Weak immersions of surfaces with L^2-bounded second fundamental form

Lecture 1

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Tristan Rivière

Lecture Notes written by Nora Loose

1 Introduction

The first appearance of the *Willmore functional* is in the work of Sophie Germain on elastic surfaces. Following the main lines of J. Bernoulli and Euler's studies of the mechanics of sticks in the first half of the XVIII$^{\text{th}}$ century, Germain formulated what she called the *fundamental hypothesis*: *at one point of the surface the elastic force which counterbalances the external forces is proportional to the sum of the principal curvature at this point*, i.e., what we now call the *Mean curvature*. Although Germain's work was controversial at the time and was criticized by contemporaries including Simeon Poisson, the *Willmore energy* was finally established on physical grounds by G. Kirchhoff in 1850 [Ki] as the free energy of an elastic membrane modulo the addition of a null Lagrangian (see for instance [LaLi]).

The *Willmore energy* was probably first considered in geometry in the early XX$^{\text{th}}$ century by Wilhelm Blaschke in his efforts to merge *minimal surface theory* with *conformal invariance*. He proved that the Willmore energy of a closed surface remains invariant under conformal transformations of the ambient space. As proved quite recently [MoN], this Lagrangian is the unique one possessing this property modulo the addition of a multiple of the Gauss curvature.

Because of its simplicity and fundamental nature, the Willmore energy appears in many areas of science and technology. Beyond *non-linear elasticity* and *conformal geometry*, one sees it for instance in *general relativity*, where the *Willmore energy* is the main term in the so called *Hawking Mass*. It arises also in *cell biology* as the main term of the free elastic energy of lipid bilayer membranes and is called the *Helfrich Energy* there. The *Willmore Energy* in its umbilic form also arises in the design of multifocal optical elements (see for instance [KaRu]), ...etc

For many years, the only surfaces known to be critical for the Willmore Lagrangian were the minimal surfaces and their images under conformal transformation. This perhaps explains why there was essentially no work on the variational theory of the Willmore functional during the several decades after Blaschke's seminal work. In the short but decisive paper [Wi] in 1965, Tom Willmore reintroduced this Lagrangian, which now bears his name. He formulated there what is now called the *Willmore Conjecture*, which asserts that the Willmore energy of a compact surface of non-zero genus in three-dimensional Euclidian space is at least $2\pi^2$, and this minimal energy is only achieved by the stereographic projection of the *Clifford Torus* in S^3 (and its conformal transformations). This famous conjecture was proved in 2011 by Fernando Codá Marques and André Neves in [MN] and their proof is the subject of one of the mini-course given at the present Park City Summer School 2013.

In the present series of five lectures, we develop some fundamental tools in the analytic study of the Willmore Lagrangian motivated by the following set of questions:

i) Does there exists a minimizer of the Willmore functional among all smooth immersions of a fixed surface Σ^2? If there is, can one estimate the energy and special properties of such a minimizer?

ii) Does there exist a minimizer of the Willmore functional among a more restricted class of immersions, for example, considering only the conformal immersions relative to a fixed conformal class c on Σ, or among all immersions of Σ into $\mathbb{R}^3$ which enclose a domain of given volume and which have a fixed area?

iii) How stable is the Willmore equation? In other words, does a sequence of "approximately Willmore" surfaces, e.g. Palais-Smale sequences for the Willmore energy, necessarily converge to a Willmore surface?

iv) Can one produce Willmore Surfaces using *min-max arguments* ? More specifically can one apply fundamental variational principles such as *Ekeland's variational Principles* or the *Mountain pass lemma* to the Willmore functional?

Question i) was first considered by Leon Simon in [Sim86] and completely solved when Σ is the torus T^2 in [Sim93]. Later, in [BK03], a condition shown in [Sim93] to be sufficient to guarantee the existence of minimizing Willmore surfaces was proved to hold for every oriented closed surface. Simon's approach has two main features. First, it is an *ambient approach*: namely, surfaces are viewed as subsets[1] of the ambient Euclidian space. The second is the use of *biharmonic approximation*, where almost-minimizing surfaces are "improved" by replacing small pieces by *biharmonic graphs*. The main drawback of this strategy is that it seems to be only well-suited for the study of minimizing surfaces, but not for questions of the form iii) and iv) for non-minimizing critical points. Simon's *ambient approach* was also used in [KS] to give partial answers to question ii).

In 2010, the author introduced the notion of *weak immersions* in order to provide a suitable framework in which general variations of the Willmore Lagrangian (see [Riv10]) are well posed. The goal of this mini-course is to present fundamental proper-

[1]More precisely, this approaches considers immersed surfaces as varifolds, i.e., Radon measures in the Grassman manifold of 2-planes of Euclidian space corresponding to the tangent bundle of these surfaces immersed into the tangent bundle the euclidian ambient space.

ties of this approach, which turns out to be a fundamental tool for addressing all of the questions i)...iv) . To illustrate this, we explain in the last lecture how to use these properties to give a new proof of Simon's existence result. More completely, we address in this mini-course the following questions:

a) Does a weak immersion define a smooth conformal structure ?

b) What happens to a sequence of weak immersions of a given surface Σ^2 for which the Willmore energy is uniformly bounded? Does it convergence in some sense to a weak immersion ?

c) Is there a weak formulation of the Willmore equation which is compatible with the notion of weak immersion ?

d) Are weak immersions which are solutions to the Willmore equation necessarily smooth ?

The five lectures of this mini-course correspond to a presentation of the 2006 and 2010 papers [Riv06] and [Riv10], published respectively in 2008 and 2014. Since that time, weak immersions have played a crucial role in answering questions i)...iv) see [Riv10], [Riv15], [KMR] and [Riv16].

2 Notations and fundamental results on the differential geometry of surfaces

2.1 Notations

For a smooth map $\vec{e}\colon D^2 \to \mathbb{R}^m$, denote

$$\nabla\vec{e} := \begin{pmatrix} \partial_{x_1}\vec{e} \\ \partial_{x_2}\vec{e} \end{pmatrix}. \tag{2.1}$$

The associated 1-form is denoted by $d\vec{e}$ and in local coordinates given by

$$d\vec{e} = \partial_{x_1}\vec{e}\; dx_1 + \partial_{x_2}\vec{e}\; dx_2.$$

Further, define

$$\nabla^\perp\vec{e} := \begin{pmatrix} -\partial_{x_2}\vec{e} \\ \partial_{x_1}\vec{e} \end{pmatrix}. \tag{2.2}$$

It corresponds to the 1-form $*d\vec{e}$, where $*$ is the Hodge operator from $\wedge^1\mathbb{R}^2$ to $\wedge^1\mathbb{R}^2$. Indeed, in local coordinates we have

$$*d\vec{e} = \partial_{x_1}\vec{e} * dx_1 + \partial_{x_2}\vec{e} * dx_2 = -\partial_{x_2}\vec{e}\; dx_1 + \partial_{x_1}\vec{e}\; dx_2.$$

For smooth maps $\vec{e}, \vec{f}\colon D^2 \to \mathbb{R}^m$, denote

$$\langle\vec{e}, \nabla\vec{f}\rangle := \begin{pmatrix} \langle\vec{e}, \partial_{x_1}\vec{f}\rangle \\ \langle\vec{e}, \partial_{x_2}\vec{f}\rangle \end{pmatrix}, \quad \langle\vec{e}, \nabla^\perp\vec{f}\rangle := \begin{pmatrix} -\langle\vec{e}, \partial_{x_2}\vec{f}\rangle \\ \langle\vec{e}, \partial_{x_1}\vec{f}\rangle \end{pmatrix}, \tag{2.3}$$

$$\vec{e}\times\nabla\vec{f} := \begin{pmatrix} \vec{e}\times\partial_{x_1}\vec{f} \\ \vec{e}\times\partial_{x_2}\vec{f} \end{pmatrix}, \quad \vec{e}\times\nabla^\perp\vec{f} := \begin{pmatrix} -\vec{e}\times\partial_{x_2}\vec{f} \\ \vec{e}\times\partial_{x_1}\vec{f} \end{pmatrix}, \tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ is the inner product of $\mathbb{R}^m$ and we use the $\times$-operation, if $m = 3$. Similarly, for $\lambda\colon D^2 \to \mathbb{R}$ and $\vec{e}\colon D^2 \to \mathbb{R}^m$ we write

$$\nabla\lambda\cdot\vec{e} := \begin{pmatrix} \partial_{x_1}\lambda\;\vec{e} \\ \partial_{x_2}\lambda\;\vec{e} \end{pmatrix}, \quad \nabla^\perp\lambda\cdot\vec{e} := \begin{pmatrix} -\partial_{x_2}\lambda\;\vec{e} \\ \partial_{x_1}\lambda\;\vec{e} \end{pmatrix}. \tag{2.5}$$

As notation suggests, we write for instance

$$\langle \nabla^{\perp}\vec{e}, \nabla\vec{f}\rangle = \langle \partial_{x_1}\vec{e}, \partial_{x_2}\vec{f}\rangle - \langle \partial_{x_2}\vec{e}, \partial_{x_1}\vec{f}\rangle, \tag{2.6}$$

and if $m = 3$,

$$\nabla\vec{e} \times \nabla^{\perp}\vec{f} = -\partial_{x_1}\vec{e} \times \partial_{x_2}\vec{f} + \partial_{x_2}\vec{e} \times \partial_{x_1}\vec{f}.$$

Naturally, denote

$$|\nabla\vec{e}|^2 := \langle \nabla\vec{e}, \nabla\vec{e}\rangle.$$

Similarly,

$$\nabla\lambda \cdot \nabla\vec{e} \; := \partial_{x_1}\lambda \; \partial_{x_1}\vec{e} + \partial_{x_2}\lambda \; \partial_{x_2}\vec{e}, \tag{2.7}$$

$$\nabla^{\perp}\lambda \cdot \nabla\vec{e} \; := -\partial_{x_2}\lambda \; \partial_{x_1}\vec{e} + \partial_{x_1}\lambda \; \partial_{x_2}\vec{e}. \tag{2.8}$$

Applying the divergence operator *div* to a vector field

$$\vec{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \end{pmatrix}$$

with components $\vec{X}_i \colon D^2 \to \mathbb{R}^m$ (for instance $\vec{X}$ of the form (2.1), (2.2), (2.3), (2.4), (2.5)) is to be understood as

$$div\,\vec{X} = \partial_{x_1}\vec{X}_1 + \partial_{x_2}\vec{X}_2.$$

Note that

$$div\langle \vec{e}, \nabla^{\perp}\vec{f}\rangle = \langle \nabla\vec{e}, \nabla^{\perp}\vec{f}\rangle = -\langle \nabla^{\perp}\vec{e}, \nabla\vec{f}\rangle \tag{2.9}$$

and for $m = 3$,

$$\begin{aligned} div\left[\vec{e} \times \nabla^{\perp}\vec{f}\right] &= -\partial_{x_1}\vec{e} \times \partial_{x_2}\vec{f} + \partial_{x_2}\vec{e} \times \partial_{x_1}\vec{f} \\ &= \nabla\vec{f} \times \nabla^{\perp}\vec{e}. \end{aligned} \tag{2.10}$$

Similarly, we use the above introduced notation for 1-forms. For instance, $\langle \vec{e}, d\vec{f} \rangle$ is the 1-form, which is in coordinates given by

$$\langle \vec{e}, d\vec{f} \rangle := \langle \vec{e}, \partial_{x_1} \vec{f} \rangle \, dx_1 + \langle \vec{e}, \partial_{x_2} \vec{f} \rangle \, dx_2$$

and thus the associated 1-form to the vector on the left hand side of (2.3). Finally, define the 2-form

$$\langle d\vec{e}, d\vec{f} \rangle := d\langle \vec{e}, d\vec{f} \rangle, \tag{2.11}$$

which in local coordinates is given by

$$\langle d\vec{e}, d\vec{f} \rangle = \Big(\langle \partial_{x_1} \vec{e}, \partial_{x_2} \vec{f} \rangle - \langle \partial_{x_2} \vec{e}, \partial_{x_1} \vec{f} \rangle \Big) \, dx_1 \wedge dx_2. \tag{2.12}$$

Observe that the Jacobian occurring in (2.12) is (2.6).

2.2 Immersions and their geometry

A smooth map $\vec{\Phi} \colon M \to N$ between two smooth manifolds M and N is called *immersion* if $d\vec{\Phi}_p \colon T_pM \to T_{f(p)}N$ is injective for all $p \in M$.

In this course, Σ denotes a smooth 2-*dimensional closed oriented manifold* and we usually consider immersions (later understood in a weak sense) from Σ into $\mathbb{R}^m$.

2.2.1 Submanifolds of $\mathbb{R}^3$

We start to consider a particularly easy class of immersions, namely the class of 2-dimensional submanifolds of $\mathbb{R}^3$ (the immersion being provided by the inclusion map).

Let S be a 2-dimensional submanifold of $\mathbb{R}^3$. We assume S to be oriented and we denote by $\vec{n}$ the associated *Gauss map*: the unit normal giving this orientation.

