma VII.1 is proved. □

Lemma VII.2. *Let $m\in\mathbb{N}^*$, and $1\le p<+\infty$. There exists $\varepsilon(m,p)>0$ such that for any sequence of maps $\mathcal{A}_k\in W^{1,2p}(D^2,M_m(\mathbb{R}))$ satisfying*

$$
\int_{D^2}|\nabla\mathcal{A}_k|^{2p}\le\varepsilon(m,p)
\tag{VII.19}
$$

and weakly converging to $\mathcal{A}_\infty$ in $W^{1,2p}$, for any sequence of maps $\vec\varphi_k$ weakly converging in $W^{1,\frac{2p}{2p-1}}(D^2,\mathbb{R}^m)$ and any sequence of maps $\vec F_k$ strongly converging to $\vec F_\infty$ in $L^{\frac{2p}{2p-1}}(D^2,\mathbb{R}^2\otimes\mathbb{R}^m)$ to a limit $\vec F_\infty$ and satisfying

$$
-\Delta\vec\varphi_k=\nabla\mathcal{A}_k\cdot\nabla^\perp\vec\varphi_k+div\,\vec F_k\qquad in\quad \mathcal{D}'(D^2)\quad ,
\tag{VII.20}
$$

then $\vec\varphi_k$ strongly converges in $W^{1,\frac{2p}{2p-1}}_{loc}(D^2,\mathbb{R}^m)$.[15] □

Proof of lemma VII.2. We consider the case $p=1$ which is the most delicate. We first prove the claim assuming that $\vec F_k\to\vec F_\infty$ in $W^{1,2}(D^2,\mathbb{R}^2\otimes\mathbb{R}^m)$. Under this assumption we claim that there exists[16] $q>1$ such that for any $\Omega\subset\subset D^2$

$$
\limsup_{k\to+\infty}\|\vec\phi_k\|_{W^{2,q}(\Omega)}<+\infty
\tag{VII.21}
$$

This last claim implies, using Rellich Kondrachov, that $\vec\phi_k\to\vec\phi_\infty$ strongly in $W^{1,2}_{loc}(D^2)$.

Proof of (VII.21). Let $\rho<1$, we prove that there exists $\gamma>0$ such that

$$
\limsup_{k\to+\infty}\ \sup_{x_0\in B_\rho(0)}\ \sup_{r<1-\rho}\ r^{-\gamma}\int_{B_r(x_0)}|\nabla\vec\varphi_k|^2\ dx^2
\tag{VII.22}
$$

Let $x_0\in B_\rho(0)$ and $r<1-\rho$. On $B_\rho(0)$ we decompose $\vec\varphi_k=\vec\psi_k+\vec v_k$ where

$$
\left\{
\begin{array}{l}
-\Delta\vec\psi_k=\nabla\mathcal{A}_k\cdot\nabla^\perp\vec\varphi_k+div\vec F_k\qquad\text{in }B_r(x_0)\\[3mm]
\vec\psi_k=0\qquad\text{on}\quad\partial B_r(x_0)
\end{array}
\right.
$$

Using Wente estimate we obtain

$$
\int_{B_r(x_0)}|\nabla\vec\psi_k|^2\ dx^2\le C\,\sqrt{\varepsilon(2,m)}\int_{B_r(x_0)}|\nabla\vec\varphi_k|^2\ dx^2+C\,r^2\ \|\vec F_k\|^2_{W^{1,2}(D^2)}\quad .
\tag{VII.23}
$$

[15] If one assume further $p>1$ the smallness condition (VII.19) is not needed for the same result to hold.

