Calc. Var. (2017) 56:117
DOI 10.1007/s00526-017-1215-8

**Calculus of Variations**

CrossMark

# The regularity of conformal target harmonic maps

**Tristan Rivière**[1]

Received: 17 November 2016 / Accepted: 15 June 2017 / Published online: 14 July 2017


**Abstract** We establish that any weakly conformal $W^{1,2}$ map from a Riemann surface $S$ into a closed oriented sub-manifold $N^n$ of an euclidian space $\mathbb{R}^m$ realizes, for almost every sub-domain, a stationary varifold if and only if it is a smooth conformal harmonic map form $S$ into $N^n$.

**Mathematics Subject Classification** 58E20 · 49Q05 · 53A10 · 49Q15 · 49Q20

## 1 Introduction

In [5] the author developed a viscosity method in order to produce closed minimal 2 dimensional surfaces into any arbitrary closed oriented sub-manifolds $N^n$ of any euclidian spaces $\mathbb{R}^m$ by min-max type arguments. The method consists in adding to the area of an immersion $\vec{\Phi}$ of a surface $\Sigma$ into $N^n$ a more coercive term such as the $L^{2p}$ norm of the second fundamental form preceded by a small parameter $\sigma^2$:

$$A^\sigma(\vec{\Phi}) := \text{Area}(\vec{\Phi}) + \sigma^2 \int_\Sigma \left[1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2\right]^p \, dvol_{g_{\vec{\Phi}}}$$

where $\vec{\mathbb{I}}_{\vec{\Phi}}$ is the second fundamental form of the immersion $\vec{\Phi}$ and $dvol_{g_{\vec{\Phi}}}$ is the volume form associated to the induced metric. For $p > 1$ and $\sigma > 0$ one proves that the Lagrangians $A^\sigma$ are Palais–Smale in some ad-hoc Finsler bundle of immersions complete for the Palais distance. By applying the now classical *Palais–Smale deformation theory* in infinite dimensional space one can then produce critical points $\vec{\Phi}_\sigma$ to $A^\sigma$. It is proved in [4] that, for a sequence of parameters $\sigma_k \to 0$, the sequence of integer rectifiable varifolds associated to

Communicated by L. Ambrosio.

✉ Tristan Rivière
riviere@math.ethz.ch

1 Department of Mathematics, ETH Zentrum, 8093 Zurich, Switzerland

the immersion of $\Sigma$ by $\vec{\Phi}_{\sigma_k}$ does not necessarily converge[1] to a stationary integer rectifiable varifold. However, by applying *Struwe's monotonicity trick* one can always select a sequence $\sigma_j \to 0$ such that the following additional "entropy estimate" holds

$$\sigma_j \int_\Sigma \left[1 + |\vec{\mathbb{I}}_{\vec{\Phi}_{\sigma_j}}|^2\right]^p \, dvol_{g_{\vec{\Phi}_{\sigma_j}}} = o\left(\frac{1}{\sigma_j \, \log \sigma_j^{-1}}\right)$$

Assuming this additional estimate, the main achievement of [5] is to prove that the immersion of $\Sigma$ by $\vec{\Phi}_{\sigma_j}$ *varifold converges* to a *stationary integer rectifiable varifold* given by the image of a smooth Riemann surface $S$ by a weakly conformal $W^{1,2}$ map $\vec{\Phi}$ into $N^n$ equipped by an integer multiplicity. The main result of the paper is to prove that, when this multiplicity is constant, such a map is smooth and satisfies the harmonic map equation. To state our main result we need two definitions.

**Definition 1.1** A property is said to hold for **almost every smooth domain** in $\Sigma$, if for any smooth domain $\Omega$ and any smooth function $f$ such that $f^{-1}(0) = \partial\Omega$ and $\nabla f \neq 0$ on $\partial\Omega$ then for almost every $t$ close enough to zero and regular value for $f$ the property holds for the domain contained in $\Omega$ or containing $\Omega$ and bounded by $f^{-1}(\{t\})$. □

Precisely we define the notion of *target harmonic map* as follows.

**Definition 1.2** Let $(\Sigma, h)$ be a smooth closed Riemann surface equipped with a metric compatible with the complex structure. A map $\vec{\Phi} \in W^{1,2}(\Sigma, N^n)$ is *target harmonic* if for almost every smooth domain $\Omega \subset \Sigma$ and any smooth function $F$ supported in the complement of an open neighborhood[2] of $\vec{\Phi}(\partial\Omega)$ we have

$$\int_\Omega \left\langle d(F(\vec{\Phi})), d\vec{\Phi}\right\rangle_h - F(\vec{\Phi}) \; A(\vec{\Phi})(d\vec{\Phi}, d\vec{\Phi})_h \; dvol_h = 0 \tag{1.1}$$

where $h$ is any[3] metric compatible with the chosen conformal structure on $S$ and where $A(\vec{q})(\vec{X}, \vec{Y})$ denotes the second fundamental form of $N^n$ at the point $\vec{q}$ and acting on the pair of vectors $(\vec{X}, \vec{Y})$ and by an abuse of notation we write

$$A(\vec{\Phi})(d\vec{\Phi}, d\vec{\Phi})_h := \sum_{i,j=1}^{2} h_{ij} \; A(\vec{\Phi})(\partial_{x_i}\vec{\Phi}, \partial_{x_j}\vec{\Phi}).$$

□

Observe that the main difference with the general definition of being *harmonic* is that, for *target harmonic* one restricts (1.1) to test functions $F$ supported in the target while for the definition of *harmonic*, one requires (1.1) to hold for any $W^{1,2}$ test function defined on the domain. Therefore, being *harmonic* implies to be *target harmonic* and the proof of the reverse is the goal of the present work. For a weakly conformal $W^{1,2}$ map into $N^n$ the condition for $\vec{\Phi}$ to be *target harmonic* is equivalent to saying that the mapping of $\Sigma$ in $N^n$ defines a *stationary integer rectifiable varifold* (see proposition A. 1). Our main result in the present paper is the following.

[1] Even modulo extraction of subsequences and in a weak sense such as the *varifold distance* topology.

[2] Observe that for almost every domain $\Omega$ the restriction of $\vec{\Phi}$ to $\partial\Omega$ is Hölder continuous and $\vec{\Phi}(\partial\Omega)$ is then closed.

[3] One observe that the condition (1.1) is conformally invariant and hence independent of the choice of the metric $h$ within the conformal class given by $\Sigma$.

**Theorem 1.1** *Any weakly conformal target harmonic map into an arbitrary closed submanifold $N^n$ of $\mathbb{R}^m$ in two dimension is smooth and satisfy the harmonic map equation.* □

It is not known if, without the conformality assumption, a *target harmonic map* is an harmonic map in the classical sense. Following the main lines of the proof below one can prove that this is indeed the case in one dimension.

## 2 The partial regularity

Since the problem is local we shall work in a chart and hence consider maps in $W^{1,2}(D^2, N^n)$ exclusively. Indeed, being *target harmonic* in $S$ implies to be *target harmonic* in any subdomain of $S$. Hence if one proves that *target harmonic* on a disc implies *harmonic* on that disc in the classical sense we deduce that *target harmonic* on $S$ implies *harmonic* on $S$. Therefore we can reduce to the case $S = D^2$.

Let $\vec{\Phi}$ be a $W^{1,2}$ map from the disc $D^2$ into $N^n$ satisfying the *weak conformality condition*

$$|\partial_{x_1}\vec{\Phi}|^2 = |\partial_{x_2}\vec{\Phi}|^2 \quad \text{and} \quad \partial_{x_1}\vec{\Phi}\cdot\partial_{x_2}\vec{\Phi} = 0 \quad \text{a.e. on } D^2 \tag{2.1}$$

For any $A > 1$ we introduce

$$\mathcal{G}^A := \left\{ \begin{array}{c} x \text{ is a Lebesgue point for } \vec{\Phi} \text{ and } \nabla\vec{\Phi} \\ x \text{ is a point of } L^2 \text{ app. differentiability} \\ A^{-1} < |\nabla\vec{\Phi}|(x) < A \end{array} \right\}$$

We are going to prove the following lemma

**Lemma 2.1** *Let $\vec{\Phi}$ be a target harmonic map on $D^2$. Under the previous notations we have that*

$$\mathcal{G} := \cup_{A>1}\mathcal{G}^A$$

*is an open subset of $D^2$, $\vec{\Phi}$ is smooth and harmonic into $N^n$ in the strong sense on $\mathcal{G}$. Moreover*

$$\mathcal{H}^2\left(\vec{\Phi}(D^2\setminus\mathcal{G})\right) = \frac{1}{2}\int_{D^2\setminus\mathcal{G}} |\nabla\vec{\Phi}|^2(y)\, dy^2 = 0. \tag{2.2}$$