The *first fundamental form* is the induced metric on S that we denote by g: For any $p \in S$ and $\vec{X}, \vec{Y} \in T_pS$, it is given by

$$g_p(\vec{X}, \vec{Y}) := \langle \vec{X}, \vec{Y} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the canonical inner product in $\mathbb{R}^3$. The *volume form* associated to g on S is locally given by

$$dvol_g := \sqrt{det(g(\partial_{x_i}, \partial_{x_j}))}\; dx_1 \wedge dx_2,$$

where (x_1, x_2) are arbitrary local positive coordinates.[2]

The *second fundamental form* at $p \in S$ is the bilinear map which assigns to a pair of vectors $\vec{X}, \vec{Y}$ in T_pS an orthogonal vector to T_pS that we shall denote by $\vec{\mathbb{I}}(\vec{X}, \vec{Y})$. This normal vector expresses how much the Gauss map varies along these directions $\vec{X}$ and $\vec{Y}$. Precisely, it is given by

$$\begin{aligned} \vec{\mathbb{I}}_p \colon T_pS \times T_pS \;&\longrightarrow\; N_pS \\ (\vec{X}, \vec{Y}) \;&\mapsto\; -\langle d\vec{n}_p(\vec{X}), \vec{Y} \rangle\; \vec{n}(p). \end{aligned} \tag{2.13}$$

Extending smoothly $\vec{X}$ and $\vec{Y}$ first locally on S and then in a neighborhood of p in $\mathbb{R}^3$, one has, using $\langle \vec{n}, \vec{Y} \rangle = 0$ on S,

$$\begin{aligned} \vec{\mathbb{I}}(\vec{X}, \vec{Y}) &= \langle \vec{n}, d\vec{Y}(\vec{X}) \rangle\; \vec{n} \;= d\vec{Y}(X) - \nabla_{\vec{X}} \vec{Y} \\ &= \overline{\nabla}_{\vec{X}} \vec{Y} - \nabla_{\vec{X}} \vec{Y}. \end{aligned} \tag{2.14}$$

Here, ∇ is the Levi-Civita connection on S generated by g, and we have $\nabla_{\vec{X}} \vec{Y} = \pi_T(d\vec{Y}(X))$, where π_T is the orthogonal projection onto TS. $\overline{\nabla}$ is the Levi-Civita connection associated to the flat metric and is simply given by $\overline{\nabla}_{\vec{X}} \vec{Y} = d\vec{Y}(X)$.

[2]Local coordinates, denoted by (x_1, x_2), are provided by a diffeomorphism x from an open set in $\mathbb{R}^2$ onto an open set in Σ. For any point q in this open set of S we shall denote by $x_i(q)$ the canonical coordinates in $\mathbb{R}^2$ of $x^{-1}(q)$. Finally ∂_{x_i} is the vector-field on S given by $\partial x / \partial x_i$.

An elementary but fundamental property of the second fundamental form says that it is symmetric[3]. It can then be diagonalized in an orthonormal basis and the two eigenvalues κ_1 and κ_2 are called the *principal curvatures* of the surface at p. The *mean curvature* is then given by

$$H := \frac{\kappa_1 + \kappa_2}{2}$$

and the *mean curvature vector* by

$$\vec{H} := H\ \vec{n} = \frac{1}{2}\,\mathrm{tr}(g^{-1}\ \vec{\mathbb{I}}) = \frac{1}{2}\sum_{i,j=1}^{2} g^{ij}\ \vec{\mathbb{I}}(\partial_{x_i}, \partial_{x_j}), \tag{2.15}$$

where (x_1, x_2) are arbitrary local coordinates. Here, $(g^{ij})_{ij}$ denotes the inverse matrix to $(g_{ij}) := (g(\partial_{x_i}, \partial_{x_j}))$. In particular, if $(\vec{e}_1, \vec{e}_2)$ is an orthonormal basis of T_pS, (2.15) becomes

$$\vec{H} = \frac{\vec{\mathbb{I}}(\vec{e}_1, \vec{e}_2) + \vec{\mathbb{I}}(\vec{e}_2, \vec{e}_2)}{2}. \tag{2.16}$$

The Gauss curvature is given by

$$K := \frac{det\left(\langle \vec{n}, \vec{\mathbb{I}}(\partial_{x_i}, \partial_{x_j})\rangle\right)}{det(g_{ij})} = \kappa_1\ \kappa_2. \tag{2.17}$$

The *Willmore functional* of the surface Σ is defined by

$$W(S) = \int_S |\vec{H}|^2\ dvol_g = \frac{1}{4}\int_S |\kappa_1 + \kappa_2|^2\ dvol_g.$$

The *Gauss-Bonnet theorem*[4] asserts that the integral of $K\ dvol_g$

[3]This can be seen combining equation (2.14) with the fact that the two Levi-Civita connections ∇ and $\overline{\nabla}$ are torsion-free, i.e. we have

$$\nabla_{\vec{X}}\vec{Y} - \nabla_{\vec{Y}}\vec{X} = [X, Y]$$

and

$$\overline{\nabla}_{\vec{X}}\vec{Y} - \overline{\nabla}_{\vec{Y}}\vec{X} = [X, Y].$$

[4]See for instance [dC76]

is proportional to a topological invariant of S: $\chi(S)$, the Euler characteristic of S. Precisely, one has

$$\begin{aligned}\int_S K\ dvol_g = \int_S \kappa_1\ \kappa_2\ dvol_g\ &= 2\pi\ \chi(S)\\ &= 4\pi\ (1-g(S)),\end{aligned}\tag{2.18}$$

where $g(S)$ denotes the genus of S. Combining the definition of W and this last identity one obtains[5]

$$\begin{aligned}W(S)-\pi\ \chi(S)\ &= \frac{1}{4}\int_S(\kappa_1^2+\kappa_2^2)\ dvol_g\\ &= \frac{1}{4}\int_S|\vec{\mathbb{I}}|^2_g\ dvol_g = \frac{1}{4}\int_S|d\vec{n}|^2_g\ dvol_g.\end{aligned}\tag{2.19}$$

Hence modulo the addition of a topological term, the Willmore energy corresponds to the *Sobolev homogeneous $\dot{H}^1-$energy of the Gauss map* for the induced metric g.

2.2.2 Immersions of an abstract surface into $\mathbb{R}^m$.

Let us now consider the general case that $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$ is a immersion of an abstract 2-dimensional oriented closed manifold Σ into $\mathbb{R}^m$.

The *first fundamental form* associated to the immersion is the metric $g := \vec{\Phi}^* g_{\mathbb{R}^m}$ induced by $\vec{\Phi}$, where $g_{\mathbb{R}^m}$ is the canonical metric on $\mathbb{R}^m$: For all $p \in \Sigma$ and $X, Y \in T_p\Sigma$,

$$g_p(X,Y) := \langle d\vec{\Phi}_p(X), d\vec{\Phi}_p(Y)\rangle$$

[5] The last identity comes from the fact that at a point p, taking an orthonormal basis $(\vec{e}_1,\vec{e}_2)$ of T_pS, one has:

$$|d\vec{n}|^2_g = \sum_{i,j=1}^2 \langle d\vec{n}(\vec{e}_i),\vec{e}_j\rangle^2 = \sum_{i,j=1}^2 |\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_j)|^2 = |\vec{\mathbb{I}}|^2,$$

since $\langle d\vec{n},\vec{n}\rangle = 0$.

where $\langle \cdot, \cdot \rangle$ is the canonical inner product in $\mathbb{R}^m$. The *volume form* associated to g on Σ is locally given by

$$dvol_g := \sqrt{det(g(\partial_{x_i}, \partial_{x_j}))}\; dx_1 \wedge dx_2,$$

where (x_1, x_2) are arbitrary local positive coordinates.

We shall denote by $\vec{e}$ the map which to a point in Σ assigns the oriented 2-plane[6], given by the push-forward by $\vec{\Phi}$ of the oriented tangent space $T_p\Sigma$. Using a positive orthonormal basis $(\vec{e}_1, \vec{e}_2)$ of $\vec{\Phi}_* T_p\Sigma$, an explicit expression of $\vec{e}$ is given by

$$\vec{e} = \vec{e}_1 \wedge \vec{e}_2.$$

With these notations the *Gauss map* which to every point p assigns the oriented $(m-2)$-orthogonal plane to $\vec{\Phi}_* T_p\Sigma$ is given by

$$\vec{n} = \star \vec{e} = \vec{n}_1 \wedge \cdots \wedge \vec{n}_{m-2}.$$

Here, $\star$ is the Hodge operator[7] from $\wedge^2 \mathbb{R}^m$ to $\wedge^{m-2}\mathbb{R}^m$ and $(\vec{n}_1, \ldots, \vec{n}_{m-2})$ is a positive orthonormal basis of the oriented normal plane to $\vec{\Phi}_* T_p\Sigma$. (Consequently, $(\vec{e}_1, \vec{e}_2, \vec{n}_1, \cdots, \vec{n}_{m-2})$ is an orthonormal basis of $\mathbb{R}^m$.)

We shall denote by $\pi_{\vec{n}}$ the orthogonal projection onto the $(m-2)$-plane at p given by $\vec{n}(p)$.

[6]We denote by $\tilde{G}_p(\mathbb{R}^m)$, the Grassman space of oriented p-planes in $\mathbb{R}^m$ that we interpret as the space of unit simple p-vectors in $\mathbb{R}^m$ which is included in the Grassmann algebra $\wedge^p \mathbb{R}^m$.

[7]The Hodge operator on $\mathbb{R}^m$ is the linear map from $\wedge^p\mathbb{R}^m$ to $\wedge^{m-p}\mathbb{R}^m$ which to a $p-$vector α assigns the $(m-p)$-vector $\star\alpha$ on $\mathbb{R}^m$, which is characterized by the following property: for any p-vector β in $\wedge^p\mathbb{R}^m$,

$$\beta \wedge \star\alpha = \langle \beta, \alpha \rangle\; \varepsilon_1 \wedge \cdots \wedge \varepsilon_m,$$

where $(\varepsilon_1, \cdots, \varepsilon_m)$ is the canonical orthonormal basis of $\mathbb{R}^m$ and $\langle \cdot, \cdot \rangle$ is the canonical inner product on $\wedge^p\mathbb{R}^m$.

The *second fundamental form* associated to the immersion $\vec{\Phi}$ is the following map:

$$\begin{aligned} \vec{\mathbb{I}}_p \colon T_pS \times T_pS &\longrightarrow (\vec{\Phi}_* T_p\Sigma)^\perp \\ (\vec{X}, \vec{Y}) &\mapsto \pi_{\vec{n}}\left(d^2\vec{\Phi}(X,Y)\right), \end{aligned}$$

where X and Y are extended smoothly into local smooth vector fields around p. One easily verifies that, although $d^2\vec{\Phi}(X,Y)$ might depend on these extensions, $\pi_{\vec{n}}(d^2\vec{\Phi}(X,Y))$ does not and we have then defined a tensor.

Let $\vec{X} := d\vec{\Phi}(X)$ and $\vec{Y} := d\vec{\Phi}(Y)$. Denote also by π_T the orthogonal projection onto $\vec{\Phi}_* T_q\Sigma$.

$$\begin{aligned} \pi_{\vec{n}}\left(d^2\vec{\Phi}(X,Y)\right) &= d(d\vec{\Phi}(X))(Y) - \pi_T\left(d(d\vec{\Phi}(X))(Y)\right) \\ &= d\vec{X}(Y) - \nabla_Y X \qquad (2.20) \\ &= \overline{\nabla}_{\vec{Y}}\vec{X} - \nabla_Y X, \end{aligned}$$

where $\overline{\nabla}$ is the Levi-Civita connection in $\mathbb{R}^m$ for the canonical metric and ∇ is the Levi-Civita connection on $T\Sigma$ induced by the metric g. Here again, as in the 3-d case in the previous subsection, from the fact that Levi-Civita connections are torsion-free we can deduce the symmetry of the second fundamental form.

Similarly to the 3-d case, the *mean curvature vector*[8] is given by

$$\vec{H} := \frac{1}{2}\,\mathrm{tr}(g^{-1}\,\vec{\mathbb{I}}) = \frac{1}{2}\sum_{i,j=1}^{2} g^{ij}\,\vec{\mathbb{I}}(\partial_{x_i}, \partial_{x_j}), \qquad (2.21)$$

[8]observe that the notion of mean curvature H does not make sense any more in codimension larger than 1 unless a normal direction is given.

where (x_1, x_2) are arbitrary local coordinates in Σ and $(g^{ij})_{ij}$ is the inverse matrix to $(g(\partial_{x_i}, \partial_{x_j}))$.

We can now give the general formulation of the *Willmore energy* of an immersion $\vec{\Phi}$ in $\mathbb{R}^m$ of an abstract surface Σ:

$$W(\vec{\Phi}) := \int_\Sigma |\vec{H}|^2 \, dvol_g.$$

A fundamental theorem by Gauss gives an expression of the intrinsic Gauss curvature in terms of the second fundamental form of any immersion of the surface in $\mathbb{R}^m$. Precisely this theorem says (see theorem 2.5 chapter 6 of [dC92])

$$K = \langle \vec{\mathbb{I}}(e_1, e_1), \vec{\mathbb{I}}(e_2, e_2) \rangle - \langle \vec{\mathbb{I}}(e_1, e_2), \vec{\mathbb{I}}(e_1, e_2) \rangle, \tag{2.22}$$

where (e_1, e_2) is an arbitrary orthonormal basis of $T_p\Sigma$.