[16] In fact under the assumptions this is true for any $q<2$

Since $\vec{v}_k$ is harmonic, the monotonicity formula gives for any $t<1$

$$\int_{B_{tr}(x_0)} |\nabla\vec{v}_k|^2\ dx^2 \le t^2 \int_{B_r(x_0)} |\nabla\vec{v}_k|^2\ dx^2 \le t^2 \int_{B_r(x_0)} |\nabla\vec{\varphi}_k|^2\ dx^2 \quad , \tag{VII.24}$$

where we used the fact that the harmonic extension minimizes the Dirichlet energy. Combining (VII.23) and (VII.24) gives then

$$\int_{B_{2^{-1}\,r}(x_0)} |\nabla\vec{\varphi}_k|^2\ dx^2 \le\ [2\,C\,\sqrt{\varepsilon(2,m)}+2^{-1}] \int_{B_r(x_0)} |\nabla\vec{\varphi}_k|^2\ dx^2 + C\,r^2\ \|\vec{F}_k\|^2_{W^{1,2}(D^2)}$$

By choosing $2\,C\,\sqrt{\varepsilon(2,m)}<1/4$ we obtain

$$\int_{B_{2^{-1}\,r}(x_0)} |\nabla\vec{\varphi}_k|^2\ dx^2 \le \frac{3}{4} \int_{B_r(x_0)} |\nabla\vec{\varphi}_k|^2\ dx^2 + C\,r^2\ \|\vec{F}_k\|^2_{W^{1,2}(D^2)} \quad . \tag{VII.25}$$

The iteration of (VII.25) gives (VII.22). Inserting (VII.22) in the right hand side of the equation (VII.20) gives

$$\limsup_{k\to+\infty}\ \sup_{x_0\in B_\rho(0)}\ \sup_{r<1-\rho}\ r^{-\gamma/2}\int_{B_r(x_0)} |\Delta\vec{\varphi}_k|\ dx^2$$

where we have used $\|div(\vec{F}_k)\|_{L^1(B_r(x_0))}\le r\,\|\vec{F}_k\|_{W^{1,2}(D^2)}$. Using Adams estimates we deduce the existence of $s>2$ such that

$$\limsup_{k\to+\infty}\|\vec{\varphi}_k\|_{W^{1,s}(B_{1-2\rho}(0)}<+\infty \quad .$$

Inserting this bound in the right hand side of the equation (VII.20) gives (VII.21).

We consider now the general case : $\vec{F}_k\to\vec{F}_\infty$ strongly in $L^2(D^2)$. The weak convergence in $W^{1,2}$ respectively of $\vec{\phi}_k$ towards $\vec{\phi}_\infty$ and of $\mathcal{A}_k$ towards $\mathcal{A}_\infty$ implies that the limits, due to the jacobian structure of the r.h.s., satisfy the equation

$$-\Delta\vec{\varphi}_\infty = \nabla\mathcal{A}_\infty\cdot\nabla^\perp\vec{\varphi}_\infty + \mathrm{div}\ \vec{F}_\infty \qquad \text{in} \quad \mathcal{D}'(D^2) \quad ,$$

Let $\vec{F}^{\,s}_\infty := \vec{F}_\infty\star\chi^s$ where $\chi^s(x):=s^{-2}\,\chi(x/s)$. Let $\vec{\varphi}^{\,s}_\infty$ be the unique solution of

$$\left\{\begin{array}{l} -\Delta\vec{\varphi}^{\,s}_\infty = \nabla\mathcal{A}_\infty\cdot\nabla^\perp\vec{\varphi}^{\,s}_\infty + \mathrm{div}\ \vec{F}^{\,s}_\infty \qquad \text{in} \quad D^2 \\[5mm] \vec{\varphi}^{\,s}_\infty = \vec{\varphi}_\infty \qquad \text{on} \quad \text{on } \partial D^2 \quad . \end{array}\right.$$

We claim that $\vec{\varphi}^s_\infty$ strongly converges to $\vec{\varphi}_\infty$ in $W^{1,2}(D^2)$. We have indeed

$$\left\{\begin{array}{l} -\Delta(\vec{\varphi}^{\,s}_\infty-\vec{\varphi}_\infty) = \nabla\mathcal{A}_\infty\cdot\nabla^\perp(\vec{\varphi}^{\,s}_\infty-\vec{\varphi}_\infty) + \mathrm{div}\ (\vec{F}^{\,s}_\infty-\vec{F}_\infty) \qquad \text{in} \quad D^2 \\[5mm] \vec{\varphi}^{\,s}_\infty-\vec{\varphi}_\infty = 0 \quad . \end{array}\right.$$