□

*Proof of lemma 2.1* Let $A > 0$. For such an $x \in \mathcal{G}^A$ we shall denote

$$e^{\lambda(x)} := |\partial_{x_1}\vec{\Phi}|(x) = |\partial_{x_2}\vec{\Phi}|(x) \quad \text{and} \quad \vec{e}_i(x) := e^{-\lambda(x)}\,\partial_{x_i}\vec{\Phi}(x).$$

We have in particular

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |\nabla\vec{\Phi}(y)|^2\, dy^2 > A^2 + o(1)$$

and hence

$$\lim_{r\to 0}\frac{\displaystyle\int_{B_r(x)} |\nabla\vec{\Phi}(y) - \nabla\vec{\Phi}(x)|^2\, dy^2}{\displaystyle\int_{B_r(x)} |\nabla\vec{\Phi}(y)|^2\, dy^2} = 0. \tag{2.3}$$

And

$$\lim_{r\to 0}\frac{\displaystyle\int_{B_r(x)} r^{-2}\,|\vec{\Phi}(y)-\vec{\Phi}(x)-\partial_{x_1}\vec{\Phi}(x)(y_1-x_1)-\partial_{x_2}\vec{\Phi}(x)(y_2-x_2)|^2\,dy^2}{\displaystyle\int_{B_r(x)}|\nabla\vec{\Phi}(y)|^2\,dy^2}=0 \tag{2.4}$$

For any $\varepsilon>0$ there exists $r_0>0$ such that for any $r<r_0$

$$\frac{\displaystyle\int_{B_r(x)}|\nabla\vec{\Phi}(y)-\nabla\vec{\Phi}(x)|^2+r^{-2}\,|\vec{\Phi}(y)-\vec{\Phi}(x)-\partial_{x_1}\vec{\Phi}(x)(y_1-x_1)-\partial_{x_2}\vec{\Phi}(x)(y_2-x_2)|^2\,dy^2}{\displaystyle\int_{B_r(x)}|\nabla\vec{\Phi}(y)|^2\,dy^2}<\varepsilon^2$$

Using *Fubini theorem* together with the *mean value Theorem*, for any such $r$ there exists $\rho_{r,x}\in[r/2,r]$ such that

$$\begin{aligned}&\frac{r}{2}\int_{\partial B_{\rho_{r,x}(x)}}|\nabla\vec{\Phi}(y)-\nabla\vec{\Phi}(x)|^2+r^{-2}\,|\vec{\Phi}(y)-\vec{\Phi}(x)-\partial_{x_1}\vec{\Phi}(x)(y_1-x_1)\\&\quad-\partial_{x_2}\vec{\Phi}(x)(y_2-x_2)|^2\,dl_{\partial B_{\rho_{r,x}}}\\&\qquad<\varepsilon^2\int_{B_r(x)}|\nabla\vec{\Phi}(y)|^2\,dy^2=\pi\,r^2\,\varepsilon^2e^{2\lambda(x)}(1+o_r(1))\end{aligned}\tag{2.5}$$

then

$$\begin{aligned}&\|\vec{\Phi}(\rho_{r,x},\theta)-\vec{\Phi}(\rho_{r,x},0)-e^{\lambda(x)}\,\rho_{r,x}\,[\cos\theta-1]\,\vec{e}_1-e^{\lambda(x)}\,\rho_{r,x}\,\sin\theta\,\vec{e}_2\|_{L^\infty([0,2\pi])}\\&\le\int_0^{2\pi}\left|\partial_\theta\left(\vec{\Phi}(\rho_{r,x},\theta)-\vec{\Phi}(\rho_{r,x},0)-e^{\lambda(x)}\,\rho_{r,x}\,[\cos\theta-1]\,\vec{e}_1-e^{\lambda(x)}\,\rho_{r,x}\,\sin\theta\,\vec{e}_2\right)\right|\,d\theta\\&=\int_{\partial B_{\rho_{r,x}}(x)}\left|\frac{1}{\rho_{r,x}}\frac{\partial\vec{\Phi}}{\partial\theta}(y)+\partial_{x_1}\vec{\Phi}(x)\,\sin\theta-\partial_{x_2}\vec{\Phi}(x)\,\cos\theta\right|\,dl_{\partial B_{\rho_{r,x}}}\\&\le\int_{\partial B_{\rho_{r,x}}(x)}|\nabla\vec{\Phi}(y)-\nabla\vec{\Phi}(x)|\,dl_{\partial B_{\rho_{r,x}}}<2\pi\,\varepsilon\,\rho_{r,x}\,e^{\lambda(x)}(1+o_r(1))\end{aligned}\tag{2.6}$$

Denote $\vec{L}_x(y):=\vec{\Phi}(x)+\partial_{x_1}\vec{\Phi}(x)(y_1-x_1)+\partial_{x_2}\vec{\Phi}(x)(y_2-x_2)$. Considering (2.5), we have also chosen $\rho_{r,x}$ in such a way that

$$\begin{aligned}&\left|\fint_{\partial B_{\rho_{r,x}}(x)}\vec{\Phi}(y)\,dl_{\partial B_{\rho_{r,x}}}-\vec{\Phi}(x)\right|^2=\left|\fint_{\partial B_{\rho_{r,x}}(x)}\vec{\Phi}(y)\,dl_{\partial B_{\rho_{r,x}}}-\fint_{\partial B_{\rho_{r,x}}(x)}\vec{L}_x(y)\,dl_{\partial B_{\rho_{r,x}}}\right|^2\\&\qquad\le\fint_{\partial B_{\rho_{r,x}}(x)}|\vec{\Phi}(y)-\vec{L}_x(y)|^2\,dl_{\partial B_{\rho_{r,x}}}\le 2\,\varepsilon^2\,\rho_{r,x}^2\,e^{2\lambda(x)}(1+o_r(1))\end{aligned}\tag{2.7}$$

We have

$$\fint_{\partial B_{\rho_{r,x}}(x)}\vec{\Phi}(y)\,dl_{\partial B_{\rho_{r,x}}}=\frac{1}{2\pi}\int_0^{2\pi}\vec{\Phi}(\rho_{r,x},\theta)\,d\theta\tag{2.8}$$

We have moreover

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\left[\vec{\Phi}(\rho_{r,x},0)+e^{\lambda(x)}\,\rho_{r,x}\,[\cos\theta-1]\,\vec{e}_1+e^{\lambda(x)}\,\rho_{r,x}\,\sin\theta\,\vec{e}_2\right]\,d\theta\\&\quad=\vec{\Phi}(\rho_{r,x},0)-e^{\lambda(x)}\,\rho_{r,x}\,\vec{e}_1\end{aligned}\tag{2.9}$$

Thus combining (2.6), ...,(2.9) we have

$$|\vec{\Phi}(x) - \vec{\Phi}(\rho_{r,x}, 0) + e^{\lambda(x)}\, \rho_{r,x}\; \vec{e}_1| \leq\; [2\pi + \sqrt{2}]\; \varepsilon\; \rho_{r,x}\; e^{\lambda(x)}\; (1 + o_r(1)). \quad (2.10)$$

Hence combining (2.6) and (2.10) we obtain

$$\begin{aligned}&\|\vec{\Phi}(\rho_{r,x}, \theta) - \vec{\Phi}(x) - e^{\lambda(x)}\, \rho_{r,x}\; \cos\theta\; \vec{e}_1 - e^{\lambda(x)}\, \rho_{r,x}\; \sin\theta\; \vec{e}_2\|_{L^\infty([0,2\pi])}\\ &\leq 5\,\pi\; \varepsilon\; \rho_{r,x}\; e^{\lambda(x)}\; (1 + o_r(1))\end{aligned} \quad (2.11)$$

Denote by $\Gamma_{\rho_{r,x}}$ the circle in $\mathbb{R}^m$ given by $\vec{L}_x(\partial B_{\rho_{r,x}}(x))$ and

$$\Gamma^{\varepsilon}_{\rho_{r,x}} = \left\{ y \in N^n \quad \text{s. t.} \quad \text{dist}(y, \Gamma_{\rho_{r,x}}) > 6\,\pi\; \varepsilon\; \rho_{r,x}\; e^{\lambda(x)} \right\} \quad (2.12)$$