Note that in particular, the Gauss curvature is controlled by the second fundamental form as follows:

$$\begin{aligned} |K| &\le |\vec{\mathbb{I}}(\vec{e}_1, \vec{e}_1)||\vec{\mathbb{I}}(\vec{e}_2, \vec{e}_2)| + |\vec{\mathbb{I}}(\vec{e}_1, \vec{e}_2)|^2 \\ &\le \frac{1}{2} \left(|\vec{\mathbb{I}}(\vec{e}_1, \vec{e}_1)|^2 + |\vec{\mathbb{I}}(\vec{e}_2, \vec{e}_2)|^2 + 2|\vec{\mathbb{I}}(\vec{e}_1, \vec{e}_2)|^2 \right) \\ &= \frac{1}{2} |\vec{\mathbb{I}}|_g^2. \end{aligned} \tag{2.23}$$

From identity (2.22) we furthermore deduce easily

$$|\vec{\mathbb{I}}|_g^2 = 4|\vec{H}|^2 - 2K. \tag{2.24}$$

Hence, using Gauss-Bonnet theorem, we obtain the following expression of the Willmore energy of an immersion into $\mathbb{R}^m$ of an arbitrary closed surface:

$$W(\vec{\Phi}) = \frac{1}{4} \int_\Sigma |\vec{\mathbb{I}}|_g^2 \, dvol_g + \pi \ \chi(\Sigma). \tag{2.25}$$

Let us take a local *normal frame* around $p \in \Sigma$: a smooth map $(\vec{n}_1, \cdots, \vec{n}_{m-2})$ from a neighborhood $U \subset \Sigma$ into $(S^{m-1})^{m-2}$ such that for any point $q \in U$, $(\vec{n}_1(q), \cdots, \vec{n}_{m-2}(q))$ realizes a positive orthonormal basis of $(\vec{\Phi}_* T_q\Sigma)^\perp$. Then

$$\pi_{\vec{n}}\left(d^2\vec{\Phi}(X,Y)\right) = \sum_{\alpha=1}^{m-2} \langle d^2\vec{\Phi}(X,Y), \vec{n}_\alpha\rangle\ \vec{n}_\alpha,$$

from which we deduce the following expression, which is the natural extension of (2.13):

$$\vec{\mathbb{I}}(X,Y) = -\sum_{\alpha=1}^{m-2} \langle d\vec{n}_\alpha(X), \vec{Y}\rangle\ \vec{n}_\alpha, \tag{2.26}$$

where we denote $\vec{Y} := d\vec{\Phi}(Y)$. Let (e_1, e_2) be an orthonormal basis of $T_p\Sigma$. Then the previous expression of the second fundamental form implies

$$|\vec{\mathbb{I}}|_g^2 = \sum_{i,j=1}^{2} \sum_{\alpha=1}^{m-2} |\langle d\vec{n}_\alpha(e_i), \vec{e}_j\rangle|^2 = \sum_{i=1}^{2} \sum_{\alpha=1}^{m-2} |\langle d\vec{n}_\alpha, \vec{e}_i\rangle|^2. \tag{2.27}$$

Observe that

$$\begin{aligned} d\vec{n} &= \sum_{\alpha=1}^{m-2} (-1)^{\alpha-1} d\vec{n}_\alpha \wedge_{\beta\neq\alpha} \vec{n}_\beta \\ &= \sum_{i=1}^{2} \sum_{\alpha=1}^{m-2} (-1)^{\alpha-1} \langle d\vec{n}_\alpha, \vec{e}_i\rangle\ \vec{e}_i \wedge_{\beta\neq\alpha} \vec{n}_\beta. \end{aligned} \tag{2.28}$$

$(\vec{e}_i \wedge_{\beta\neq\alpha} \vec{n}_\beta)$ for $\alpha = 1\ldots, m-2$ and $i = 1, 2$ realizes a linearly independent family of $2(m-2)$ orthonormal vectors in $\wedge^{m-2}\mathbb{R}^m$. Hence

$$|d\vec{n}|_g^2 = \sum_{i=1}^{2} \sum_{\alpha=1}^{m-2} |\langle d\vec{n}_\alpha, \vec{e}_i\rangle|^2 = |\vec{\mathbb{I}}|_g^2. \tag{2.29}$$

Combining (2.25) and (2.29), we obtain

$$W(\vec{\Phi}) = \frac{1}{4}\int_\Sigma |d\vec{n}|_g^2 \ dvol_g + \pi \ \chi(\Sigma), \tag{2.30}$$

which generalizes identity (2.19) to arbitrary immersions of closed 2-dimensional surfaces.

The *negative Laplace-Beltrami operator* Δ_g associated to the metric $g = g_{\vec{\Phi}}$, acting on a smooth function $f\colon \Sigma \to \mathbb{R}$, is in local coordinates given by

$$\Delta_g f = \frac{1}{\sqrt{\det(g_{ij})}}\,\partial_{x_i}\left(g^{ij}\sqrt{\det(g_{ij})}\partial_{x_j} f\right), \tag{2.31}$$

where we use the Einstein summation convention.

2.2.3 Immersions into a Riemannian manifold

We now want to have a short glance at a more general framework, namely immersions mapping into a Riemannian manifold:

Let $\vec{\Phi}$ be an immersion from a 2-dimensional closed, oriented manifold Σ into an arbitrary oriented Riemannian manifold $(M^m, \overline{g})$ of dimension $m \geq 3$.

The first fundamental form g of $\vec{\Phi}$ is the pull-back of the metric $\overline{g}$ by $\vec{\Phi}$: For all $p \in \Sigma$ and $X, Y \in T_p\Sigma$, it is given by

$$g_p(X,Y) := \overline{g}\left(d\vec{\Phi}_p(X), d\vec{\Phi}_p(Y)\right).$$

For defining the second fundamental form one formally replaces the exterior differential d with the Levi-Civita connection $\overline{\nabla}$ of $(M, \overline{g})$. Precisely, the second fundamental form associated to the immersion $\vec{\Phi}$ at a point $p \in \Sigma$ is the following map:

$$\begin{aligned} \vec{\mathbb{I}}_p : T_p\Sigma \times T_p\Sigma \ &\longrightarrow \ (\vec{\Phi}_* T_p\Sigma)^\perp \\ (X,Y) \ &\mapsto \ \pi_{\vec{n}}\left(\overline{\nabla}_{\vec{Y}}(d\vec{\Phi}(X))\right), \end{aligned}$$

where $\pi_{\vec{n}}$ denotes the orthogonal projection from $T_{\vec{\Phi}(p)}M$ onto $(\vec{\Phi}_* T_p\Sigma)^\perp$, the space orthogonal to $\vec{\Phi}_*(T_p\Sigma)$ with respect to the metric $\overline{g}$. As before we use the notation $\vec{Y} = d\vec{\Phi}(Y)$.

As before, the *mean curvature vector* is given by

$$\vec{H} = \frac{1}{2}\,\mathrm{tr}(g^{-1}\vec{\mathbb{I}}) = \frac{1}{2}\sum_{i,j=1}^{2} g^{ij}\ \vec{\mathbb{I}}(\partial_{x_i}, \partial_{x_j}), \tag{2.32}$$

where we are using local coordinates (x_1, x_2) on Σ. The Willmore energy of $\vec{\Phi}$ is defined as

$$W(\vec{\Phi}) := \int_\Sigma |\vec{H}|^2\ dvol_g.$$

2.3 Conformal invariance of the Willmore Energy

Definition 2.1. *i) Two metrics g and h on a smooth manifold M are said to be* conformal *(or conformally equivalent), if there exists a smooth function $\mu\colon M \to \mathbb{R}$ such that*

$$h = e^{2\mu} g.$$

ii) An immersion $\vec{\Phi}\colon M \to N$ between two Riemannian manifolds (M, g) and (N, k) is called conformal *if the pull-back metric $h := \vec{\Phi}^* k$ is conformally equivalent to g on M.*

Example.

i) Let $U \subset \mathbb{C}$ be an open subset and $\vec{\Phi}\colon U \to \mathbb{C}$ a diffeomorphism onto ints image. $\vec{\Phi}$ is conformal if and only if it is either holomorphic (if it is orientation-preserving) or antiholomorphic (if it is orientation-reversing). In particular, the space of conformal transformations of $\mathbb{C}$ is infinite-dimensional.

ii) The group of conformal transformations in dimension $m \geq 3$ reduces to a finite-dimensional group. Precisely, Liouville's theorem gives that any conformal diffeomorphism $\vec{\Phi}$ from $U \subset \mathbb{R}^m$ into $\mathbb{R}^m$ is of the form

$$\vec{\Phi}(x) = a + \alpha \, \frac{A(x - x_0)}{|x - x_0|^{\varepsilon}},$$

for some $a \in \mathbb{R}^m$, $x_0 \in \mathbb{R}^m \setminus U$, $\alpha \in \mathbb{R}$, an orthogonal matrix A and $\varepsilon \in \{0, 2\}$.

iii) The space of conformal maps from 2D into 3D is infinite-dimensional. It contains non-analytic mappings.

The conformal invariance of the Willmore energy was known since the work of Blaschke [Bla29] in 3 dimensions, in the general case it is a consequence of the following theorem due to Bang Yen Chen [Che74].

Theorem 2.2. *Let $\vec{\Phi}$ be the immersion of a 2-dimensional oriented manifold Σ into a Riemannian manifold (M, g).*

Let h be a conformally equivalent metric to g, given by

$$h := e^{2\mu} g.$$

We denote by $\vec{H}^g$ and $\vec{H}^h$ the mean curvature vectors of the immersion $\vec{\Phi}$ in (M, g) and (M, h) respectively. We also denote by K^g and K^h the scalar curvatures of $(\Sigma, \vec{\Phi}^* g)$ and $(\Sigma, \vec{\Phi}^* h)$ respectively. Furthermore, $\overline{K}^g$ and $\overline{K}^h$ denote the sectional curvatures of the subspace $\vec{\Phi}_* T_p \Sigma$ in the manifold (M, g) and (M, h) respectively. With the previous notations the following pointwise identity holds:

$$e^{2\mu \circ \vec{\Phi}} \, \left(|\vec{H}^h|_h^2 - K^h + \overline{K}^h \right) = |\vec{H}^g|_g^2 - K^g + \overline{K}^g. \qquad (2.33)$$

Proof of theorem 2.2. See [Riv].

The conformal invariance of the Willmore energy is a corollary of theorem 2.2.

Corollary 2.3. *Let Σ be a smooth 2-dimensional closed oriented manifold and let $\vec{\Phi}$ be an immersion of Σ into an oriented Riemannian manifold (M^m, g).*

Let Ψ be a positive conformal diffeomorphism from (M^m, g) into another oriented Riemannian manifold (N^m, k).

Then the following equality holds:

$$W(\vec{\Phi}) + \int_\Sigma \overline{K}^g \, dvol_{\vec{\Phi}^* g} = W(\Psi \circ \vec{\Phi}) + \int_\Sigma \overline{K}^k \, dvol_{(\Psi\circ\vec{\Phi})^* k}, \quad (2.34)$$

where $\overline{K}^g$ (resp. $\overline{K}^k$) is the sectional curvature of the 2-plane $\vec{\Phi}_* T\Sigma$ in (M^m, g) (resp. of the 2-plane $\Psi_* \vec{\Phi}_* T\Sigma^2$ in (N^m, k)).

Proof of corollary 2.3. By definition Ψ realizes an isometry between $(M, \Psi^* k)$ and (N, k). Let $\mu \colon M \to \mathbb{R}$ such that

$$e^{2\mu} g = \Psi^* k.$$

We can apply the previous theorem with $h = \Psi^* k$ and obtain

$$\left[|\vec{H}^{\vec{\Phi}^* g}|^2 - K^{\vec{\Phi}^* g} + \overline{K}^g \right] = e^{2\mu\circ\vec{\Phi}} \ \left[|\vec{H}^{(\Psi\circ\vec{\Phi})^* k}|^2 - K^{(\Psi\circ\vec{\Phi})^* k} + \overline{K}^k \right].$$

Furthermore, we have

$$dvol_{\vec{\Phi}^* g} = e^{-2\mu\circ\vec{\Phi}} \, dvol_{(\Psi\circ\vec{\Phi})^* k}.$$

Hence combining the two last facts gives the following pointwise identity everywhere on Σ:

$$\begin{aligned} &\left[|\vec{H}^{\vec{\Phi}^* g}|^2 - K^{\vec{\Phi}^* g} + \overline{K}^g \right] \ dvol_{\vec{\Phi}^* g} \\ &\qquad = \left[|\vec{H}^{(\Psi\circ\vec{\Phi})^* k}|^2 - K^{(\Psi\circ\vec{\Phi})^* k} + \overline{K}^k \right] \ dvol_{(\Psi\circ\vec{\Phi})^* k}, \end{aligned} \quad (2.35)$$

(2.34) is obtained by integrating (2.35) over Σ, the scalar curvature terms canceling each other on both sides of the identity due to Gauss-Bonnet theorem. $\square$

Example.

i) Let $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$ be an immersion into $\mathbb{R}^m$. Define Ψ to be the inverse of the stereographic projection

$$\pi\colon S^m \setminus \{(0,\ldots,0,1)\} \to \mathbb{R}^m.$$

Recall that π is conformal, thus applying Corollary 2.3 gives

$$\int_\Sigma |\vec{H}^{\vec{\Phi}^* g_{\mathbb{R}^m}}|^2 dvol_{\vec{\Phi}^* g_{\mathbb{R}^m}} = \int_\Sigma \left(|\vec{H}^{(\Psi\circ\vec{\Phi})^* k}|^2 + 1 \right) dvol_{(\Psi\circ\vec{\Phi})^* k},$$

where k denotes the round metric on S^m.

ii) For $a \in \mathbb{R}^m$, consider the inversion i_a at a, which for $x \in \mathbb{R}^m \setminus \{a\}$ is given by

$$i_a(x) = \frac{x-a}{|x-a|^2}.$$

i_a, restricted to $\mathbb{R}^m \setminus B_\delta(a)$, is a conformal diffeomorphism. Given a smooth immersion $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$ with $a \notin \vec{\Phi}(\Sigma)$, we have thus

$$W(i_a \circ \vec{\Phi}) = W(\vec{\Phi}),$$

due to Corollary 2.3.