Multiplying by $\vec{\varphi}^{\,s}_\infty-\vec{\varphi}_\infty$ and integrating by parts gives

$$\begin{array}{l} \displaystyle\int_{D^2} |\nabla(\vec{\varphi}^{\,s}_\infty-\vec{\varphi}_\infty)|^2\ dx^2 \le \int_{D^2} (\vec{\varphi}^{\,s}_\infty-\vec{\varphi}_\infty)\cdot\nabla\mathcal{A}_\infty\cdot\nabla^\perp(\vec{\varphi}^{\,s}_\infty-\vec{\varphi}_\infty)\ dx^2 \\[5mm] \displaystyle\qquad -\int_{D^2} \vec{F}^{\,s}_\infty-\vec{F}_\infty\cdot\nabla(\vec{\varphi}^{\,s}_\infty-\vec{\varphi}_\infty)\ dx^2 \end{array} \tag{VII.26}$$

Recall the Wente inequality

$$\forall\, a,b \in W^{1,2}(D^2) \quad \text{and} \quad \forall\, c \in W^{1,2}_0(D^2) \qquad \left|\int_{D^2} c\ \nabla a\cdot\nabla^\perp b\right| \le C\ \|\nabla a\|_2\ \|\nabla b\|_2\ \|\nabla c\|_2$$

Applying this inequality to the first term in the r.h.s. of (VII.26)

$$\|\vec{\varphi}^s_\infty - \vec{\varphi}_\infty\|_{W^{1,2}(D^2)} \le C\ \varepsilon(m,2)\ \|\vec{\varphi}^s_\infty - \vec{\varphi}_\infty\|_{W^{1,2}(D^2)} + \|\vec{F}^s_\infty - \vec{F}_\infty\|_2$$

Choosing $\varepsilon(m,1)$ small enough in such a way $C\ \varepsilon(m,1) < 1/2$ we obtain

$$\|\vec{\varphi}^s_\infty - \vec{\varphi}_\infty\|_{W^{1,2}(D^2)} \le\ 2\ \|\vec{F}^s_\infty - \vec{F}_\infty\|_2 \quad , \tag{VII.27}$$

which implies

$$\vec{\varphi}^s_\infty \longrightarrow \vec{\varphi}_\infty \quad \text{strongly in } W^{1,2}(D^2) \quad .$$

Let $\vec{\varphi}^s_k$ be the unique solution of

$$\left\{\begin{array}{l} -\Delta\vec{\varphi}^s_k = \nabla\mathcal{A}_k\cdot\nabla^\perp\vec{\varphi}^s_k + \text{div}\ (\vec{F}_k\star\chi^s) \qquad \text{in} \quad D^2 \\[5mm] \vec{\varphi}^s_\infty = \vec{\varphi}_k \qquad \text{on} \qquad \text{on } \partial D^2 \quad . \end{array}\right.$$

For any fixed $s>0$ and any $\Omega\subset\subset D^2$, using the first part of the proof we have the existence of $q>1$ such that

$$\limsup_{k\rightarrow+\infty} \|\vec{\varphi}^s_k\|_{W^{2,q}(\Omega)} < +\infty \quad .$$

Hence

$$\lim_{k\rightarrow+\infty} \|\vec{\varphi}^s_k - \vec{\varphi}^s_\infty\|_{W^{1,2}} = 0 \quad . \tag{VII.28}$$

Similarly as in the proof of (VII.27) we have for $\varepsilon(m,1)$ chosen as above

$$\|\vec{\varphi}^s_k - \vec{\varphi}_k\|_{W^{1,2}(D^2)} \le\ 2\ \|\vec{F}^s_k - \vec{F}_k\|_{L^2(D^2)} \quad . \tag{VII.29}$$

Using the triangular inequality and Young inequality we have

$$\begin{aligned} \|\vec{F}^s_k - \vec{F}_k\|_{L^2(D^2)} &\le \|(\vec{F}_k - \vec{F}_\infty)\star\chi^s\|_{L^2(D^2)} + \|\vec{F}^s_\infty - \vec{F}_\infty\|_{L^2(D^2)} + \|\vec{F}_k - \vec{F}_\infty\|_{L^2(D^2)} \\ &\le 2\,\|\vec{F}_k - \vec{F}_\infty\|_{L^2(D^2)} + \|\vec{F}^s_\infty - \vec{F}_\infty\|_{L^2(D^2)} \quad . \end{aligned} \tag{VII.30}$$