Because of our assumptions, $\Sigma_{\rho_{r,x}} := \vec{\Phi}(B_{\rho_{r,x}}(x)) \cap \Gamma^{\varepsilon}_{\rho_{r,x}}$ defines a rectifiable integer stationary varifold in $N^n \cap \Gamma^{\varepsilon}_{\rho_{r,x}}$. Leon Simon monotonicity formula in $\mathbb{R}^m$ gives

$$\begin{aligned}&\rho_{r,x}^{-2}\; \mathcal{H}^2 \left( \Sigma_{\rho_{r,x}} \cap B^m_{e^{\lambda(x)}\;\rho_{r,x}}(\vec{\Phi}(x)) \right) - \pi\; e^{2\,\lambda(x)} \;\geq -\frac{1}{4} \int_{B^m_{e^{\lambda(x)}\;\rho_{r,x}}(\vec{\Phi}(x))} |\vec{H}_{\mathbb{R}^m}|^2 d\mathcal{H}^2 \llcorner \Sigma_{\rho_{r,x}}\\ &-\frac{1}{\rho_{r,x}^2} \int_{\vec{q}\in B^m_{e^{\lambda(x)}\;\rho_{r,x}}(\vec{\Phi}(x))} |\vec{H}_{\mathbb{R}^m}||\vec{q} - \vec{\Phi}(x)| d\mathcal{H}^2 \llcorner \Sigma_{\rho_{r,x}}\end{aligned} \quad (2.13)$$

where $\vec{H}_{\mathbb{R}^m}$ is the generalized curvature of the varifold given by $\vec{\Phi}$ in $\mathbb{R}^m$. The generalized curvature is by definition given by

$$\int \vec{H}_{\mathbb{R}^m} \cdot F(\vec{q})\; d\mathcal{H}^2 \;= -\int_{\Sigma} d(F(\vec{\Phi})) \cdot d\vec{\Phi}\; dvol_{g_{\vec{\Phi}}}$$

Using the stationarity in $N^n$ we have (A. 2) and then we have

$$\vec{H}_{\mathbb{R}^m} := -\,A(\vec{\Phi})(d\vec{\Phi}, d\vec{\Phi})_{g_{\vec{\Phi}}}.$$

Hence, since $|d\vec{\Phi}|_{g_{\vec{\Phi}}} = 1$ we deduce that $|\vec{H}_{\mathbb{R}^m}| \leq \|A\|_{L^\infty(N^n)}$. Inserting this bound in (2.13) we obtain

$$\rho_{r,x}^{-2}\; \mathcal{H}^2 \left( \Sigma_{\rho_{r,x}} \cap B^m_{e^{\lambda(x)}\;\rho_{r,x}}(\vec{\Phi}(x)) \right) - \pi\; e^{2\,\lambda(x)} \;\geq -\,C\; \rho_{r,x}^{-1}\; \mathcal{H}^2 \left( \Sigma_{\rho_{r,x}} \cap B^m_{e^{\lambda(x)}\;\rho_{r,x}}(\vec{\Phi}(x)) \right) \quad (2.14)$$

Thus

$$\rho_{r,x}^{-2}\; \mathcal{H}^2 \left( \Sigma_{\rho_{r,x}} \cap B^m_{e^{\lambda(x)}\;\rho_{r,x}}(\vec{\Phi}(x)) \right) \geq \pi\; e^{2\,\lambda(x)}\; (1 - C\; \rho_{r,x}) \quad (2.15)$$

Thus having chosen $r_0 < \varepsilon^2$, we have

$$\mathcal{H}^2 \left( \Sigma_{\rho_{r,x}} \cap B^m_{e^{\lambda(x)}\;\rho_{r,x}}(\vec{\Phi}(x)) \right) \geq \pi\; e^{2\,\lambda(x)}\; \rho_{r,x}^2\; \left(1 - C\varepsilon^2\right). \quad (2.16)$$

In the mean time, due to (2.3) we have

$$\mathcal{H}^2 \left( \Sigma_{\rho_{r,x}} \right) \leq \frac{1}{2} \int_{B_{\rho_{r,x}}(x)} |\nabla \vec{\Phi}(y)|^2\; dy^2 \leq \pi\; e^{2\lambda(x)}\; \rho_{r,x}^2\; (1 + C\; \varepsilon^2). \quad (2.17)$$

The *tilt excess* of $\Sigma_{\rho_{r,x}}$ in the sense of Allard [1] is given by

$$\begin{aligned}&E \left( \Sigma_{\rho_{r,x}}, \vec{e}_1 \wedge \vec{e}_2, \vec{\Phi}(x), \rho_{r,x} \right)\\ &= \frac{1}{\rho_{r,x}^2} \int_{B_{\rho_{r,x}}(x) \cap \mathcal{G}} |e^{-2\lambda(y)} \partial_{x_1} \vec{\Phi}(y) \wedge \partial_{x_2} \vec{\Phi}(y) - \vec{e}_1 \wedge \vec{e}_2|^2\; e^{2\lambda(y)}\; dy^2\end{aligned}$$

Since we have the pointwize bound

$$2\,|e^{\lambda(y)}-e^{\lambda(x)}|=||\nabla\vec{\Phi}|(x)-|\nabla\vec{\Phi}|(y)|\le|\nabla\vec{\Phi}(x)-\nabla\vec{\Phi}(y)| \tag{2.18}$$

hence we deduce from (2.5) that[4]

$$\int_{B_{\rho_{r,x}}(x)\cap\mathcal{G}}|e^{\lambda(y)}-e^{\lambda(x)}|^2\,dy^2\le\pi\,\varepsilon^2\,\rho_{r,x}^2e^{2\lambda(x)}$$

Hence, denoting

$$E_\varepsilon:=\left\{y\in B_{\rho_{r,x}}(x)\cap\mathcal{G}\,;\ |\lambda(y)-\lambda(x)|>\sqrt{\varepsilon}\right\},$$

we have

$$|E_\varepsilon|\le\varepsilon\,|B_{\rho_{r,x}}(x)|. \tag{2.19}$$

Observe

$$\begin{aligned}\int_{E_\varepsilon}|\nabla\vec{\Phi}(y)|^2\,dy^2&\le 2\int_{B_{\rho_{r,x}}(x)}|\nabla\vec{\Phi}(y)-\nabla\vec{\Phi}(x)|^2\,dy^2+2\int_{E_\varepsilon}|\nabla\vec{\Phi}(x)|^2\,dy^2\\ &\le 4\,\pi\,[\varepsilon+\varepsilon^2]\,\rho_{r,x}^2\,e^{2\lambda(x)}\le 3\,\pi\,\varepsilon\,\rho_{r,x}^2\,e^{2\lambda(x)}\end{aligned} \tag{2.20}$$

Observe that we have the pointwize bound

$$\begin{aligned}|e^{-2\lambda(y)}\partial_{x_1}\vec{\Phi}(y)\wedge\partial_{x_2}\vec{\Phi}(y)-\vec{e}_1\wedge\vec{e}_2|^2\,e^{2\lambda(y)}&=|\vec{e}_1(y)\wedge\partial_{x_2}\vec{\Phi}-\vec{e}_1(x)\wedge\vec{e}_2(x)\,e^{\lambda(y)}|^2\\ &\le 2\,|\vec{e}_1(x)-\vec{e}_1(y)|^2\,e^{2\lambda(y)}+2\,|\vec{e}_2(x)-\vec{e}_2|^2\,e^{2\lambda(y)}\le 4\,|e^{\lambda(x)}-e^{\lambda(y)}|^2\\ &\quad+2\,|\nabla\vec{\Phi}(x)-\nabla\vec{\Phi}(y)|^2\\ &\le 4\,|\nabla\vec{\Phi}(x)-\nabla\vec{\Phi}(y)|^2\end{aligned}$$

where we have also used (2.18). We have using one more time (2.5) together with (2.20) and the previous pointwize inequality

$$\begin{aligned}E\left(\Sigma_{\rho_{r,x}},\vec{e}_1\wedge\vec{e}_2,\vec{\Phi}(x),\rho_{r,x}\right)&\le\frac{4}{\rho_{r,x}^2}\int_{E_\varepsilon}|\nabla\vec{\Phi}(y)|^2\,dy^2\\ &\quad+\frac{1}{\rho_{r,x}^2}\int_{B_{\rho_{r,x}}(x)\cap\mathcal{G}\setminus E_\varepsilon}|e^{-2\lambda(y)}\partial_{x_1}\vec{\Phi}(y)\wedge\partial_{x_2}\vec{\Phi}(y)-\vec{e}_1\wedge\vec{e}_2|^2\,e^{2\lambda(y)}\,dy^2\\ &\le 12\,\pi\,\varepsilon\,A^2+\varepsilon\,\frac{1}{\rho_{r,x}^2}\int_{B_{\rho_{r,x}}(x)}|\nabla\vec{\Phi}(y)|^2\,dy^2\le 14\,\pi\,\varepsilon\,A^2.\end{aligned} \tag{2.21}$$