If $a \in \vec{\Phi}(\Sigma)$, then the situation is different. In fact, in this case we have

$$W(i_a \circ \vec{\Phi}) = W(\vec{\Phi}) + 4\pi \cdot \mathrm{Card}\{\vec{\Phi}^{-1}(a)\}.$$

2.4 Two-dimensional geometry in isothermal charts

Definition 2.4. *Let* $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$ *be an immersion of* Σ*. A chart* $\psi\colon D^2 \to \Sigma$ *is called* isothermal *or* conformal *for* $\vec{\Phi}$*, if*

$$\begin{cases} \langle \partial_{x_1}(\vec{\Phi}\circ\psi), \partial_{x_2}(\vec{\Phi}\circ\psi)\rangle = 0 & \text{in } D^2 \\ \\ |\partial_{x_1}(\vec{\Phi}\circ\psi)| = |\partial_{x_2}(\vec{\Phi}\circ\psi)| & \text{in } D^2. \end{cases} \tag{2.36}$$

Here, $\langle \partial_{x_1}(\vec{\Phi}\circ\psi), \partial_{x_2}(\vec{\Phi}\circ\psi)\rangle$ *denotes the usual inner product in* $\mathbb{R}^m$*.*

Note that a different way of formulating (2.36) would be to say that the map $\vec{\Phi}\circ\psi\colon D^2 \to \mathbb{R}^m$ is conformal (cf. Definition 2.1 ii)).

In isothermal charts many objects defined in subsection 2.2 take an easier form, which we want to explore now.

Let $\psi\colon D^2 \to \Sigma$ be an isothermal chart for the immersion $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$.

The first fundamental form in the coordinates provided by ψ is

$$\psi^* g_{\vec{\Phi}} = e^{2\lambda}(dx_1^2 + dx_2^2),$$

where $e^\lambda = |\partial_{x_1}(\vec{\Phi}\circ\psi)| = |\partial_{x_2}(\vec{\Phi}\circ\psi)|$. The volume element is given by

$$dvol_g := e^{2\lambda}\ dx_1 \wedge dx_2 \quad .$$

Moreover, for the second fundamental form, we have

$$\vec{\mathbb{I}}_{ij} := \vec{\mathbb{I}}(\partial_{x_i}, \partial_{x_j}) = \pi_{\vec{n}}\left(\partial_{x_j}\partial_{x_i}(\vec{\Phi}\circ\psi)\right). \tag{2.37}$$

Note that

$$\langle \partial_{x_1}^2(\vec{\Phi}\circ\psi), \partial_{x_1}(\vec{\Phi}\circ\psi)\rangle$$

$$= \partial_{x_1} \underbrace{\langle \partial_{x_1}(\vec{\Phi} \circ \psi), \partial_{x_1}(\vec{\Phi} \circ \psi) \rangle}_{=e^{2\lambda}} - \langle \partial_{x_1}(\vec{\Phi} \circ \psi), \partial^2_{x_1}(\vec{\Phi} \circ \psi) \rangle$$

$$= \frac{1}{2}\partial_{x_1}\left(e^{2\lambda}\right) = \frac{1}{2}\partial_{x_1}\langle \partial_{x_2}(\vec{\Phi} \circ \psi), \partial_{x_2}(\vec{\Phi} \circ \psi) \rangle \tag{2.38}$$

$$= \langle \partial_{x_1}\partial_{x_2}(\vec{\Phi} \circ \psi), \partial_{x_2}(\vec{\Phi} \circ \psi) \rangle$$

$$= \partial_{x_2} \underbrace{\langle \partial_{x_1}(\vec{\Phi} \circ \psi), \partial_{x_2}(\vec{\Phi} \circ \psi) \rangle}_{=0} - \langle \partial^2_{x_2}(\vec{\Phi} \circ \psi), \partial_{x_1}(\vec{\Phi} \circ \psi) \rangle$$

$$= -\langle \partial^2_{x_2}(\vec{\Phi} \circ \psi), \partial_{x_1}(\vec{\Phi} \circ \psi) \rangle.$$

Similarly, one obtains that

$$\langle \partial^2_{x_1}(\vec{\Phi} \circ \psi), \partial_{x_2}(\vec{\Phi} \circ \psi) \rangle = -\langle \partial^2_{x_2}(\vec{\Phi} \circ \psi), \partial_{x_2}(\vec{\Phi} \circ \psi) \rangle. \tag{2.39}$$

(2.37), (2.38) and (2.39) together imply that

$$\vec{\mathbb{I}}_{11} + \vec{\mathbb{I}}_{22} = \pi_{\vec{n}}\left(\Delta(\vec{\Phi} \circ \psi)\right) = \Delta(\vec{\Phi} \circ \psi), \tag{2.40}$$

where $\Delta = \partial^2_{x_1} + \partial^2_{x_2}$ denotes the negative flat Laplacian.

From expression (2.31) of the intrinsic negative Laplace-Beltrami operator Δ_g in coordinates the following relation follows immediately:

$$\Delta_g = e^{-2\lambda}\Delta. \tag{2.41}$$

Using (2.40) and (2.41), the mean curvature vector takes the following form:

$$\vec{H} = \frac{1}{2}\,\mathrm{tr}(g^{-1}\ \vec{\mathbb{I}}) = \frac{e^{-2\lambda}}{2}\sum_{i,j=1}^{2}\delta^{ij}\ \vec{\mathbb{I}}_{ij} = \frac{e^{-2\lambda}}{2}\Delta(\vec{\Phi} \circ \psi) \tag{2.42}$$

$$= \frac{1}{2}\Delta_g(\vec{\Phi}\circ\psi). \tag{2.43}$$

The Willmore functional has therefore the expression

$$W(\vec{\Phi}) = \frac{1}{4}\int_\Sigma |\Delta_g\vec{\Phi}|^2 \; dvol_g. \tag{2.44}$$

Induced orthonormal moving frame. Given an isothermal chart $\psi\colon D^2 \to \Sigma$ for an immersion $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$, we can look at the frame

$$(\vec{e}_1, \vec{e}_2) = e^{-\lambda}\; (\partial_{x_1}(\vec{\Phi}\circ\psi), \partial_{x_2}(\vec{\Phi}\circ\psi)).$$

This realizes a *tangent orthonormal moving frame*, i.e. a mapping from D^2 to $\vec{\Phi}_*(TD^2\times TD^2)$ such that at every $(x_1, x_2) \in D^2$, the pair $(\vec{e}_1, \vec{e}_2)(x_1, x_2)$ realizes a positive orthonormal basis of $\vec{\Phi}_*(T_{(x_1,x_2)}D^2)$.

A simple computation, similarly as the one in (2.38) shows

$$\langle \vec{e}_1, \nabla\vec{e}_2\rangle = -\nabla^\perp\lambda. \tag{2.45}$$

In particular it follows

$$div\langle \vec{e}_1, \nabla\vec{e}_2\rangle = 0. \tag{2.46}$$

Note that this identity can be writen independently of the parametrization as

$$d^{*_g}\langle \vec{e}_1, d\vec{e}_2\rangle = 0, \tag{2.47}$$

which one refers to as the *Coulomb condition.* A frame $(\vec{e}_1, \vec{e}_2)$ satisfying (2.47) is called a *Coulomb frame.*

The following identity of 2-forms on D^2 holds:

$$d\langle \vec{e}_1, d\vec{e}_2\rangle = K\; dvol_g, \tag{2.48}$$

where K is the Gauss curvature of (D^2, g).

To see this, let e_i be the vector field on D^2 given by $d\vec{\Phi}(e_i) = \vec{e}_i$ for $i = 1, 2$. Further, denote

$$D_{e_i}\vec{e}_j := \pi_{\vec{n}}(d\vec{e}_j(e_i)).$$

Then, using Cartan's formula[9], we obtain

$$\begin{aligned}
&d\langle \vec{e}_1, d\vec{e}_2\rangle(e_1, e_2)\\
&= d\big(\langle \vec{e}_1, d\vec{e}_2(e_2)\rangle\big)(e_1) - d\big(\langle \vec{e}_1, d\vec{e}_2(e_1)\rangle\big)(e_2) - \langle \vec{e}_1, d\vec{e}_2([e_1, e_2])\rangle\\
&= \langle d\vec{e}_1(e_1), d\vec{e}_2(e_2)\rangle - \langle d\vec{e}_1(e_2), d\vec{e}_2(e_1)\rangle\\
&\quad \underbrace{+\langle \vec{e}_1, d(d\vec{e}_2(e_2))(e_1)\rangle - \langle \vec{e}_1, d\left(d\vec{e}_2(e_1)\right)(e_2)\rangle - \langle \vec{e}_1, d\vec{e}_2([e_1, e_2])\rangle}_{=\langle \vec{e}_1, \vec{R}(\vec{e}_1, \vec{e}_2, \vec{e}_2)\rangle = 0}\\
&\overset{(*)}{=} \langle D_{e_1}\vec{e}_1, D_{e_2}\vec{e}_2\rangle - \langle D_{e_2}\vec{e}_1, D_{e_1}\vec{e}_2\rangle\\
&= \langle \vec{\mathbb{I}}(e_1, e_1), \vec{\mathbb{I}}(e_2, e_2)\rangle - \left|\vec{\mathbb{I}}(e_1, e_2)\right|^2\\
&= K.
\end{aligned} \tag{2.50}$$

In the last identity we have made use of the Gauss theorem (2.22). By $\vec{R}$ we denote the Riemannian $(3, 1)$-curvature tensor of the exterior differential d, the Levi-Civita connection associated to $(\mathbb{R}^m, \langle\cdot,\cdot\rangle)$.

---

[9]The Cartan formula for the exterior differential of a 1-form α on a differentiable manifolfd M^m says that for any pair of vector fields X, Y on this manifold the following identity holds:

$$d\alpha(X, Y) = d(\alpha(Y))(X) - d(\alpha(X))(Y) - \alpha([X, Y]), \tag{2.49}$$

see Corollary 1.122 chapter I of [GHL04].

In $(*)$ we used that

$$\langle d\vec{e}_1(e_1), d\vec{e}_2(e_2)\rangle = \langle D_{e_1}\vec{e}_1, D_{e_2}\vec{e}_2\rangle + \langle \nabla_{e_1}\vec{e}_1, \nabla_{e_2}\vec{e}_2\rangle,$$

where the second summand vanishes, since $\nabla_{e_1}\vec{e}_1$ ($\nabla_{e_2}\vec{e}_2$ resp.) is oriented along $\vec{e}_2$ ($\vec{e}_1$ resp.):

$$\langle \nabla_{e_i}\vec{e}_i, \vec{e}_i\rangle = e_i(\underbrace{\langle \vec{e}_i, \vec{e}_i\rangle}_{\equiv 1}) - \langle \nabla_{e_i}\vec{e}_i, \vec{e}_i\rangle = 0,$$

for $i = 1, 2$. Similarly, one shows that

$$\langle \nabla_{e_1}\vec{e}_2, \vec{e}_2\rangle = \langle \nabla_{e_2}\vec{e}_1, \vec{e}_1\rangle = 0$$

such that

$$\langle \nabla_{e_2}\vec{e}_1, \nabla_{e_1}\vec{e}_2\rangle = 0.$$

From (2.50), the claimed identity in (2.48) follows since

$$d\langle \vec{e}_1, d\vec{e}_2\rangle = d\langle \vec{e}_1, d\vec{e}_2\rangle(e_1, e_2)\; e_1^* \wedge e_2^*$$

and $e_1^* \wedge e_2^* = dvol_g$.

Combining (2.45) and (2.48) gives

$$-\Delta\lambda = - * d * d\lambda = *d\langle \vec{e}_1, d\vec{e}_2\rangle = e^{2\lambda}K,$$

where Δ denotes the negative flat Laplacian. Recall from Section 2.1 that $*d\lambda$ is the associated 1-form to $\nabla^\perp\lambda$.

The previous identity is the well-known expression of the Gauss curvature in isothermal coordinates in terms of the conformal factor λ, called *Liouville equation.* We have proven the following lemma.

Lemma 2.5. *Let $\vec{\Phi}\colon D^2 \to \mathbb{R}^m$ be a conformal immersion with*

$$|\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}| = e^\lambda.$$

Then
$$-\Delta\lambda = e^{2\lambda}K, \tag{2.51}$$
where K is the Gauss curvature of $(D^2, \vec{\Phi}^* g_{\mathbb{R}^m})$ and Δ is the negative flat Laplacian on D^2.

One easily obtains the following generalization.

Theorem 2.6. *Let $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$ be a smooth immersion and $g := \vec{\Phi}^* g_{\mathbb{R}^m}$ the induced first fundamental form. Let h be a conformally equivalent metric on Σ satisfying*
$$g = e^{2\alpha} h.$$
Then
$$-\Delta_h \alpha = e^{2\alpha} K_g - K_h, \tag{2.52}$$
where K_g and K_h are the Gauss curvatures of (Σ, g) and (Σ, h) respectively.

Proof of Theorem 2.6. In the next subsection we will show that there exist isothermal coordinates for $\vec{\Phi}$, i.e. locally g is of the form
$$g = e^{2\lambda}(dx_1^2 + dx_2^2). \tag{2.53}$$
Hence, we have
$$h = e^{2\sigma}(dx_1^2 + dx_2^2), \tag{2.54}$$
where $\sigma := \lambda - \alpha$.