From the triangular inequality we have for any $s>0$ and $\Omega\subset\subset D^2$ we have combining (VII.27), (VII.29) and (VII.30)

$$\begin{aligned} \|\nabla(\vec{\varphi}_k - \vec{\varphi}_\infty)\|_{L^2(\Omega)} &\le \|\nabla(\vec{\varphi}_k - \vec{\varphi}^s_k)\|_{L^2(\Omega)} + \|\nabla(\vec{\varphi}^s_k - \vec{\varphi}^s_\infty)\|_{L^2(\Omega)} + \|\nabla(\vec{\varphi}_\infty - \vec{\varphi}^s_\infty)\|_{L^2(\Omega)} \\ &\le 2\ \|\vec{F}^s_k - \vec{F}_k\|_{L^2(D^2)} + \|\nabla(\vec{\varphi}^s_k - \vec{\varphi}^s_\infty)\|_{L^2(\Omega)} + 2\,\|\vec{F}^s_\infty - \vec{F}_\infty\|_{L^2(D^2)} \\ &\le 4\,\|\vec{F}_k - \vec{F}_\infty\|_{L^2(D^2)} + \|\nabla(\vec{\varphi}^s_k - \vec{\varphi}^s_\infty)\|_{L^2(\Omega)} + 4\,\|\vec{F}^s_\infty - \vec{F}_\infty\|_{L^2(D^2)} \quad . \end{aligned}$$

Let $\delta>0$, there exists $s>0$ such that $\|\vec{F}^s_\infty - \vec{F}_\infty\|_{L^2(D^2)} \le \delta/8$. Once s is fixed, using (VII.28), we have the existence of k_δ such that

$$\forall k > k_\delta \quad \|\vec{F}_k - \vec{F}_\infty\|_{L^2(D^2)} < \delta/16 \quad \text{and} \quad \|\nabla(\vec{\varphi}^s_k - \vec{\varphi}^s_\infty)\|_{L^2(\Omega)} < \delta/16 \quad .$$

Hence for $k>k_\delta$ we have $\|\nabla(\vec{\varphi}_k - \vec{\varphi}_\infty)\|_{L^2(\Omega)} < \delta$. This implies the lemma for $p=1$.

□

Lemma VII.3. *For any $C_0 > 0$ there exists $\varepsilon > 0$ such that for any conformal weak immersion in $\mathcal{E}_{\Sigma,2}(D^2)$ satisfying*

$$\|\nabla\lambda\|_{L^{2,\infty}(D^2)} \le C_0 \quad \text{and} \quad \int_{D^2} |\nabla\vec{n}|^2\, dx^2 < \varepsilon \quad ,$$

then

$$\sigma^2 \int_{D^2_{1/2}} |\nabla\vec{n}|^4\, e^{-2\lambda}\, dx^2 \le C\, \sigma^2 \int_{D^2} H^4\, e^{2\lambda}\, dx^2 + C\, \sigma^2\, e^{-2\overline{\lambda}} \left[\int_{D^2} |\nabla\vec{n}|^2\, dx^2\right]^2 \qquad \text{(VII.31)}$$

where $e^\lambda = |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$ and $\overline{\lambda} = |D^2_{1/2}|^{-1} \int_{D^2_{1/2}} \lambda(x)\, dx^2$. □

Proof of lemma VII.3. Arguing as in the beginning of the proof of lemma IV.1 we have

$$\|\alpha - \overline{\alpha}\|_{L^\infty(D^2_{5/6})} \le C \quad . \qquad \text{(VII.32)}$$

We have also

$$\Delta\vec{\Phi} = 2\, e^{2\lambda}\, \vec{H} = 2\, e^{2\overline{\lambda}}\, e^{2\,(\lambda-\overline{\lambda})}\, \vec{H} \quad \text{in } D^2 \quad . \qquad \text{(VII.33)}$$