where we recall that $x$ has been chosen in such a way that $2\,e^{2\lambda(x)}\le A^2$. Combining (2.17) and (2.21) we can apply Allard main regularity result and precisely we obtain for $\varepsilon$ small enough the existence of $\gamma<1$ such that $\mathcal{S}_{r,x}:=\Sigma_{\rho_{r,x}}\cap B^m_{\gamma\,e^{\lambda(x)}\,r}(\vec{\Phi}(x))$ is a smooth minimal sub-manifold. Moreover, because of the *upper semi-continuity* of the density function for a stationarity varifold we can assume that all points $\vec{q}\in\mathcal{S}_{r,x}$ have multiplicity 1 (See for instance the presentation of the absence of hole in [2] section 7). In other words we have that

$$\int_{B^m_{\gamma\,e^{\lambda(x)}\,r}(\vec{\Phi}(x))}d\mathcal{H}^2\llcorner\Sigma_{\rho_{r,x}}=\text{Area}(\mathcal{S}_{r,x})=\int_{\vec{\Phi}^{-1}(\mathcal{S}_{r,x})}dvol_{g_{\vec{\Phi}}}=\frac{1}{2}\int_{\vec{\Phi}^{-1}(\mathcal{S}_{r,x})}|\nabla\vec{\Phi}|^2\,dx^2 \tag{2.22}$$

[4] We are restricting to the integration on $B_{\rho_{r,x}}(x)\cap\mathcal{G}$ because one requires $\lambda$ to be defined as a measurable function.

and, in the area formula, the counting function

$$\mathcal{H}^0(\vec{\Phi}^{-1}(\mathcal{S}_{r,x}) \cap \vec{\Phi}^{-1}\{\vec{q}\}) = 1 \qquad \text{for} \quad \mathcal{H}^2 - \text{ a. e. } \ \vec{q} \in \mathcal{S}_{r,x}. \tag{2.23}$$

We claim that for $s$ small enough $\vec{\Phi}(B_s(x)) \subset \mathcal{S}_{r,x}$. Assuming there is point $\vec{p} \notin B^m_{\gamma r\ e^{\lambda}(x)}(\vec{\Phi}(x))$ but $\vec{p} \in \vec{\Phi}(B_s(x))$. As $s$ goes to zero we have in one hand, using (2.5),

$$\int_{B_s(x)} |\nabla\vec{\Phi}|^2 \, dy^2 \leq 2\pi \, s^2 \, (1+\varepsilon) \, e^{2\lambda(x)} \tag{2.24}$$

but the stationarity of $\Sigma_{\rho_{s,x}} := \vec{\Phi}(B_{\rho_{s,x}}(x)) \cap \Gamma^{\varepsilon}_{\rho_{2\,s,x}}$ in $N^n \cap \Gamma^{\varepsilon}_{\rho_{2\,s,x}}$ together with the monotonicity formula would imply

$$\begin{aligned} \pi(\gamma r - s)^2 \, e^{2\lambda(x)} \, (1 - o_r(1)) &\leq \pi(\operatorname{dist}(\vec{p}, \Gamma^{\varepsilon}_{\rho_{2\,s,x}}) \, (1 - o_r(1)) \leq \mathcal{H}^2(\Sigma_{\rho_{2\,s,x}}) \\ &\leq \frac{1}{2} \int_{B_s(x)} |\nabla\vec{\Phi}|^2 \, dy^2 \end{aligned}$$

which contradicts[5] (2.24) for $s$ very small compared to $r$. Hence, for $s$ small enough

$$\vec{\Phi}(B_s(x)) \subset \mathcal{S}_{r,x} = \Sigma_{\rho_{r,x}} \cap B^m_{\gamma\, e^{\lambda(x)}\, r}(\vec{\Phi}(x))$$

Let $\Psi$ be a smooth conformal diffeomorphism from $\Sigma_{\rho_{r,x}} \cap B^m_{\gamma\, e^{\lambda(x)}\, r}(\vec{\Phi}(x))$ into the disc $D^2$. Denote by $e^{2\mu}\, dz^2 = \Psi^* g_{\mathbb{R}^m}$ and consider $f := \Psi \circ \vec{\Phi}$. This is in particular a $W^{1,2}$ weakly conformal map from $B_s(x)$ into $\mathbb{C}$. Consider any map $g \in W^{1,2}(B_{\rho_{s,x}}(x))$ such that $g = f$ on $\partial B_{\rho_{s,x}}(x)$. The union of $f$ and $g$ realizes a $W^{1,2} \cap L^\infty$ map from $S^2$ into $\mathbb{C}$ that we denote $\tilde{f}$. The cycle $\tilde{f}_*[S^2]$ has then an *algebraic covering number* equal to 0 $\mathcal{H}^2-$almost everywhere in $\mathbb{C}$ that is to say[6]

$$\int_{S^2} \tilde{f}^* dz_1 \wedge dz_2 = 0.$$

Away from the compact 1 rectifiable set $f(\partial B_{\rho_{s,x}}(x))$, because of (2.23), the image $f(B_{\rho_{s,x}}(x))$ has covering number $+1$, $-1$ or 0. Hence, $g$ must have an odd covering number almost everywhere in $\mathbb{C} \setminus f(\partial B_{\rho_{s,x}}(x))$ whenever the covering number of $f$ is non zero. This homological fact implies

$$\begin{aligned} \frac{1}{2} \int_{B_{\rho_{s,x}}(x))} e^{2\mu(f)} \, |\nabla f|^2 \, dx^2 &= \int_{B_{\rho_{s,x}}(x))} e^{2\mu(f)} |\partial_{x_1} f \times \partial_{x_2} f| \, dx_1 \wedge dx_2 \\ &\leq \int_{B_{\rho_{s,x}}(x))} e^{2\mu(g)} |\partial_{x_1} g \times \partial_{x_2} g| \, dx_1 \wedge dx_2 \leq \frac{1}{2} \int_{B_{\rho_{s,x}}(x))} e^{2\mu(g)} \, |\nabla g|^2 \, dx^2 \end{aligned}$$

Hence by the *Dirichlet principle* $f$ coincides with it's harmonic extension[7] in $(D^2, e^{2\mu}\, dz^2)$. The map $\Psi \circ \vec{\Phi}$ is then smooth (holomorphic or anti-holomorphic) on $B_{\rho_{s,x}}(x))$. This implies Lemma 2.1. □

---

[5] Observe that based on similar arguments one could prove directly that $\vec{\Phi}$ is continuous on the whole $\Sigma$ without using Allard's result but this is not needed.

[6] Indeed due to the $W^{1,2}$ nature of $\tilde{f}$ and since we are in 2 dimension we have that $d$ and $*$ commute (this is clearly not the case in higher dimension) and we have in particular $\tilde{f}^* dz_1 \wedge dz_2 = d(\tilde{f}_1 \, d\tilde{f}_2)$

[7] The harmonic extension is unique for r small enough since one is taking value in a convex geodesic ball of $(D^2, e^{2\mu}\, dz^2)$.

# 3 The harmonicity of conformal target harmonic maps

## 3.1 The integration by part formula

The goal of the present subsection is to prove the following *integration by parts formula.*

**Lemma 3.1** *Let $F(\vec{p}) = (F_{ij}(\vec{p}))_{1\le i\le m\ 1\le j\le k}$ be a $C^1$ map on $N^n$ taking values into $m\times k$ real matrices. Then, for any smooth compactly supported function $\varphi$ in $D^2$ and for almost every regular value $t>0$ of $\varphi$ one has*

$$\sum_{i=1}^m \int_{\partial\Omega_t} F(\vec{\Phi})ij\,\frac{\partial\vec{\Phi}_i}{\partial\nu}\,dl_{\partial\Omega_t} - \int_{\Omega_t}\nabla(F(\vec{\Phi})ij)\,\nabla\vec{\Phi}_i\,dy^2 + \int_{\Omega_t} F(\vec{\Phi})_{ij}\,A_i(\vec{\Phi})(d\vec{\Phi},d\vec{\Phi})\,dy^2 = 0 \tag{3.1}$$

*where $\Omega_t = \varphi^{-1}((t,+\infty))$ and $\nu$ is the exterior unit normal to the level set $\nu := \nabla\varphi/|\nabla\varphi|$.* □

In order to prove lemma 3.1, we will first establish some intermediate results.