Then we can apply Lemma 2.5 to (2.53) and (2.54) respectively and obtain
$$K_g = -e^{-2\lambda}\Delta\lambda \tag{2.55}$$
and
$$K_h = -e^{-2(\lambda-\alpha)}\Delta(\lambda - \alpha) \tag{2.56}$$

Combining (2.55) and (2.56) yields

$$K_h = e^{2\alpha}\left(K_g + e^{-2\lambda}\Delta\alpha\right) = e^{2\alpha}K_g + \Delta_h\alpha,$$

where we used (2.41) in the last identity. This finishes the proof. $\square$

Finally, we want to take advantage of the fact that for a conformal immersion $\vec{\Phi}$ from D^2 into $\mathbb{R}^3$, $(\vec{e}_1, \vec{e}_2, \vec{n})$ is an orthonormal frame of $\vec{\Phi}^*(\mathbb{R}^3)$, where $\vec{e}_i := e^{-\lambda}\ \partial_{x_i}\vec{\Phi}$ and $\vec{n}$ is the Gauss map of $\vec{\Phi}$. The following two lemmas dealing with conformal immersions into the 3-dimensional space are easy consequences of this.

Lemma 2.7. *Let $\vec{\Phi}\colon D^2 \to \mathbb{R}^3$ be a conformal immersion. Then*

$$K\ \vec{n} = -\frac{e^{-2\lambda}}{2}\ \nabla\vec{n} \times \nabla^{\perp}\vec{n} = -\frac{e^{-2\lambda}}{2}\ div\,[\vec{n} \times \nabla^{\perp}\vec{n}], \tag{2.57}$$

where K is the Gauss curvature of $\vec{\Phi}$.

Proof. Note that

$$\nabla\vec{n} \times \nabla^{\perp}\vec{n} = div\,[\vec{n} \times \nabla^{\perp}\vec{n}] = -2\ \partial_{x_1}\vec{n} \times \partial_{x_2}\vec{n}.$$

Since

$$\langle \partial_{x_1}\vec{n}, \vec{n}\rangle = \langle \partial_{x_2}\vec{n}, \vec{n}\rangle = 0,$$

for proving (2.57), we have to show that

$$e^{2\lambda}K = |\partial_{x_1}\vec{n} \times \partial_{x_2}\vec{n}|. \tag{2.58}$$

For the Gauss curvature, we can write

$$K := \frac{det\left(\langle \vec{n}, \vec{\mathbb{I}}(\partial_{x_i}, \partial_{x_j})\rangle\right)}{det(g_{ij})} = e^{-2\lambda}\ det\left(-\langle \partial_{x_i}\vec{n}, \vec{e}_j\rangle\right), \tag{2.59}$$

where $\vec{e}_i := e^{-\lambda}\ \partial_{x_i}\vec{\Phi}$. The parallelogram spanned by $\partial_{x_1}\vec{n}$ and $\partial_{x_2}\vec{n}$ is contained in the tangent bundle, of which $(\vec{e}_1, \vec{e}_2)$ is an orthonormal frame, hence we have

$$|\partial_{x_1}\vec{n} \times \partial_{x_2}\vec{n}| = det\left(\langle \partial_{x_i}\vec{n}, \vec{e}_j\rangle\right).$$

This, together with (2.59), implies (2.58) and thus the result. $\square$

Lemma 2.8. *Let $\vec{\Phi}\colon D^2 \to \mathbb{R}^3$ be a conformal immersion. Then the following identity holds:*

$$-2H\ \nabla\vec{\Phi} = \nabla\vec{n} + \vec{n} \times \nabla^{\perp}\vec{n}. \tag{2.60}$$

Proof. Consider the tangent frame $(\vec{e}_1, \vec{e}_2)$, given by

$$\vec{e}_i := e^{-\lambda}\ \partial_{x_i}\vec{\Phi},$$

where $e^{\lambda} = |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$. The oriented Gauss map $\vec{n}$ is given by

$$\vec{n} = \vec{e}_1 \times \vec{e}_2.$$

We have

$$\begin{aligned} \langle \vec{e}_1, \vec{n} \times \nabla^{\perp}\vec{n}\rangle &= -\langle \nabla^{\perp}\vec{n}, \vec{e}_2\rangle, \\ \langle \vec{e}_2, \vec{n} \times \nabla^{\perp}\vec{n}\rangle &= \langle \nabla^{\perp}\vec{n}, \vec{e}_1\rangle. \end{aligned}$$

From this we deduce

$$\left\{ \begin{aligned} -\vec{n} \times \partial_{x_2}\vec{n} &= \langle \partial_{x_2}\vec{n}, \vec{e}_2\rangle\ \vec{e}_1 - \langle \partial_{x_2}\vec{n}, \vec{e}_1\rangle\ \vec{e}_2, \\ \vec{n} \times \partial_{x_1}\vec{n} &= -\langle \partial_{x_1}\vec{n}, \vec{e}_2\rangle\ \vec{e}_1 + \langle \partial_{x_1}\vec{n}, \vec{e}_1\rangle\ \vec{e}_2. \end{aligned} \right.$$

Thus,

$$\left\{ \begin{aligned} \partial_{x_1}\vec{n} - \vec{n} \times \partial_{x_2}\vec{n} &= \left[\langle \partial_{x_2}\vec{n}, \vec{e}_2\rangle + \langle \partial_{x_1}\vec{n}, \vec{e}_1\rangle\right]\ \vec{e}_1, \\ \partial_{x_2}\vec{n} + \vec{n} \times \partial_{x_1}\vec{n} &= \left[\langle \partial_{x_2}\vec{n}, \vec{e}_2\rangle + \langle \partial_{x_1}\vec{n}, \vec{e}_1\rangle\right]\ \vec{e}_2. \end{aligned} \right.$$

Since
$$H = -\frac{1}{2} e^{-\lambda} \left[\langle \partial_{x_1} \vec{n}, \vec{e}_1 \rangle + \langle \partial_{x_2} \vec{n}, \vec{e}_2 \rangle\right],$$
we deduce (2.60) and Lemma 2.8 is proven. $\square$

2.5 Existence of isothermal coordinates and the Chern moving frame method

In this subsection we want to prove the existence of isothermal coordinates for an immersion $\vec{\Phi}$ of Σ into $\mathbb{R}^m$. More precisely, we show that for any $p \in \Sigma$ there exists a neighborhood U of p and a diffeomorphism $\psi\colon D^2 \to U$ such that ψ is an isothermal chart for $\vec{\Phi}$, i.e. satisfying (2.36).

By picking any chart $\varphi\colon D^2 \to U$ around p, it suffices to prove the following theorem, which can then be applied to $\vec{\Phi} \circ \varphi$.

Theorem 2.9. *Let $\vec{\Phi}$ be an immersion of the disc D^2 into $\mathbb{R}^m$. Then there exists a diffeomorphism ψ of D^2 such that the map $\vec{\Phi} \circ \psi\colon D^2 \to \mathbb{R}^m$ is conformal, i.e. satisfies* (2.36).

To obtain this result we will make use of the Riemann mapping theorem (see for instance [Rud87], chapter 14), which we want to recall now.

Theorem 2.10. *If $U \subsetneq \mathbb{C}$ is non-empty, open and simply connected, then there exists a biholomorphic mapping f from U onto the open unit disk $D^2 = \{z \in \mathbb{C} : |z| < 1\}$.*

In the last subsection we have seen how any isothermal chart generates a Coulomb frame in the tangent bundle. S.S. Chern observed that this is in fact an exact matching, the converse being also true: starting from a Coulomb frame one can generate isothermal coordinates. This is our strategy for the proof of Theorem 2.9.

Proof of Theorem 2.9. Let $\vec{\Phi}\colon D^2 \to \mathbb{R}^m$ be an immersion. We are looking for a *Coulomb tangent orthonormal moving frame* $(\vec{e}_1, \vec{e}_2)$: a tangent moving frame from D^2 to $\vec{\Phi}_*(TD^2 \times TD^2)$ such that the Coulomb condition (2.47) is satisfied.

Note that the passage from an arbitrary tangent orthonormal moving frame $(\vec{f}_1, \vec{f}_2)$ to another one $(\vec{e}_1, \vec{e}_2)$ is realized through a change of gauge which corresponds to the action of an $SO(2)$ rotation $e^{i\theta}$ on the tangent space $\vec{\Phi}_*(T_{(x_1,x_2)}D^2)$:

$$\vec{e}_1 + i\vec{e}_2 = e^{i\theta}\ (\vec{f}_1 + i\vec{f}_2). \tag{2.61}$$

One shows that the following gauge change formula holds:

$$\langle \vec{e}_1, d\vec{e}_2\rangle = \langle \vec{f}_1, d\vec{f}_2\rangle + d\theta. \tag{2.62}$$

Thus, construcing a Coulomb tangent orthonormal moving frame is the same as finding the change of gauge $\theta \in W^{1,2}(D^2)$ which minimizes the energy

$$\int_{D^2} \left| d\theta + \langle \vec{f}_1, d\vec{f}_2\rangle \right|_g^2\ dvol_g. \tag{2.63}$$

In fact, there exists a unique minimum satisfying

$$\left\{\begin{array}{rcll} d^{*_g}\left[d\theta + \langle \vec{f}_1, d\vec{f}_2\rangle\right] & = & 0 & \text{in } D^2 \\[2mm] \iota^*_{\partial D^2}\left(*_g\left[d\theta + \langle \vec{f}_1, d\vec{f}_2\rangle\right]\right) & = & 0 & \text{on } \partial D^2, \end{array}\right.$$

where $\iota_{\partial D^2}$ is the canonical inclusion of ∂D^2 into $\overline{D^2}$. Denote by $(\vec{e}_1, \vec{e}_2)$ the corresponding Coulomb frame, i.e. the frame obtained by $\vec{e}_1 + i\vec{e}_2 = e^{i\theta}(\vec{f}_1 + i\vec{f}_2)$.

By Poincaré's Lemma there exists λ satisfying

$$\left\{\begin{array}{rcl} d\lambda & = & *_g\langle \vec{e}_1, d\vec{e}_2\rangle \\[2mm] \displaystyle\int_{\partial D^2} \lambda & = & 0. \end{array}\right. \tag{2.64}$$

Denote moreover $e_i := d\vec{\Phi}^{-1}(\vec{e}_i)$ and let (e_1^*, e_2^*) be the dual basis to (e_1, e_2). The Cartan formula for the exterior differential of a 1-form implies

$$\begin{aligned} de_i^*(e_1, e_2) &= d(e_i^*(e_2))(e_1) - d(e_i^*(e_1))(e_2) - e_i^*([e_1, e_2]) \\ &= -e_i^*([e_1, e_2]) \\ &= -((\vec{\Phi}^{-1})^* e_i^*)([d\vec{\Phi}(e_1), d\vec{\Phi}(e_2)]) \\ &= -((\vec{\Phi}^{-1})^* e_i^*)([\vec{e}_1, \vec{e}_2]). \end{aligned} \tag{2.65}$$

The Levi-Civita connection ∇ on $\vec{\Phi}_* TD^2$ is given by $\nabla_X \vec{Y} := \pi_T(d\vec{Y}(X))$ where π_T is the orthogonal projection onto the tangent plane. Since the Levi-Civita connection is torsion-free, we have

$$[\vec{e}_1, \vec{e}_2] = \nabla_{e_1} \vec{e}_2 - \nabla_{e_2} \vec{e}_1 = \pi_T(d\vec{e}_2(e_1) - d\vec{e}_1(e_2)). \tag{2.66}$$

Since $\vec{e}_1$ and $\vec{e}_2$ have unit length, the tangential projection of $d\vec{e}_1$ (resp. $d\vec{e}_2$) are oriented along $\vec{e}_2$ (resp. $\vec{e}_1$). So we have

$$\left\{ \begin{array}{l} \pi_T(d\vec{e}_2(e_1)) = \langle d\vec{e}_2, \vec{e}_1 \rangle (e_1)\ \vec{e}_1 \\[2mm] \pi_T(d\vec{e}_1(e_2)) = \langle d\vec{e}_1, \vec{e}_2 \rangle (e_2)\ \vec{e}_2. \end{array} \right. \tag{2.67}$$

Combining (2.65), (2.66) and (2.67) gives then

$$\left\{ \begin{array}{l} de_1^*(e_1, e_2) = -\langle d\vec{e}_2, \vec{e}_1 \rangle (e_1) \\[2mm] de_2^*(e_1, e_2) = \langle d\vec{e}_1, \vec{e}_2 \rangle (e_2). \end{array} \right. \tag{2.68}$$

Equation (2.64) gives

$$\left\{ \begin{array}{l} -\langle d\vec{e}_2, \vec{e}_1 \rangle (e_1) = *_g d\lambda(e_1) = -d\lambda(e_2) \\[2mm] \langle d\vec{e}_1, \vec{e}_2 \rangle (e_2) = *_g d\lambda(e_2) = d\lambda(e_1). \end{array} \right. \tag{2.69}$$

Thus combining (2.68) and (2.69) yields

$$\begin{cases} de_1^* = -d\lambda(e_2)\ e_1^* \wedge e_2^* = d\lambda \wedge e_1^* \\ \\ de_2^* = d\lambda(e_1)\ e_1^* \wedge e_2^* = d\lambda \wedge e_2^*. \end{cases} \tag{2.70}$$

We have thus finally proven that

$$\begin{cases} d\left(e^{-\lambda} e_1^*\right) = 0 \\ \\ d\left(e^{-\lambda} e_2^*\right) = 0. \end{cases} \tag{2.71}$$

Apply Poincaré's Lemma and obtain functions ϕ_1, ϕ_2 with average 0 on the disc D^2 such that

$$d\phi_i := e^{-\lambda} e_i^*. \tag{2.72}$$

Note that (2.72) implies that ϕ is a local homeomorphism. We can thus make the following local calculations envolving ϕ^{-1}: From (2.72) we have

$$\partial_{y_i} \phi^{-1} = e^{\lambda \circ \phi^{-1}} e_i$$

and consequently,

$$e^{-\lambda \circ \phi^{-1}}\ g(e_j, \partial_{y_i} \phi^{-1}) = e^{-\lambda \circ \phi^{-1}}\ e_j^*(\partial_{y_i} \phi^{-1}) = \delta_{ij}.$$

This implies

$$g(\partial_{y_i} \phi^{-1}, \partial_{y_j} \phi^{-1}) = e^{2\lambda \circ \phi^{-1}} \delta_{ij} \tag{2.73}$$

or in other words

$$\langle \partial_{y_i}(\vec{\Phi} \circ \phi^{-1}), \partial_{y_j}(\vec{\Phi} \circ \phi^{-1}) \rangle = e^{2\lambda \circ \phi^{-1}}\ \delta_{ij}. \tag{2.74}$$

This says in particular that there exists an open disc D^2_ε around 0 on which ϕ is a homeomorphism (and thus a diffeomorphism) and such that $\vec{\Phi} \circ \phi^{-1} \colon \phi(D^2_\varepsilon) \to \mathbb{R}^m$ is a conformal immersion. The Riemann Mapping theorem 2.10 gives the existence of a biholomorphic diffeomorphism h from D^2 into $\phi(D^2_\varepsilon)$. Thus $\vec{\Phi} \circ \phi^{-1} \circ h$ realizes a conformal immersion from D^2 onto $\vec{\Phi}(D^2_\varepsilon)$ and $\tilde{\psi} := \phi^{-1} \circ h$ is an isothermal chart.