This gives that $\nabla^2\vec{\Phi} \in L^4(D^2_{3/4})$ and hence $\nabla\vec{n} \in L^4(D^2_{3/4})$ Let $\vec{a}$ satisfying

$$\left\{\begin{array}{l} \Delta\vec{a} = \mathrm{div}(\vec{n}\times\nabla\vec{n}) \quad \text{in } D^2 \\[3mm] \vec{a} = 0 \qquad \text{on } \partial D^2 \quad , \end{array}\right. \qquad \text{(VII.34)}$$

and let $\vec{b}$ such that $\vec{n}\times\nabla\vec{n} = \nabla\vec{a} + \nabla^\perp\vec{b}$. Using classical elliptic estimates we have

$$\int_{D^2} |\nabla\vec{a}|^2 + |\nabla\vec{b}|^2\, dx^2 \le C \int_{D^2} |\nabla\vec{n}|^2\, dx^2 \le C\,\varepsilon \quad . \qquad \text{(VII.35)}$$

Let $\rho \in [1/2, 3/4]$ such that

$$\int_{\partial D^2_\rho} |\nabla\vec{a}|^2 + |\nabla\vec{b}|^2\, dl \le 4 \int_{D^2} |\nabla\vec{n}|^2\, dx^2 \le 4\varepsilon \qquad \text{(VII.36)}$$

Observe that $W^{1,2}(\partial D^2_\rho) \hookrightarrow W^{1-1/4,4}(\partial D^2_\rho)$. Hence we have

$$\|\vec{a}\|_{W^{1-1/4,4}(\partial D^2_\rho)} + \|\vec{b}\|_{W^{1-1/4,4}(\partial D^2_\rho)} \le C \left[\int_{D^2} |\nabla\vec{n}|^2\, dx^2\right]^{1/2} \qquad \text{(VII.37)}$$

Recall now from [40] the following general formula

$$\vec{n}\times\nabla\vec{n} = 2\, H\, \nabla^\perp\vec{\Phi} + \nabla^\perp\vec{n} \qquad \text{(VII.38)}$$

Hence we have in particular on D^2_ρ

$$\Delta\vec{a} = 2\,\mathrm{div}(H\,\nabla^\perp\vec{\Phi}) \qquad \text{(VII.39)}$$

Classical elliptic estimates give then, using (VII.37),

$$\|\nabla\vec{a}\|_{L^4(D^2_\rho)} \le C\, e^{\overline{\lambda}} \left[\int_{D^2_\rho} H^4\, dx^2\right]^{1/4} + C \left[\int_{D^2} |\nabla\vec{n}|^2\, dx^2\right]^{1/2} \qquad \text{(VII.40)}$$

We have also on D^2_ρ

$$\Delta\vec{b} = \nabla^\perp\vec{n}\times\nabla\vec{n} \quad . \tag{VII.41}$$

Classical elliptic estimates imply (using $H^{1/2}(D^2_\rho)\hookrightarrow L^4(D^2_\rho)$).

$$\|\nabla\vec{b}\|_{L^4(D^2_\rho)} \le C\ \|\Delta\vec{b}\|_{L^{4/3}(D^2_\rho)} + C\,\|\nabla\vec{b}\|_{L^2(\partial D^2_\rho)} \quad .$$

Hence using (VII.36) and (VII.41) we obtain

$$\begin{aligned}\|\nabla\vec{b}\|_{L^4(D^2_\rho)} \le C\ &\left[\int_{D^2_\rho}|\nabla\vec{n}|^{4/3}\,|\nabla\vec{n}|^{4/3}\ dx^2\right]^{3/4}\\ &+C\ \left[\int_{D^2}|\nabla\vec{n}|^2\ dx^2\right]^{1/2} \quad .\end{aligned} \tag{VII.42}$$

Hence

$$\|\nabla\vec{b}\|_{L^4(D^2_\rho)} \le C\ \left[\int_{D^2_\rho}|\nabla\vec{n}|^2\ dx^2\right]^{1/2}\ \|\nabla\vec{n}\|_{L^4(D^2_\rho)} + C\,\|\nabla\vec{n}\|_{L^2(D^2)} \quad . \tag{VII.43}$$