Let $x\in D^2$ and choose $r$ such that

$$\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} ds\int_{\partial B_s(x)}|\nabla\vec{\Phi}|^2\,dl_{\partial B_s} = \int_{\partial B_r(x)}|\nabla\vec{\Phi}|^2\,dl_{\partial B_r} < +\infty \tag{3.2}$$

Hence the restriction of $\vec{\Phi}$ to $\partial B_r(x)$ is $W^{1,2}$ and the continuous image of $\partial B_r(x)$ by $\vec{\Phi}$, $\Gamma_{r,x} := \vec{\Phi}(\partial B_r(x))$ has finite length and is rectifiable and compact.

We denote $\mathcal{B} := D^2\backslash\mathcal{G}$. Because of the previous lemma 2.1 we have that $\mathcal{B}$ is closed. Hence

$$\mathcal{B} = \bigcap_{\varepsilon>0}\mathcal{B}_\varepsilon \quad\text{where}\quad \mathcal{B}_\varepsilon := \left\{x\in D^2\ ;\ \text{dist}(x,\mathcal{B})\le\varepsilon\right\}$$

Since the integral of $|\nabla\vec{\Phi}|^2$ over $\mathcal{B}$ is zero, we clearly have

$$\lim_{\varepsilon\to 0}\int_{\mathcal{B}_\varepsilon}|\nabla\vec{\Phi}|^2(y)\,dy^2 = 0 \tag{3.3}$$

We denote also

$$\mathcal{G}_\varepsilon := D^2\backslash\mathcal{B}_\varepsilon$$

Finally we denote by $\mathfrak{B}_\varepsilon$ the *rectifiable image* of $\mathcal{B}_\varepsilon$ by $\vec{\Phi}$ that is the image by the approximate continuous representative of $\vec{\Phi}$ of the intersection of $\mathcal{B}_\varepsilon$ with the points of approximate differentiability of $\vec{\Phi}$. We claim the following.

**Lemma 3.2** *Under the previous notations we have*

$$\lim_{\varepsilon\to 0}\int_{D^2\cap\vec{\Phi}^{-1}(\mathfrak{B}_\varepsilon)}|\nabla\vec{\Phi}|^2(y)\,dy^2 = 0 \tag{3.4}$$

*Proof of lemma 3.2* Identity (3.3) implies that

$$\lim_{\varepsilon\to 0}\mathcal{H}^2(\mathfrak{B}_\varepsilon) = 0 \tag{3.5}$$

Hence for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ and for any $\varepsilon < \varepsilon_\delta$ there exists a covering of $\mathfrak{B}_\varepsilon$ by balls $(B^m_{r_l}(\vec{p}_l))_{l\in\Lambda}$ such that

$$\sum_{l\in\Lambda} r_l^2 < \delta$$

Using the monotonicity formula for stationary varifolds we obtain

$$\int_{D^2\cap\vec{\Phi}^{-1}(\mathfrak{B}_\varepsilon)} |\nabla\vec{\Phi}|^2(y)\, dy^2 \le \sum_{i\in I}\int_{\vec{\Phi}^{-1}(B^m_{r_l}(\vec{p}_l))} |\nabla\vec{\Phi}|^2(y)\, dy^2 \le C\sum_{l\in\Lambda} r_l^2 < C\,\delta$$

which implies the lemma. □

We shall prove the following.

**Lemma 3.3** *Under the previous notations we have*

$$\lim_{s\to 0}\int_{\mathcal{G}_\varepsilon}\left[1+|\nabla\vec{\Phi}|^2(y)\right]\left|\frac{\mathcal{H}^2\left(\mathfrak{B}_\varepsilon\cap B^m_s(\vec{\Phi}(y))\right)}{\pi\, s^2}-\theta^2_\varepsilon(\vec{\Phi}(y))\right|\, dy^2 = 0 \tag{3.6}$$

*where*

$$\theta^2_\varepsilon(\vec{p}) := \lim_{s\to 0}\frac{\mathcal{H}^2\left(\mathfrak{B}_\varepsilon\cap B^m_s(\vec{p})\right)}{\pi\, s^2}$$

*exists $\mathcal{H}^2$ almost everywhere since $\mathfrak{B}_\varepsilon\subset\cup_{\eta>0}\vec{\Phi}(\mathcal{G}_\eta)$ is 2-rectifiable. Moreover since $\vec{\Phi}$ is a smooth immersion on $\mathcal{G}_\varepsilon$, $\theta^2_\varepsilon(\vec{\Phi}(y))$ is a well defined measurable function on $\mathcal{G}_\varepsilon$.*

*Proof of lemma 3.3* Since $|\nabla\vec{\Phi}|$ is uniformly bounded on $\mathcal{G}_\varepsilon$ we have

$$\begin{aligned}&\int_{\mathcal{G}_\varepsilon}\left[1+|\nabla\vec{\Phi}|^2(y)\right]\left|\frac{\mathcal{H}^2\left(\mathfrak{B}_\varepsilon\cap B^m_s(\vec{\Phi}(y))\right)}{\pi\, s^2}-\theta^2_\varepsilon(\vec{\Phi}(y))\right|\, dy^2\\ &\quad\le\left[1+\|\nabla\vec{\Phi}\|_{L^\infty(\mathcal{G}_\varepsilon)}\right]\int_{\mathcal{G}_\varepsilon}\left|\frac{\mathcal{H}^2\left(\mathfrak{B}_\varepsilon\cap B^m_s(\vec{\Phi}(y))\right)}{\pi\, s^2}-\theta^2_\varepsilon(\vec{\Phi}(y))\right|\, dy^2\end{aligned}$$

Since $\vec{\Phi}$ is a smooth immersion on $\mathcal{G}_\varepsilon$, $\theta^2_\varepsilon(\vec{\Phi}(y))$ is a well defined measurable function on $\mathcal{G}_\varepsilon$ and we have for almost $y$

$$\lim_{s\to 0}\frac{\mathcal{H}^2\left(\mathfrak{B}_\varepsilon\cap B^m_s(\vec{\Phi}(y))\right)}{\pi\, s^2} = \theta^2_\varepsilon(\vec{\Phi}(y))$$

The monotonicity formula for the stationary varifold given by the image of $D^2$ by $\vec{\Phi}$ gives

$$\sup_{s>0;\ y\in D^2}\frac{\mathcal{H}^2\left(\mathfrak{B}_\varepsilon\cap B^m_s(\vec{\Phi}(y))\right)}{\pi\, s^2}\le\sup_{s>0;\ y\in D^2}\frac{\mathcal{H}^2\left(\vec{\Phi}(D^2)\cap B^m_s(\vec{\Phi}(y))\right)}{\pi\, s^2}\le C$$

for some $C$. The lemma follows by a direct application of dominated convergence. □

*Proof of lemma 3.1* In order to simplify the presentation we shall restrict to $\Omega_t$ to be balls. Because of (3.4), for every $x \in D^2$ and almost every $r > 0$ we have

$$\lim_{\varepsilon\to 0}\int_{\vec{\Phi}^{-1}(\mathfrak{B}_\varepsilon)\cap\partial B_r(x)} |\nabla\vec{\Phi}|(y)\, dl_{\partial B_r} = 0 \tag{3.7}$$

We choose $r$ such that (3.7) holds true and such that also

$$\lim_{s\to 0}\int_{\mathcal{G}_\varepsilon\cap\partial B_r(x)} \left[1+|\nabla\vec{\Phi}|^2(y)\right]\left|\frac{\mathcal{H}^2\left(\mathfrak{B}_\varepsilon\cap B^m_s(\vec{\Phi}(y))\right)}{\pi\, s^2}-\theta^2_\varepsilon(\vec{\Phi}(y))\right| dl_{\partial B_r} = 0. \tag{3.8}$$

Hence in particular $\mathcal{H}^1(\vec{\Phi}(\mathcal{B}_\varepsilon\cap\partial B_r(x)))$ is converging to zero as $\varepsilon$ goes to zero. Since $\vec{\Phi}$ is continuous on $\partial B_r(x)$ and since $\mathcal{B}_\varepsilon$ is a closed set we have that $\vec{\Phi}(\mathcal{B}_\varepsilon\cap\partial B_r(x))$ is a compact subset of $N^n$. Because of the previous, for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon < \varepsilon_\delta$, we can include $\vec{\Phi}(\mathcal{B}_\varepsilon\cap\partial B_r(x))$ in finitely many balls $(B^m_{r_l}(\vec{p}_l))_{l\in\Lambda}$ such that

$$\sum_{l\in\Lambda} r_l < \delta. \tag{3.9}$$

We also choose $\varepsilon_\delta > 0$ such that for any $\varepsilon < \varepsilon_\delta$

$$\int_{\vec{\Phi}^{-1}(\mathfrak{B}_\varepsilon)\cap\partial B_r(x)} |\nabla\vec{\Phi}|(y)\, dl_{\partial B_r} < \delta. \tag{3.10}$$