We have been looking for a diffeomorphism ψ from D^2 to D^2 (not into D^2_ε), though, such that $\vec{\Phi} \circ \psi$ is conformal. To construct such a ψ, we will replace ϕ by its associated quasiconformal homeomorphism f, introduced in Section 2.6: A computation shows that (2.72) implies

$$\langle \partial_{x_i} \phi, \partial_{x_j} \phi \rangle = 2e^{-2\lambda} g_{ij} =: h_{ij}. \tag{2.75}$$

Note that we have

$$c^{-1}(\delta_{ij}) \leq (h_{ij}) \leq c\,(\delta_{ij}). \tag{2.76}$$

We can thus apply Corollary 2.16 to ϕ and deduce that there exists a (quasiconformal) map $f \colon D^2 \to \mathbb{C}$, homeomorphic onto its image, such that $g := \phi \circ f^{-1} \colon f(D^2) \to \phi(D^2)$ is a holomorphic function. Since ϕ is a local homeomorphism and f is a homeomorphism, we have

$$\partial_z g \neq 0 \qquad \text{on } f(D^2).$$

In particular g is conformal with

$$|\partial_{y_1} g| = |\partial_{y_2} g| = \frac{1}{\sqrt{2}} |\partial_z g|.$$

Note that the following computation is again a local one, which is why we can use (2.74).

$$\langle \partial_{y_i} (\vec{\Phi} \circ f^{-1}), \partial_{y_j} (\vec{\Phi} \circ f^{-1}) \rangle$$

$$= \langle d(\vec{\Phi} \circ \phi^{-1})(\partial_{y_i} g), d(\vec{\Phi} \circ \phi^{-1})(\partial_{y_j} g) \rangle$$

$$= \frac{1}{2} |\partial_z g|^2 \langle \partial_{y_i}(\vec{\Phi} \circ \phi^{-1}), \partial_{y_j}(\vec{\Phi} \circ \phi^{-1}) \rangle$$

$$= \frac{1}{2} |\partial_z g|^2 e^{2\lambda \circ \phi^{-1}} \delta_{ij}.$$

Applying the Riemann Mapping theorem 2.10 gives the existence of a biholomorphic diffeomorphism h from D^2 into $f(D^2)$. Thus $\vec{\Phi} \circ f^{-1} \circ h$ realizes a conformal immersion from D^2 onto $\vec{\Phi}(D^2)$ and $\psi := f^{-1} \circ h$ is the isothermal chart we have been looking for. $\square$

2.6 Some facts on Riemann surfaces

This section collects some useful facts on Riemann surfaces that we shall need during the course. We follow [Jos97] and [IT92]. In the sequel, a *surface* denotes a 2-dimensional smooth manifold.

Definition 2.11. *Let Σ be a surface. An atlas on Σ with charts $\varphi_i \colon D^2 \to U_i \subset S$ is called* conformal *if the transition maps*

$$\varphi_j^{-1} \circ \varphi_i \colon \ \varphi_i^{-1}(U_i \cap U_j) \subset \mathbb{C} \to \mathbb{C}$$

are biholomorphic. A maximal conformal atlas is called a conformal structure *on S.*

A Riemann surface *is a surface together with a conformal structure.*

Recall Example i) in Section 2.3 and note that Definition 2.11 provides that the transition maps of a conformal atlas are conformal and preserve orientation. In particular, a Riemann surface is oriented.

Remark 2.12. *In other words, a Riemann surface is a complex manifold of real dimension* 2*. Note that any holomorphic atlas of a complex manifold M induces an* almost complex structure *J on M, that is a section in* $\mathrm{End}(TM)$ *satisfying $J^2 = -1$, via*

$$J\partial_{x_i} = \partial_{y_i} \quad \text{and} \quad J\partial_{y_i} = -\partial_{x_i}. \tag{2.77}$$

Conversely, an almost complex structure J on an even-real-dimensional manifold M is not necessarily integrable*, i.e. the differentiable structure of M cannot necessarily be defined by a holomorphic atlas satisfying* (2.77)*. However, this is true in real dimension* 2*, such that any almost complex manifold of real dimension* 2 *is in fact a Riemann surface.*

Riemannian metrics and Beltrami coefficients.

Theorem 2.13. *Let (Σ, g) be an oriented surface with a Riemannian metric. Then (Σ, g) admits a conformal structure.*

For proving Theorem 2.13, the goal is to find for each coordinate neighborhood $(U_j, (x_j, y_j))$ from an atlas $\{(U_j, (x_j, y_j))\}_j$ isothermal coordinates (u_j, v_j) for the Riemannian metric g such that it is represented as

$$\rho_j \left(du_j^2 + dv_j^2\right), \tag{2.78}$$

for ρ_j being some positive smooth function. Setting $w_j = u_j + iv_j$, the atlas $\{(U_j, w_j)\}_j$ defines then a conformal structure on Σ. Note that the Chern moving frame method, as used in the proof of Theorem 2.9, is one way to find isothermal coordinates.

We want to give the rough idea of an alternative proof for finding isothermal coordinates, which is related to *Beltrami coefficients*. If the Riemannian metric g is in local coordinates

$(U,(x,y))$ given by

$$g_{11}\,dx^2 + 2g_{12}\,dxdy + g_{22}\,dy^2,$$

setting $z = x + iy$ yields the representation

$$\sigma|dz + \mu\, d\bar{z}|^2, \tag{2.79}$$

where

$$\sigma = \frac{1}{4}\left(g_{11} + g_{22} + 2\sqrt{g_{11}g_{22} - g_{12}^2}\right)$$

is a positive smooth function on U and

$$\mu = \frac{g_{11} - g_{22} + 2ig_{12}}{g_{11} + g_{22} + 2\sqrt{g_{11}g_{22} - g_{12}^2}} \tag{2.80}$$

is a complex-valued smooth function with

$$\|\mu\|_{L^\infty(U)} < 1.$$

To find an isothermal coordinate $w = u + iv$, note that such satisfies

$$\rho\left(du^2 + dv^2\right) = \rho\,|dw|^2 = \rho\,|w_z|^2\left|dz + \frac{w_{\bar{z}}}{w_z}\,d\bar{z}\right|^2, \tag{2.81}$$

using the notation from (2.78), i.e. $\rho > 0$ being a smooth function. Comparing (2.79) and (2.81), we deduce that an isothermal coordinate is an (a.e.) diffeomorphic solution w to the *Beltrami differential equation*

$$w_{\bar{z}} = \mu\, w_z, \tag{2.82}$$

where μ with $\|\mu\|_{L^\infty(U)} < 1$ is given by (2.80) and is called the *Beltrami coefficient induced by the Riemannian metric* g. The following theorem gives that (2.82) has a quasiconformal solution for any Beltrami coefficient.

Definition 2.14. *A quasiconformal map $f\colon \mathbb{C} \to \mathbb{C}$ is an orientation-preserving homeomorphism of $\mathbb{C}$*

i) such that f_z and $f_{\bar z}$ are in $L^1_{\mathrm{loc}}(\mathbb{C})$;

ii) for which there exists a constant $0 \le k < 1$ such that

$$|f_{\bar z}| \le k|f_z| \qquad a.e.\ on\ \mathbb{C}.$$

Theorem 2.15 (Existence of quasiconformal mappings with complex dilation μ, [IT92], Theorem 4.30)**.** *Let $\mu \in L^\infty(\mathbb{C},\mathbb{C})$ satisfy*

$$\|\mu\|_{L^\infty(\mathbb{C})} < 1.$$

Then there exists a homeomorphism $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which is a quasiconformal mapping of $\mathbb{C}$ with complex dilation μ, i.e.

$$f_{\bar z} = \mu\, f_z.$$

f is uniquely determined by the normalization conditions

$$f(0) = 0, \quad f(1) = 1, \quad f(\infty) = f(\infty).$$

We are now in the position to fill the gap we left in the proof of Theorem 2.9, using the existence of quasiconformal mappings for arbitrary Beltrami coefficients. We first make the following observation.

Given $\phi \in W^{1,\infty}(D^2,\mathbb{C})$, define

$$h_{ij} := \langle \partial_{x_i}\phi, \partial_{x_j}\phi\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical inner product in $\mathbb{R}^2$. If the symmetric 2×2-matrix h satisfies

$$c^{-1}(\delta_{ij}) \le (h_{ij}) \le c\,(\delta_{ij}), \tag{2.83}$$

$h = |d\phi|^2$ defines a metric. From (2.79) and (2.80) we obtain

$$|d\phi|^2 = \sigma |dz + \mu d\bar{z}|^2, \tag{2.84}$$

where

$$\frac{\partial_{\bar{z}}\phi}{\partial_z \phi} = \mu = \frac{h_{11} - h_{22} + 2ih_{12}}{h_{11} + h_{22} + 2\sqrt{h_{11}h_{22} - h_{12}^2}}. \tag{2.85}$$

Recall that μ is a Beltrami coefficient with $\|\mu\|_{L^\infty(D^2)} < 1$.

Corollary 2.16. *Let $\phi \in W^{1,\infty}(D^2, \mathbb{C})$ such that*

$$h_{ij} := \langle \partial_{x_i}\phi, \partial_{x_j}\phi \rangle$$

satisfies

$$c^{-1}(\delta_{ij}) \leq (h_{ij}) \leq c\,(\delta_{ij}).$$

Let μ be the associated Beltrami coefficient, given by (2.85) and f be the quasiconformal mapping with complex dilation μ (extended by 0 outside D^2), given by Theorem 2.15.

Then $\phi \circ f^{-1} \colon f(D^2) \to \phi(D^2)$ is holomorphic.

Proof. In [IT92], Proposition 4.13, it is computed that the complex dilation $\mu_{f^{-1}}$ is of the form

$$\mu_{f^{-1}} = -\left(\frac{f_z}{\overline{f_z}}\,\mu_f\right) \circ f^{-1}.$$

Consequently,

$$\begin{aligned}
\partial_{\bar{z}}(\phi \circ f^{-1}) &= \partial_w \phi \circ f^{-1} \cdot (f^{-1})_{\bar{z}} + \partial_{\bar{w}}\phi \circ f^{-1} \cdot \overline{(f^{-1})_z} \\
&= \partial_w \phi \circ f^{-1} \cdot \mu_{f^{-1}}\,(f^{-1})_z + (\mu_f\,\partial_w\phi) \circ f^{-1} \cdot \overline{(f^{-1})_z} \\
&= (\mu_f\,\partial_w\phi) \circ f^{-1} \left(-\frac{f_z}{\overline{f_z}} \circ f^{-1}\,(f^{-1})_z + \overline{(f^{-1})_z}\right) \\
&= 0
\end{aligned}$$

and thus $\phi \circ f^{-1} \colon f(D^2) \to \mathbb{C}$ is holomorphic. $\square$

The classification of Riemann surfaces.

Theorem 2.17 (Uniformization Theorem)**.** *Let Σ be a compact Riemann surface of genus p. Then there exists a conformal diffeomorphism*

$$\Psi \colon \Sigma \to \Sigma',$$

where Σ' is

i) a compact Riemann surface of the form $\mathbb{H}/\Gamma$, where Γ is a subgroup of $PSL(2,\mathbb{R})$ acting freely and properly discontinuously on $\mathbb{H}$, if $p \geq 2$;

ii) a compact Riemann surface $\mathbb{C}/a\mathbb{Z} + b\mathbb{Z}$, where $a, b \in \mathbb{C}$, if $p = 1$;

iii) the Riemann sphere S^2, if $p = 0$.

A corollary of Theorem 2.17 is the following.

Theorem 2.18. *Let Σ be a compact Riemann surface, where the conformal structure is induced by the Riemannian metric g. Then there exists a conformally equivalent Riemannian metric h on Σ,*

$$g = e^{2\alpha} h,$$

with

- *constant (sectional) curvature K_h and*
- *unit volume: $vol_h(\Sigma) = 1$.*

Such h is unique if the genus of the surface is larger than one.

Moreover, we have

$$K_h \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} \quad \textit{if and only if } genus(\Sigma) \begin{cases} \geq 2 \\ = 1 \\ = 0 \end{cases} .$$

Weak immersions of surfaces with L^2-bounded second fundamental form

Lecture 2

PCMI Graduate Summer School 2013

Tristan Rivière

Lecture Notes written by Nora Loose

3 The space of weak immersion with L^2-bounded second fundamental form

3.1 Definition

Let Σ be a smooth 2-dimensional closed oriented manifold. Let g_0 be a smooth reference metric on Σ.