Combining (VII.40) with (VII.43) together with the fact that $\|\nabla\vec{n}\|_{L^2(D^2)}<\varepsilon$ gives

$$\begin{aligned}\|\nabla\vec{n}\|_{L^4(D^2_\rho)} \le\ & C\,e^{\overline{\lambda}}\left[\int_{D^2_\rho}H^4\,dx^2\right]^{1/4} + C\,\varepsilon\ \|\nabla\vec{n}\|_{L^4(D^2_\rho)}\\ &+C\,\|\nabla\vec{n}\|_{L^2(D^2)} \quad .\end{aligned} \tag{VII.44}$$

Hence for ε small enough we finally obtain

$$\sigma^2\int_{D^2_\rho}|\nabla\vec{n}|^4\ e^{-2\lambda}\,dx^2 \le C\ \sigma^2\int_{D^2_\rho}H^4\,e^{2\lambda}\ dx^2 + \sigma^2\ e^{-2\overline{\lambda}}\left[\int_{D^2}|\nabla\vec{n}|^2\ dx^2\right]^2 \quad . \tag{VII.45}$$

□

Lemma VII.4. *For any $\gamma\in(0,1)$ there exists $\varepsilon>0$ such that for any $\vec{\phi}$ in $W^{1,2}(D^2)$ satisfying*

$$\Delta\vec{\phi} = \nabla^\perp\vec{n}\times\nabla\vec{\phi} + div\,\vec{F} \tag{VII.46}$$

where

$$\int_{D^2}|\nabla\vec{n}|^2\ dx^2 \le \varepsilon \quad \text{and} \quad \sup_{B_r(x)\subset D^2} r^{-\gamma}\int_{B_r(x)}|\vec{F}|^2\ dx^2 < +\infty \tag{VII.47}$$

then

$$\sup_{B_r(x)\subset D^2_{1/2}} r^{-\gamma}\int_{B_r(x)}|\nabla\vec{\phi}|^2\ dx^2 \le C_\gamma\ \left[\sup_{B_r(x)\subset D^2} r^{-\gamma}\int_{B_r(x)}|\vec{F}|^2\ dx^2 + \int_{D^2}|\nabla\vec{\phi}|^2\ dx^2\right] \tag{VII.48}$$

where C_γ depends only on $\gamma\in(0,1)$. □

Proof of lemma VII.4 For any $x_0\in D^2_{1/2}$ and $r<1/4$ we decompose $\vec{\phi}=\vec{\psi}+\vec{v}$ in $B_r(x_0)$ where $\vec{\psi}$ is the solution of

$$\left\{\begin{array}{l}\Delta\vec{\psi} = \nabla^\perp\vec{n}\times\nabla\vec{\phi} + \mathrm{div}\,\vec{F} \quad \text{in } B_r(x_0)\\[2mm] \vec{\psi}=0 \qquad \text{on} \quad \partial B_r(x_0)\end{array}\right.$$

Using Wente inequality we have

$$\int_{B_r(x_0)} |\nabla\vec{\psi}|^2\ dx^2 \le C\,\varepsilon\ \int_{B_r(x_0)} |\nabla\vec{\phi}|^2\ dx^2 + C\int_{B_r(x_0)} |\vec{F}|^2\ dx^2 \quad .$$

Since $\vec{v}$ is harmonic we have for any $t\in(0,1)$

$$\int_{B_{tr}(x_0)} |\nabla\vec{v}|^2\ dx^2 \le t^2\ \int_{B_r(x_0)} |\nabla\vec{v}|^2\ dx^2 \le t^2\ \int_{B_r(x_0)} |\nabla\vec{\phi}|^2\ dx^2$$

Hence in particular for $t=1/2$

$$\int_{B_{2^{-1}r}(x_0)} |\nabla\vec{\phi}|^2\ dx^2 \le (2^{-1}+C\,\varepsilon)\ \int_{B_r(x_0)} |\nabla\vec{\phi}|^2\ dx^2 + C\int_{B_r(x_0)} |\vec{F}|^2\ dx^2 \tag{VII.49}$$

We choose ε such that $(2^{-1}+C\,\varepsilon)=2^{-\gamma}$ and (VII.48) is obtained by iterating (VII.49). □

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