Let $\chi$ be a cut-off function on $\mathbb{R}_+$ such that $\chi\equiv 1$ on $[2,+\infty)$ and $\chi\equiv 0$ on $[0,1]$. We introduce

$$\xi_\varepsilon(\vec{\Phi}(y)) := \prod_{l\in\Lambda}\chi\left(\frac{|\vec{\Phi}(y)-\vec{p}_l|}{r_l}\right)$$

Observe that $\xi_\varepsilon(\vec{\Phi}(y))$ is zero on $\mathcal{B}_\varepsilon\cap\partial B_r(x)$. For any $s > 0$ we also introduce

$$\eta_s(\vec{\Phi}(y)) := \chi\left(\frac{\mathrm{dist}(\vec{\Phi}(y),\Gamma_{r,x})}{s}\right)$$

where we recall that $\Gamma_{r,x} := \vec{\Phi}(\partial B_r(x))$. Using the assumption (1.1) we have for any $F$ as in the statement of the lemma

$$\begin{aligned}\sum_{i=1}^m -\int_{B_r(x)} \nabla(\xi_\varepsilon(\vec{\Phi})\,\eta_s(\vec{\Phi})\,F(\vec{\Phi})_{ij})\,\nabla\vec{\Phi}_i\, dy^2 \\ +\int_{B_r(x)} \xi_\varepsilon(\vec{\Phi})\,\eta_s(\vec{\Phi})F(\vec{\Phi})_{ij}\,A_i(\vec{\Phi})(d\vec{\Phi},d\vec{\Phi})\, dy^2 = 0\end{aligned} \tag{3.11}$$

We have

$$
\begin{aligned}
&\left|\int_{B_r(x)} \nabla(\xi_\varepsilon(\vec{\Phi}))\,\eta_s(\vec{\Phi})\,F(\vec{\Phi})_{ij}\,\nabla\vec{\Phi}_i\,dy^2\right| \\
&\leq C\,\|F\|_\infty \sum_{l\in\Lambda}\frac{1}{r_l}\int_{B_r(x)}|\chi'|\left(\frac{|\vec{\Phi}(y)-\vec{p}_l|}{r_l}\right)|\nabla\vec{\Phi}|^2(y)\,dy^2 \\
&\leq C\,\|F\|_\infty \sum_{l\in\Lambda} r_l^{-1}\int_{\vec{\Phi}^{-1}(B^m_{r_l}(\vec{p}_l))}|\nabla\vec{\Phi}|^2(y)\,dy^2 \leq\ C\,\|F\|_\infty \sum_{l\in\Lambda} r_l \leq C\,\|F\|_\infty\,\delta
\end{aligned}
\tag{3.12}
$$

where we observe that the bound is independent of $s$. We now write

$$
\int_{B_r(x)} \xi_\varepsilon(\vec{\Phi})\,\nabla(\eta_s(\vec{\Phi}))\,F(\vec{\Phi})_{ij}\,\nabla\vec{\Phi}_i\,dy^2 = \int_{B_r(x)\cap\mathcal{B}_\varepsilon}\cdots + \int_{B_r(x)\cap\mathcal{G}_\varepsilon}\cdots \tag{3.13}
$$

We have

$$
\begin{aligned}
&\left|\int_{B_r(x)\cap\mathcal{B}_\varepsilon} \xi_\varepsilon(\vec{\Phi})\,\nabla(\eta_s(\vec{\Phi}))\,F(\vec{\Phi})_{ij}\,\nabla\vec{\Phi}_i\,dy^2\right| \\
&\qquad\leq \frac{C\,\|F\|_\infty}{s}\int_{B_r(x)\cap\mathcal{B}_\varepsilon} \mathbf{1}_{\mathrm{dist}(\vec{\Phi}(y),\Gamma^\varepsilon_{r,x})<s}\,|\nabla\vec{\Phi}|^2(y)\,dy^2
\end{aligned}
\tag{3.14}
$$

Where $\Gamma^\varepsilon_{r,x}$ is the smooth immersed curve $\vec{\Phi}(\partial B_r(x))\cap\mathcal{G}_\varepsilon$ and $\mathbf{1}_{\mathrm{dist}(\vec{\Phi}(y),\Gamma^\varepsilon_{r,x})<s}(y)$ is the characteristic function of the set of $y$ such that $\vec{\Phi}(y)$ is at the distance at most $s$ to $\Gamma^\varepsilon_{r,x}$. The fact that we can restrict to $\Gamma^\varepsilon_{r,x}$ instead of $\Gamma_{r,x}$ is due to the fact that we are cutting off $\Gamma_{r,x}\setminus\Gamma^\varepsilon_{r,x}$ by multiplying by $\xi_\varepsilon(\vec{\Phi})$. Observe that since the curve $\Gamma^\varepsilon_{r,x}$ is a smooth immersion of the open subset of $\partial B_r(x)$ given by $\partial B_r(x)\cap\mathcal{G}_\varepsilon$ we have for $s$ small enough

$$
\begin{aligned}
\mathbf{1}_{\mathrm{dist}(\vec{p},\Gamma^\varepsilon_{r,x})<s} &\leq \frac{1}{s}\int_{\Gamma^\varepsilon_{r,x}} \mathbf{1}_{\mathrm{dist}(\vec{p},\vec{q})<2s}\,d\mathcal{H}^1(\vec{q}) \\
&\leq \frac{1}{s}\int_{\partial B_r(x)\cap\mathcal{G}_\varepsilon} \mathbf{1}_{\mathrm{dist}(\vec{p},\vec{\Phi}(z))<2s}\,|\nabla\vec{\Phi}|(z)\,dl_{\partial B_r}
\end{aligned}
\tag{3.15}
$$

Inserting this inequality in (3.14) gives

$$
\begin{aligned}
&\left|\int_{B_r(x)\cap\mathcal{B}_\varepsilon} \xi_\varepsilon(\vec{\Phi})\,\nabla(\eta_s(\vec{\Phi}))\,F(\vec{\Phi})_{ij}\,\nabla\vec{\Phi}_i\,dy^2\right| \\
&\leq \frac{C\,\|F\|_\infty}{s^2}\int_{\partial B_r(x)\cap\mathcal{G}_\varepsilon}|\nabla\vec{\Phi}|(z)\,dl_{\partial B_r}\int_{B_r(x)\cap\mathcal{B}_\varepsilon}\mathbf{1}_{\mathrm{dist}(\vec{\Phi}(y),\vec{\Phi}(z))<2s}\,|\nabla\vec{\Phi}|^2(y)\,dy^2 \\
&\leq C\,\|F\|_\infty\int_{\partial B_r(x)\cap\mathcal{G}_\varepsilon}|\nabla\vec{\Phi}|(z)\,\frac{\mathcal{H}^2(\mathfrak{B}_\varepsilon\cap B^m_s(\vec{\Phi}(z))}{s^2}\,dl_{\partial B_r}
\end{aligned}
\tag{3.16}
$$

Using (3.8) we then obtain

$$
\begin{aligned}
&\limsup_{s\to 0}\left|\int_{B_r(x)\cap\mathcal{B}_\varepsilon} \xi_\varepsilon(\vec{\Phi})\,\nabla(\eta_s(\vec{\Phi}))\,F(\vec{\Phi})_{ij}\,\nabla\vec{\Phi}_i\,dy^2\right| \\
&\qquad\leq C\,\|F\|_\infty\int_{\partial B_r(x)\cap\mathcal{G}_\varepsilon}|\nabla\vec{\Phi}|(z)\,\theta^2_\varepsilon(\vec{\Phi}(y))\,dl_{\partial B_r}(z)
\end{aligned}
\tag{3.17}
$$

and using the uniform bound on the density which itself comes from the monotonicity formula

$$\limsup_{s\to 0}\left|\int_{B_r(x)\cap\mathcal{B}_\varepsilon} \xi_\varepsilon(\vec{\Phi})\,\nabla(\eta_s(\vec{\Phi}))\,F(\vec{\Phi})_{ij}\,\nabla\vec{\Phi}_i\,dy^2\right|$$
$$\le\, C\,\|F\|_\infty\int_{\partial B_r(x)\cap\mathcal{G}_\varepsilon\cap\vec{\Phi}^{-1}(\mathfrak{B}_\varepsilon)}|\nabla\vec{\Phi}|(z)\,dl_{\partial B_r}(z)\le\, C\,\|F\|_\infty\,\delta \tag{3.18}$$

where we have used (3.10). Observe that since we have cut-off $\Gamma_{r,x}\setminus\Gamma^\varepsilon_{r,x}$ by multipying by $\xi_\varepsilon(\vec{\Phi})$ we have for $s$ small enough

$$\lim_{s\to 0}\int_{B_r(x)\cap\mathcal{G}_\varepsilon}\xi_\varepsilon(\vec{\Phi})\,\nabla(\eta_s(\vec{\Phi}))\,F(\vec{\Phi})_{ij}\,\nabla\vec{\Phi}_i\,dy^2$$
$$\lim_{s\to 0}\int_{B_r(x)\cap\mathcal{G}_\varepsilon}\xi_\varepsilon(\vec{\Phi})\,\nabla\left(\eta\left(\frac{\text{dist}(\vec{\Phi}(y),\Gamma^\varepsilon_{r,x})}{s}\right)\right)\,F(\vec{\Phi})_{ij}\,\nabla\vec{\Phi}_i\,dy^2 \tag{3.19}$$