One defines the Sobolev spaces $W^{k,p}(\Sigma,\mathbb{R}^m)$ of measurable maps from Σ into $\mathbb{R}^m$ in the following way:

$$W^{k,p}(\Sigma,\mathbb{R}^m) = \left\{ f\colon \Sigma \to \mathbb{R}^m \ ; \ \sum_{l=0}^{k} \int_\Sigma |\nabla^l f|_{g_0}^p \, dvol_{g_0} < +\infty \right\}.$$

Since Σ is assumed to be closed, this space is independent of the choice of g_0: For any two smooth reference metrics g_0 and g_1, one can find a constant $c > 0$ such that for all $p \in \Sigma$ and $X \in T_pM$

$$c^{-1} g_0(X,X) \le g_1(X,X) \le c\, g_0(X,X). \tag{3.1}$$

We now want to define a weak notion of immersions $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$. We want

$$g_{\vec{\Phi}}(X,Y) = \langle d\vec{\Phi}(X), d\vec{\Phi}(Y)\rangle$$

to define an L^∞-metric on Σ.

Therefore, we require $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$ to be a *Lipschitz immersion*, i.e. a map satisfying

i)
$$\vec{\Phi} \in W^{1,\infty}(\Sigma,\mathbb{R}^m),$$

ii) $\exists c > 0$ s.t. $\forall p \in \Sigma,\ X \in T_pM$
$$c^{-1} g_0(X,X) \le g_{\vec{\Phi}}(X,X) \le c\, g_0(X,X). \tag{3.2}$$

Condition ii) ensures the non-degeneracy of the metric $g_{\vec{\Phi}}$. Note that this condition is again independent of the choice of g_0, due to (3.1).

For a Lipschitz immersion we can define the Gauss map as the following measurable map in $L^\infty(\Sigma)$, taking values in the Grassmannian $Gr_{m-2}(\mathbb{R}^m)$ of oriented $(m-2)$-planes in $\mathbb{R}^m$:

$$\vec{n}_{\vec{\Phi}} := \star \frac{\partial_{x_1}\vec{\Phi} \wedge \partial_{x_2}\vec{\Phi}}{|\partial_{x_1}\vec{\Phi} \wedge \partial_{x_2}\vec{\Phi}|}$$

for an arbitrary choice of local positive coordinates (x_1, x_2).

We say that a Lipschitz immersion $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$ has L^2-*bounded second fundamental form* if

iii)

$$\vec{n} \in W^{1,2}(\Sigma, Gr_{m-2}(\mathbb{R}^m))^{10}.$$

In other words, condition iii) requires

$$\int_\Sigma |d\vec{n}|^2_{g_0}\ dvol_{g_0} < +\infty.$$

Summing up, we have the following definition.

Definition 3.1. *The space $\mathcal{E}_\Sigma$ of* Lipschitz immersions with L^2-bounded second fundamental form *is defined as*

$$\mathcal{E}_\Sigma := \left\{ \vec{\Phi}\colon \Sigma \to \mathbb{R}^m \textit{ measurable s.t. i), ii) and iii) hold} \right\}.$$

[10]The Grassman manifold $Gr_{m-2}(\mathbb{R}^m)$ can be seen as being the submanifold of the Euclidian space $\wedge^{m-2}\mathbb{R}^m$ of $(m-2)$-vectors in $\mathbb{R}^m$ made of unit simple $(m-2)$-vectors. Then one defines

$$W^{1,2}(D^2, Gr_{m-2}(\mathbb{R}^m)) := \left\{ \vec{n} \in W^{1,2}(D^2, \wedge^{m-2}\mathbb{R}^m)\ ;\ \vec{n} \in Gr_{m-2}(\mathbb{R}^m) \text{ a.e.} \right\}.$$

Definition 3.2. *For $\vec{\Phi} \in \mathcal{E}_\Sigma$, denote by*

$$\mathbb{I}(\vec{\Phi}) := \int_\Sigma |\vec{\mathbb{I}}|^2_{g_{\vec{\Phi}}} \; dvol_{g_{\vec{\Phi}}} = \int_\Sigma |d\vec{n}|^2_{g_{\vec{\Phi}}} \; dvol_{g_{\vec{\Phi}}} \tag{3.3}$$

the L^2-norm of the second fundamental form of $\vec{\Phi}$.

Note that the identity in (3.3) holds by (2.29). Thus, $\mathbb{I}(\vec{\Phi})$ can also be interpreted as the Dirichlet energy of the Gauss map $\vec{n}_{\vec{\Phi}}$. Moreover, $\mathbb{I}(\vec{\Phi})$ is finite due to condition iii) in the definition of $\mathcal{E}_\Sigma$.

3.2 Fundamental results on integrability by compensation

3.2.1 Classical results from Calderon-Zygmund theory

Recall from classical Calderon-Zygmund theory the following theorem.

Theorem 3.3. *Let $f \in L^1(D^2)$. Then there exists a unique solution in $W^{1,1}_0(D^2)$ of*

$$\begin{cases} \Delta\phi &= f \quad \text{in } D^2 \\ \phi &= 0 \quad \text{on } \partial D^2 \end{cases} \tag{3.4}$$

and $\phi \in W^{1,p}_0(D^2)$ for all $p < 2$, with

$$\|\nabla\phi\|_{L^p(D^2)} \leq C_p \; \|f\|_{L^1(D^2)}.$$

One might ask to which optimal space beside $\bigcap_{p<2} L^p(D^2)$ the function $\nabla\phi$ belongs to. It certainly does not belong to $L^2(D^2)$: if this was true in general, this would also hold for the Radon measure $f = \Delta \log r$ on D^2. But

$$\nabla \log r = \frac{1}{r}\partial_r \notin L^2(D^2).$$

Observe, however, that $\frac{1}{r}$ is an element of the *weak L^2-space* $L^{2,\infty}(D^2)$ of measurable functions $f\colon D^2 \to \mathbb{R}$ such that

$$|f|_{L^{2,\infty}(D^2)} = \sup_{\lambda>0} \left\{ \lambda \left| \{x \in D^2;\ |f(x)| > \lambda\} \right|^{1/2} \right\} < \infty.$$

The space $(L^{2,\infty}, |\cdot|_{L^{2,\infty}})$ is a quasi-Banach space, equivalent to a Banach space. This means that the quasi-norm $|\cdot|_{L^{2,\infty}}$ is equivalent to a norm $\|\cdot\|_{L^{2,\infty}}$. We have the continuous embedding $L^2 \subset L^{2,\infty}$.[11] Furthermore, it can be checked that, since D^2 is a bounded domain, for all $p < 2$,

$$L^p(D^2) \subset L^{2,\infty}(D^2) \tag{3.5}$$

continuously.

One shows that $(L^{2,\infty}, \|\cdot\|_{L^{2,\infty}})$ is the dual space to a space named *Lorentz space* $L^{2,1}$, which is made of measurable functions f such that

$$\int_0^\infty |\{x;\ |f(x)| > \lambda\}|^{1/2}\ d\lambda < \infty.$$

See [Gra08] for further details and more material on weak L^p and Lorentz spaces.

We have the following result.

Theorem 3.4. *Let $f \in L^1(D^2)$. Then for the unique solution ϕ in $W_0^{1,1}(D^2)$ of*

$$\begin{cases} \Delta\phi &= f \quad \text{in } D^2 \\ \phi &= 0 \quad \text{on } \partial D^2 \end{cases}$$

[11] More generally, for $p < \infty$ and $\Omega \subset \mathbb{R}^n$ open, one defines $L^{p,\infty}(\Omega)$ to be the space of measurable functions $f\colon \Omega \to \mathbb{R}$ such that

$$|f|_{L^{p,\infty}(\Omega)} := \sup_{\lambda>0} \left\{ \lambda\, |\{x \in \Omega;\ |f(x)| > \lambda\}|^{1/p} \right\} < \infty.$$

This space defines a quasi-Banach space for the quasi-norm $|\cdot|_{L^{p,\infty}(\Omega)}$ and for $1 < p < \infty$ this quasi norm is equivalent to a norm. $L^p(\Omega)$ embeds continuously in $L^{p,\infty}(\Omega)$.

we have $\nabla\phi \in L^{2,\infty}(D^2)$ *and*

$$\|\nabla\phi\|_{L^{2,\infty}(D^2)} \leq C\ \|f\|_{L^1(D^2)}.$$

Sketch of proof of Theorem 3.4. One checks that the gradient of the Green's function for D^2 ([GT01], Chapter 2.5) satisfies

$$\sup_y \||\nabla_x K|(x,y)\|_{L^{2,\infty}_x(D^2)} < \infty.$$

Since

$$\nabla\phi(x) = \int_{D^2} \nabla_x K(x,y) f(y) dy, \tag{3.6}$$

we have

$$\begin{aligned} \|\nabla\phi\|_{L^{2,\infty}(D^2)} &\leq \left\| \int_{D^2} |\nabla_x K(x,y)||f(y)|dy \right\|_{L^{2,\infty}(D^2)} \\ &\leq \int_{D^2} \|\nabla_x K(x,y)\|_{L^{2,\infty}_x(D^2)} |f(y)|dy \qquad (3.7) \\ &\leq \sup_y \|\nabla_x K\|_{L^{2,\infty}_x(D^2)}\ \|f\|_{L^1(D^2)}, \end{aligned}$$

which is the result.

From this proof one also derives immediately the following:

Theorem 3.5. *Let* $X = (X_1, X_2)$ *be a vector field in* $L^1(D^2)$*. Then for the solution of*

$$\begin{cases} \Delta\phi &= divX \quad in\ D^2 \\ \phi &= 0 \qquad\quad on\ \partial D^2, \end{cases}$$

we have $\phi \in L^{2,\infty}(D^2)$ *with*

$$\|\phi\|_{L^{2,\infty}(D^2)} \leq C\ \|X\|_{L^1(D^2)}.$$

Indeed, we have

$$\phi(x) = \int_{D^2} K(x,y)\ divX\ dy$$

$$= -\int_{D^2} \partial_{x_1} K(x,y)\ X_1 + \partial_{x_2} K(x,y)\ X_2\ dy$$

and the result follows as in (3.7).

Furthermore, one can proceed in a similar manner as in the proof of Theorem 3.4 and use estimates of the gradient of the Green's function K_Σ on a closed surface Σ ([Aub82], Chapter 4) satisfying

$$\Delta_x K_\Sigma = \delta_{x=y} - \frac{1}{vol_{g_0}(\Sigma)},$$

to obtain the analog of Theorem 3.4 for a closed, oriented and connected surface Σ.

Theorem 3.6. *Let Σ be a closed, oriented and connected surface. Let $f \in L^1(\Sigma)$ such that*

$$\int_\Sigma f\ dvol_{g_0} = 0.$$

Then there is a unique solution ϕ in $W^{1,1}(\Sigma)$ of

$$\Delta\phi = f \qquad in\ \mathcal{D}'(\Sigma)$$

and it satisfies

$$\|\nabla\phi\|_{L^{2,\infty}_{g_0}(\Sigma)} \le C_{g_0} \|f\|_{L^1_{g_0}(\Sigma)}.$$

One might also ask about the regularity of ϕ solving (3.4). Combining Theorem 3.3 and the Sobolev embedding

$$W_0^{1,p}(D^2) \hookrightarrow L^{p^*}(D^2), \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{2},$$

one deduces that

$$\phi \in \bigcap_{q<\infty} L^q(D^2).$$

This is not fully optimal: Using the embedding

$$\dot{W}^{1,(2,\infty)}(\mathbb{R}^2) \hookrightarrow \mathrm{BMO}(\mathbb{R}^2),$$

where $\mathrm{BMO}(\mathbb{R}^2)$ is the John Nierenberg space of functions of *Bounded Mean Oscillation*, one obtains that the extension of ϕ by 0 is BMO. Note that

$$\mathrm{BMO}(\mathbb{R}^2) \supsetneq L^\infty(\mathbb{R}^2),$$

where for instance $\log|x| \in \mathrm{BMO}(\mathbb{R}^2) \setminus L^\infty(\mathbb{R}^2)$.

3.2.2 Wente's integrability by compensation

In a famous work Henry Wente produced CMC tori immersed in $\mathbb{R}^3$ and discovered some "improvement" in the integrablility of ϕ, the solution to (3.4), as well as the one of $\nabla\phi$, when f is a Jacobian of $W^{1,2}$-functions.

Theorem 3.7. *[Wen69] Let a and b be two functions in $W^{1,2}(D^2)$, and let ϕ be the unique solution in $W_0^{1,p}(D^2)$ - for $1 \leq p < 2$ - of the equation*

$$\left\{\begin{array}{rcll} -\Delta\phi & = & \partial_{x_1} a\, \partial_{x_2} b - \partial_{x_1} b\, \partial_{x_2} a & \text{in } D^2 \\ \phi & = & 0 & \text{on } \partial D^2. \end{array}\right. \tag{3.8}$$

Then ϕ belongs to $C^0 \cap W^{1,2}(D^2)$ and

$$\|\phi\|_{L^\infty(D^2)} + \|\nabla\phi\|_{L^2(D^2)} \leq C\ \|\nabla a\|_{L^2(D^2)}\ \|\nabla b\|_{L^2(D^2)}. \tag{3.9}$$

where C is a constant independent of a and b.

Proof. We shall first assume that a and b are smooth, so as to justify the various manipulations which we will need to perform. The conclusion of the theorem for general a and b in $W^{1,2}(D^2)$ may then be reached through a simple density argument. In this fashion, we will obtain the continuity of ϕ being the uniform limit of smooth functions.