Using the coarea formula this gives

$$\lim_{s\to 0}\sum_{i=1}^m\int_{B_r(x)\cap\mathcal{G}_\varepsilon}\xi_\varepsilon(\vec{\Phi})\,\nabla\left(\eta\left(\frac{\text{dist}(\vec{\Phi}(y),\Gamma^\varepsilon_{r,x})}{s}\right)\right)\,F(\vec{\Phi})_{ij}\,\nabla\vec{\Phi}_i\,dy^2$$
$$=\lim_{s\to 0}\frac{1}{s}\int_0^{2s}\chi'\left(\frac{\sigma}{s}\right)\,d\sigma\int_{\text{dist}(\vec{p},\Gamma^\varepsilon_{r,x})=\sigma}\xi_\varepsilon(\vec{p})\,F_j(\vec{p})\cdot\nu\,d\mathcal{H}^1\llcorner\Sigma^\varepsilon_{r,x} \tag{3.20}$$

where $\Sigma^\varepsilon_{r,x}$ is the immersed sub-manifold $\vec{\Phi}(B_r(x))\cap\mathcal{G}_\varepsilon$ and $\vec{\nu}$ is the unit exterior vector in this sub-manifold orthogonal to the level set $\text{dist}(\vec{p},\Gamma^\varepsilon_{r,x})=\sigma$ and $F_j(\vec{p})\cdot\nu=\sum_{i=1}^m F_{ji}(\vec{p})\,\nu_i$. Since $\vec{\Phi}$ is an immersion in a neighborhood of $\partial B_r\cap\mathcal{G}_\varepsilon$, for $\sigma$ small enough and being a regular value of $f^\varepsilon_{r,x}(\vec{p}):=\text{dist}(\vec{p},\Gamma^\varepsilon_{r,x})$ the level set is made of the following union

$$\left(f^\varepsilon_{r,x}\right)^{-1}(\sigma)=\gamma^\varepsilon_{r,x}(\sigma)\cup_{\alpha\in A_\sigma}\partial\omega_\alpha(\sigma)$$

where $\gamma^\varepsilon_{r,x}(\sigma)$ is a smooth curve converging to $\Gamma^\varepsilon_{r,x}$ and $\omega_\alpha(\sigma)$ are subdomains of $B_r(x)$ included in $\text{dist}(\vec{\Phi}(y),\Gamma^\varepsilon_{r,x})^{-1}([0,\sigma])$. The Taylor expansion of $\vec{\Phi}$ with respect to each point $y\in\partial B_r(x)$ gives

$$\lim_{\sigma\to 0}\int_{\gamma^\varepsilon_{r,x}(\sigma)}\xi_\varepsilon(\vec{p})\,F_j(\vec{p})\cdot\nu\,d\mathcal{H}^1\llcorner\Sigma^\varepsilon_{r,x}=\int_{\partial B_r(x)\cap\mathcal{G}_\varepsilon}\xi_\varepsilon(\vec{\Phi})\,\sum_{i=1}^m F_{ij}(\vec{\Phi})\,\frac{\partial\vec{\Phi}_i}{\partial r}\,dl_{\partial B_r} \tag{3.21}$$

Hence

$$\lim_{s\to 0}\frac{1}{s}\int_0^{2s}\chi'\left(\frac{\sigma}{s}\right)\,d\sigma\int_{\gamma^\varepsilon_{r,x}(\sigma)}\xi_\varepsilon(\vec{p})\,F_j(\vec{p})\cdot\nu\,d\mathcal{H}^1\llcorner\Sigma^\varepsilon_{r,x}$$
$$=\int_{\partial B_r(x)\cap\mathcal{G}_\varepsilon}\xi_\varepsilon(\vec{\Phi})\,\sum_{i=1}^m F_{ij}(\vec{\Phi})\,\frac{\partial\vec{\Phi}_i}{\partial r}\,dl_{\partial B_r} \tag{3.22}$$

For the other contributions we have

$$\lim_{s\to 0}\frac{1}{s}\int_0^{2s}\chi'\left(\frac{\sigma}{s}\right)\,d\sigma\sum_{\alpha\in A_\sigma}\int_{\partial\omega_\alpha(\sigma)}\xi_\varepsilon(\vec{p})\,F_j(\vec{p})\cdot\nu\,d\mathcal{H}^1\llcorner\Sigma^\varepsilon_{r,x}$$
$$\lim_{s\to 0}\frac{1}{s}\int_0^{2s}\chi'\left(\frac{\sigma}{s}\right)\,d\sigma\sum_{\alpha\in A_\sigma}\int_{\omega_\alpha(\sigma)}\text{div}_{\Sigma^\varepsilon_{r,x}}\left(\xi_\varepsilon(\vec{p})\,F_j(\vec{p})\right)\,d\mathcal{H}^2\llcorner\Sigma^\varepsilon_{r,x} \tag{3.23}$$

Since $\omega_\alpha(\sigma)$ is included in a $\sigma$ neighborhood of the smooth curve $\Gamma^\varepsilon_{r,x}$, using the monotonicity formula, for $\sigma$ small enough, covering such a neighborhood by $\simeq \sigma^{-1}\,\mathcal{H}^1(\Gamma^\varepsilon_{r,x})$ balls of radius $2\sigma$ we have the following bound

$$\sum_{\alpha\in A_\sigma} \mathcal{H}^2(\omega_\alpha(\sigma)) \le C\,\sigma\,\mathcal{H}^1(\Gamma^\varepsilon_{r,x}) \sup_{t,\vec{p}} \frac{\mathcal{H}^2(\Sigma^\varepsilon_{r,x}\cap B^m_t(\vec{p}))}{t^2} \le C_\varepsilon\,\sigma \tag{3.24}$$

Hence we have

$$\lim_{s\to 0}\frac{1}{s}\int_0^{2s}\chi'\left(\frac{\sigma}{s}\right)\,d\sigma \sum_{\alpha\in A_\sigma}\int_{\partial\omega_\alpha(\sigma)} \xi_\varepsilon(\vec{p})\;F_j(\vec{p})\cdot\nu\;d\mathcal{H}^1 \llcorner \Sigma^\varepsilon_{r,x}$$

$$\le \lim_{s\to 0}\frac{C_\varepsilon}{s}\int_0^{2s}\|\chi'\|_\infty\;\|\mathrm{div}_{\Sigma^\varepsilon_{r,x}}\left(\xi_\varepsilon(\vec{p})\;F_j(\vec{p})\right)\|_{L^\infty(\Sigma^\varepsilon_{r,x})}\;\sigma\;\;d\sigma \le C_\varepsilon\,\sigma \tag{3.25}$$

Combining (3.19) …(3.25) we obtain

$$\begin{aligned}\lim_{s\to 0}&\int_{B_r(x)\cap\mathcal{G}_\varepsilon}\xi_\varepsilon(\vec{\Phi})\;\nabla(\eta_s(\vec{\Phi}))\;F(\vec{\Phi})_{ij}\;\nabla\vec{\Phi}_i\;dy^2\\ &=\int_{\partial B_r(x)\cap\mathcal{G}_\varepsilon}\xi_\varepsilon(\vec{\Phi})\;\sum_{i=1}^m F_{ij}(\vec{\Phi})\;\frac{\partial\vec{\Phi}_i}{\partial r}\;dl_{\partial B_r}\end{aligned} \tag{3.26}$$

Collecting (3.10), (3.12), (3.18) and (3.26) we obtain

$$\begin{aligned}&\left|\sum_{i=1}^m\int_{\partial B_r(x)}F(\vec{\Phi})ij\;\frac{\partial\vec{\Phi}_i}{\partial r}\;dl_{\partial B_r}-\int_{B_r(x)}\nabla(F(\vec{\Phi})ij)\;\nabla\vec{\Phi}_i\;dy^2\right.\\ &\quad\left.+\int_{B_r(x)}F(\vec{\Phi})_{ij}\;A_i(\vec{\Phi})(d\vec{\Phi},d\vec{\Phi})\;dy^2\right| < C(\|F\|_\infty)\;\delta\end{aligned} \tag{3.27}$$

This holds for any $\delta > 0$ and hence lemma 3.1 is proved. □

### 3.2 Proof of Theorem 1.1

Let $\varphi \in C^\infty_0(D^2)$, be a non negative function. The co-area formula gives

$$\int_{D^2}\Delta\varphi\;\vec{\Phi}\;dx^2 = -\int_{D^2}\nabla\varphi\cdot\nabla\vec{\Phi} = \int_0^{+\infty}dt\int_{\varphi^{-1}\{t\}}\partial_\nu\vec{\Phi}\;dl_{\varphi^{-1}\{t\}} \tag{3.28}$$

Using lemma 3.1 for $F_{ij} := \delta_{ij}$, we obtain

$$\begin{aligned}\int_{D^2}\Delta\varphi\;\vec{\Phi}\;dx^2 &= -\int_0^{+\infty}dt\int_{\varphi^{-1}((t,+\infty))}A(\vec{\Phi})(d\vec{\Phi},d\vec{\Phi})\;dy^2\\ &= -\int_0^{+\infty}\int_{D^2}\mathbf{1}_{\varphi(x)>t}\;A(\vec{\Phi})(d\vec{\Phi},d\vec{\Phi})\;dy^2\\ dt &= -\int_{D^2}\varphi(x)\;A(\vec{\Phi})(d\vec{\Phi},d\vec{\Phi})\;dy^2\end{aligned} \tag{3.29}$$

This implies that $\vec{\Phi}$ satisfies weakly the harmonic map equation into $N^n$

$$-\Delta\vec{\Phi} = A(\vec{\Phi})(d\vec{\Phi},d\vec{\Phi}) \qquad \text{in } \mathcal{D}'(D^2)$$

and using Hélein's regularity result [3], we prove theorem 1.1. □

**Acknowledgements** The author would like to thank Alexis Michelat for stimulating discussions on the notion of target harmonic maps and the regularity question for these maps.

# A Appendix

**Proposition A. 1** *Let $N^n$ be a $C^2$ sub-manifold of the euclidian space $\mathbb{R}^m$. Let $(\Sigma, h)$ be a compact Riemann surface (equipped with a metric compatible with the complex structure) possibly with boundary. Let $\vec{\Phi}$ be a map in $W^{1,2}(\Sigma, N^n)$. Assume $\vec{\Phi}$ is weakly conformal and continuous on $\partial\Sigma$. The integer rectifiable varifold associated to $(\vec{\Phi}, \Sigma)$ is stationary in $N^n\setminus\vec{\Phi}(\partial\Sigma)$ if and only if*

$$\forall\, F \in C_0^\infty(N^n\setminus\vec{\Phi}(\partial\Sigma), \mathbb{R}^m) \quad \int_\Sigma \left[\left\langle d(F(\vec{\Phi})), d\vec{\Phi}\right\rangle_h - F(\vec{\Phi})\ A(\vec{\Phi})(d\vec{\Phi}, d\vec{\Phi})_h\right] dvol_h = 0$$

*where $A(\vec{q})(\vec{X}, \vec{Y})$ denotes the second fundamental form of $N^n$ at the point $\vec{q}$ and acting on the pair of vectors $(\vec{X}, \vec{Y})$ and by an abuse of notation we write*

$$A(\vec{\Phi})(d\vec{\Phi}, d\vec{\Phi})_h := \sum_{i,j=1}^{2} h_{ij}\ A(\vec{\Phi})(\partial_{x_i}\vec{\Phi}, \partial_{x_j}\vec{\Phi}).$$

□

*Proof of proposition A. 1* For any $\vec{q} \in N^n$ one denotes by $P_T(\vec{q})$ the symmetric matrix giving the orthogonal projection onto $T_{\vec{q}}N^n$. The integer rectifiable varifold given by $(\vec{\Phi}, \Sigma)$ is by defintion the following *Radon measure* on $G_2(T\mathbb{R}^m)$ the Grassman bundle of un-oriented 2-planes over $\mathbb{R}^m$ given by

$$\forall\, \phi \in C^\infty(G_2(T\mathbb{R}^m)) \qquad \mathbf{v}_{\vec{\Phi}}(\phi) = \int_{G_2(T\mathbb{R}^m)} \phi(S, \vec{q})\, dV_{\vec{\Phi}}(S, \vec{q})$$
$$:= \int_\Sigma \phi(\vec{\Phi}_*(T_x\Sigma), \vec{\Phi}(x))\, dvol_{g_{\vec{\Phi}}}$$

By definition (see [1]), the varifold $\mathbf{v}_{\vec{\Phi}}$ is stationary in $N^n$ if

$$\forall\, F \in C_0^\infty(N^n\setminus\vec{\Phi}(\partial\Sigma), \mathbb{R}^m) \quad \int_\Sigma \mathrm{div}_S(P_T\, F)(\vec{q})\, dV_{\vec{\Phi}}(S, \vec{q}) = 0 \tag{A. 1}$$

In local conformal coordinates at a point where $|\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}| = e^\lambda$, introducing the orthonormal basis of $S := \vec{\Phi}_* T_x\Sigma$ given by $\vec{e}_i := e^{-\lambda}\partial_{x_i}\vec{\Phi}$, one has by definition

$$\mathrm{div}_{\vec{\Phi}_* T_x\Sigma}(P_T\, F)(\vec{\Phi}) := \sum_{i=1}^{2} \partial_{\vec{e}_i}(P_T\, F)(\vec{\Phi}) \cdot \vec{e}_i = \sum_{i=1}^{2}\sum_{k=1}^{m} e_i^k\ \partial_{z_k}(P_T\, F)(\vec{\Phi}) \cdot \vec{e}_i$$

where $\vec{e}_i := \sum_{k=1}^m e_i^k\ \partial_{z_k}$. Hence we have

$$\mathrm{div}_{\vec{\Phi}_* T_x\Sigma}(P_T\, F)(\vec{\Phi}) = e^{-2\lambda} \sum_{i=1}^{2}\sum_{k=1}^{m} \partial_{x_i}\Phi^k\ \partial_{z_k}(P_T\, F)(\vec{\Phi}) \cdot \partial_{x_i}\vec{\Phi}$$
$$= e^{-2\lambda}\, \nabla(F(\vec{\Phi})) \cdot \nabla\vec{\Phi} - F(\vec{\Phi}) \sum_{i=1}^{2} e^{-2\lambda}\ A(\vec{\Phi})(\partial_{x_i}\vec{\Phi}, \partial_{x_i}\vec{\Phi})$$

where we used respectively that $P_T(\vec{\Phi})\nabla\vec{\Phi} = \nabla\vec{\Phi}$ and that $A(\vec{q})(\vec{X}, \vec{Y}) = -\,\partial_{\vec{X}} P_T(\vec{q}) \cdot \vec{Y}$. Multiplying by $dvol_{g_{\vec{\Phi}}} = e^{2\lambda}\, dx_1 \wedge dx_2$ we obtain at almost every point $x$ where $\nabla\vec{\Phi}(x) \neq 0$

$$\mathrm{div}_{\vec{\Phi}_* T_x \Sigma}(P_T\, F)(\vec{\Phi})\, dvol_{g_{\vec{\Phi}}} = \left[\left\langle d(F(\vec{\Phi})), d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}} - F(\vec{\Phi})\ A(\vec{\Phi})(d\vec{\Phi}, d\vec{\Phi})_{g_{\vec{\Phi}}}\right] dvol_{g_{\vec{\Phi}}} \tag{A. 2}$$

This concludes the proof of the lemma. □

## References


1. Allard, W.K.: On the first variation of a varifold. Ann. Math. **2**(95), 417–491 (1972)
2. De Lellis, C.: Allard's interior regularity theorem, an invitation to stationary varifolds. Course University of Zürich. http://user.math.uzh.ch/delellis/index.php?id=publications
3. Hélein, F.: Harmonic Maps, Conservation Laws and Moving Frames. Cambridge Tracts in Math, vol. 150. Cambridge Univerity Press, Cambridge (2002)
4. Michelat, A., Rivière, T.: A viscosity method for the min-max construction of closed geodesics (2015). arXiv:1511.04545
5. Rivière, T.: A viscosity method in the min-max theory of minimal surfaces (2015). arXiv:1512.08918