Observe first that integration by parts and a simple application of the Cauchy-Schwarz inequality yield the estimate

$$\begin{aligned}\int_{D^2} |\nabla\phi|^2 = -\int_{D^2} \phi\,\Delta\phi \ &\leq \|\phi\|_\infty \ \|\partial_x a\,\partial_y b - \partial_x b\,\partial_y a\|_1 \\ &\leq 2\,\|\phi\|_\infty \|\nabla a\|_2 \ \|\nabla b\|_2.\end{aligned}$$

Accordingly, if ϕ lies in $L^\infty(D^2)$, then it automatically lies in $W^{1,2}(D^2)$.

Step 1. Given two functions $\tilde{a}$ and $\tilde{b}$ in $C_0^\infty(\mathbb{C})$, which is dense in $W^{1,2}(\mathbb{C})$, we first establish the estimate (3.9) for

$$\tilde{\phi} := \frac{1}{2\pi}\log\frac{1}{r} * \left[\partial_x\tilde{a}\,\partial_y\tilde{b} - \partial_x\tilde{b}\,\partial_y\tilde{a}\right]. \tag{3.10}$$

Owing to the translation-invariance, it suffices to show that

$$|\tilde{\phi}(0)| \leq C_0\ \|\nabla\tilde{a}\|_{L^2(\mathbb{C})}\ \|\nabla\tilde{b}\|_{L^2(\mathbb{C})}. \tag{3.11}$$

We have

$$\begin{aligned}\tilde{\phi}(0) &= -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log r\ \partial_x\tilde{a}\,\partial_y\tilde{b} - \partial_x\tilde{b}\,\partial_y\tilde{a} \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty}\log\, r\ \frac{\partial}{\partial r}\left(\tilde{a}\ \frac{\partial\tilde{b}}{\partial\theta}\right) - \frac{\partial}{\partial\theta}\left(\tilde{a}\ \frac{\partial\tilde{b}}{\partial r}\right)\ dr\ d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty}\tilde{a}\ \frac{\partial\tilde{b}}{\partial\theta}\ \frac{dr}{r}\ d\theta.\end{aligned}$$

Define $\overline{\tilde{a}}_r = \frac{1}{2\pi}\int_0^{2\pi} \tilde{a}(r,\theta)d\theta$, and since $\int_0^{2\pi} \frac{\partial \tilde{b}}{\partial \theta}\, d\theta = 0$, there holds

$$\tilde{\phi}(0) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty} [\tilde{a} - \overline{\tilde{a}}_r]\, \frac{\partial \tilde{b}}{\partial \theta}\, \frac{dr}{r}\, d\theta.$$

Applying successively the Cauchy-Schwarz and Poincaré inequalities on the circle S^1, we obtain

$$\begin{aligned} |\tilde{\phi}(0)| &\leq \frac{1}{2\pi}\int_0^{+\infty} \frac{dr}{r} \left(\int_0^{2\pi} |\tilde{a} - \overline{\tilde{a}}_r|^2\right)^{\frac{1}{2}} \left(\int_0^{2\pi} \left|\frac{\partial \tilde{b}}{\partial \theta}\right|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{2\pi}\int_0^{+\infty} \frac{dr}{r} \left(\int_0^{2\pi} \left|\frac{\partial \tilde{a}}{\partial \theta}\right|^2\right)^{\frac{1}{2}} \left(\int_0^{2\pi} \left|\frac{\partial \tilde{b}}{\partial \theta}\right|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Inequality (3.11) may then be inferred from the latter by applying once more the Cauchy-Schwarz inequality.

Returning to the disk D^2, we can extend a and b to $\mathbb{C}$ by setting $\tilde{a}(z) = a(\frac{z}{|z|^2})$ (and $\tilde{b}$ similarly) outside of D^2. Then

$$\|\nabla \tilde{a}\|_{L^2(\mathbb{C})} = 2\|\nabla a\|_{L^2(D^2)}, \quad \|\nabla \tilde{b}\|_{L^2(\mathbb{C})} = 2\|\nabla b\|_{L^2(D^2)}. \quad (3.12)$$

Let $\tilde{\phi}$ be the function in (3.10). The difference $\phi - \tilde{\phi}$ satisfies the equation

$$\begin{cases} \Delta(\phi - \tilde{\phi}) &= 0 \qquad \text{in } D^2 \\ \phi - \tilde{\phi} &= -\tilde{\phi} \quad \text{on } \partial D^2. \end{cases}$$

The *maximum principle* and inequalities (3.11), (3.12) imply

$$\|\phi - \tilde{\phi}\|_{L^\infty(D^2)} \leq \|\tilde{\phi}\|_{L^\infty(\partial D^2)} \leq C\|\nabla a\|_{L^2(D^2)}\, \|\nabla b\|_{L^2(D^2)}.$$

With the triangle inequality $|\|\phi\|_\infty - \|\tilde{\phi}\|_\infty| \leq \|\phi - \tilde{\phi}\|_\infty$ and inequality (3.11) again, we reach the desired L^∞-estimate of ϕ, and therefore, by the above discussion, the theorem is proved. $\square$

The following result, due to S. Chanillo and Y.Y. Li is a generalization of Wente's theorem 3.7.

Theorem 3.8 ([CL92]). *Let a and b be two functions in $W^{1,2}(D^2)$. Let $(a^{ij})_{1\leq i,j\leq 2}$ be a 2×2 symmetric matrix-valued map in $L^\infty(D^2)$ such that there exists $C>0$ for which*

$$C^{-1}\ |\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq C\ |\xi|^2$$

for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and all $x \in D^2$.

Let φ be the solution in $W^{1,p}(D^2, \mathbb{R})$ for any $1 \leq p < 2$ of

$$\left\{\begin{array}{rcll} \partial_{x_i}\left[a^{ij}\,\partial_{x_j}\lambda\right] & = & \partial_{x_1}a\,\partial_{x_2}b - \partial_{x_2}a\,\partial_{x_1}b & \text{on } D^2 \\ \varphi & = & 0 & \text{on } \partial D^2. \end{array}\right. \tag{3.13}$$

Then $\varphi \in L^\infty \cap W^{1,2}(D^2, \mathbb{R})$ and there exists $C>0$ independent of a and b such that

$$\|\varphi\|_{L^\infty(D^2)} + \|\nabla\varphi\|_{L^2(D^2)} \leq C\ \|\nabla a\|_{L^2(D^2)}\ \|\nabla b\|_{L^2(D^2)}. \tag{3.14}$$

3.2.3 Integrability by compensation on Riemann surfaces

Integration by compensation is natural in the framework of surfaces. An example provides the following generalization of Wente's inequality to Riemann surfaces, due to Topping ([Top97]) and Ge ([Ge98]).

Theorem 3.9. *Let (Σ, g) be a Riemann surface, $a, b \in W^{1,2}(\Sigma)$*

and ϕ the solution in $W^{1,p}(\Sigma)$ - for $1\le p<2$ - of

$$\left\{\begin{array}{rcll} *_g\Delta_g\phi & = & da\wedge db & \text{on } \Sigma\\ \phi & = & 0 & \text{on } \partial\Sigma,\ \text{if } \partial\Sigma\neq\emptyset\\ \phi(p) & = & 0 & \text{at a certain } p\in\Sigma,\ \text{if } \partial\Sigma=\emptyset. \end{array}\right. \tag{3.15}$$

Then ϕ belongs to $W^{1,2}\cap C^0(\Sigma)$ and satisfies

$$\|\phi\|_{L^\infty(\Sigma)}+\|\nabla\phi\|_{L^2(\Sigma)}\le C\ \|\nabla a\|_{L^2(\Sigma)}\ \|\nabla b\|_{L^2(\Sigma)}, \tag{3.16}$$

where C is a universal constant, not depending on Σ.

Note that the operator $*_g\Delta_g$ in (3.15) as well as the norms in (3.16) are conformally invariant (and thus it makes sense to consider them on a Riemann surface).

3.2.4 Some more results in integrability by compensation

Theorem 3.10 ([Bet92], [Ge99]). *Let a be such that $\nabla a\in L^{2,\infty}(D^2)$. Let $p\in(1,\infty)$ and $b\in W^{1,p}(D^2)$. Then the $W^{1,1}_0(D^2)$-solution ϕ of*[12]

$$\left\{\begin{array}{rcll} \Delta\phi & = & \partial_{x_1}a\ \partial_{x_2}b-\partial_{x_2}a\ \partial_{x_1}b & \text{in } D^2\\ \phi & = & 0 & \text{on } \partial D^2 \end{array}\right.$$

is in $W^{1,p}_0(D^2)$ and

$$\|\nabla\phi\|_{L^p(D^2)}\le C\|\nabla a\|_{L^{2,\infty}(D^2)}\|\nabla b\|_{L^p(D^2)}.$$

[12] If $p\le 2$, the Jacobian $\partial_{x_1}a\ \partial_{x_2}b-\partial_{x_2}a\ \partial_{x_1}b$ is understood in the weak sense

$$\partial_{x_1}a\ \partial_{x_2}b-\partial_{x_2}a\ \partial_{x_1}b:=div\left[a\,\nabla^\perp b\right].$$

Since $\nabla a\in L^{2,\infty}(D^2)$ we have that $a\in L^q(D^2)$ for all $q<\infty$ and hence $a\nabla^\perp b\in L^r(D^2)$ for all $r<p$.

4 Existence of isothermal coordinates in the weak framework

Given a Lipschitz immersion $\vec{\Phi} \in \mathcal{E}_\Sigma$ with L^2-bounded second fundamental form, we want to show the existence of isothermal coordinates for $\vec{\Phi}$:

For all $p \in \Sigma$ there exists a neighborhood U of p and a bi-Lipschitz diffeomorphism $\psi\colon D^2 \to U$ such that ψ is a (weakly) isothermal chart for $\vec{\Phi}$, i.e. satisfying

$$\begin{cases} \langle \partial_{x_1}(\vec{\Phi} \circ \psi), \partial_{x_2}(\vec{\Phi} \circ \psi) \rangle = 0 & \text{a.e. in } D^2 \\ \\ |\partial_{x_1}(\vec{\Phi} \circ \psi)| = |\partial_{x_2}(\vec{\Phi} \circ \psi)| & \text{a.e. in } D^2. \end{cases} \tag{4.1}$$

In the proof of Theorem 2.9 we presented the Chern moving frame method in order to show the existence of isothermal coordinates in the smooth case. We now want to apply this method in the context of Lipschitz immersions with L^2-bounded second fundamental form and prove the following analogon, where we require an additional energy restriction:

Lemma 4.1. *Let $\vec{\Phi} \in \mathcal{E}_{D^2}$ such that*

$$\int_{D^2} |\nabla \vec{n}_{\vec{\Phi}}|^2 dx_1 dx_2 < \frac{8\pi}{3}. \tag{4.2}$$

Then there exists a homeomorphism ψ of D^2, locally bi-Lipschitz, such that the map $\vec{\Phi} \circ \psi\colon D^2 \to \mathbb{R}^m$ is conformal, i.e. it satisfies (4.1).

To prove Lemma 4.1 we construct a local Coulomb tangent moving frame with controlled $W^{1,2}$-energy. We shall do so in two steps: In subsection 4.1 we will explore that under assumption (4.2) it is possible to "lift" the Gauss map and to construct a tangent moving frame with bounded $W^{1,2}$-energy. After that, in subsection 4.2, we will show how to turn the latter into a Coulomb frame with controlled energy.

Finally, in subsection 4.3, we will see that, by means of a result on integrability by compensation, we can use the Chern moving frame method also in the weak framework and conclude Lemma 4.1.

4.1 Hélein's energy controlled lifting theorem

The following **lifting theorem** was proven by F. Hélein in [Hél02].

Theorem 4.2. *Let $\vec{n} \in W^{1,2}(D^2, Gr_{m-2}(\mathbb{R}^m))$.*

Then there exists a constant $C > 0$, such that, if one assumes that

$$\int_{D^2} |\nabla \vec{n}|^2 \, dx_1 dx_2 < \frac{8\pi}{3}, \tag{4.3}$$

then there exist $\vec{e}_1, \vec{e}_2 \in W^{1,2}(D^2, S^{m-1})$ such that

$$\vec{n} = \star(\vec{e}_1 \wedge \vec{e}_2)^{13}, \tag{4.4}$$

and

$$\int_{D^2} \sum_{i=1}^{2} |\nabla \vec{e}_i|^2 \; dx_1 \, dx_2 \leq C \; \int_{D^2} |\nabla \vec{n}|^2 \; dx_1, dx_2. \tag{4.5}$$

[13] Condition (4.4), together with the facts that the $\vec{e}_i$ are S^{m-1}-valued and $\vec{n}$ has norm one, implies that $\vec{e}_1$ and $\vec{e}_2$ are orthogonal to each other.

4.2 Construction of local Coulomb frames with controlled $W^{1,2}$-energy

Proof of Theorem 4.1. Let $\vec{\Phi} \in \mathcal{E}_{D^2}$ satisfy (4.2). Due to Theorem 4.2, there exists a frame

$$(\vec{f}_1, \vec{f}_2) \in W^{1,2}(D^2, S^{m-1})^2 \tag{4.6}$$

with controlled energy. Consequently, one can proceed as in the proof of Theorem 2.9 and look for the minimizer of

$$\int_{D^2} \left| d\theta + \langle \vec{f}_1, d\vec{f}_2 \rangle \right|_g^2 \, dvol_g,$$

among all $\theta \in W^{1,2}(D^2, \mathbb{R})$, in order to produce a Coulomb frame on D^2. This Lagrangian is convex on the Hilbert space $W^{1,2}(D^2, \mathbb{R})$ and goes to $+\infty$ as $\|\theta\|_{W^{1,2}} \to +\infty$. Then there exists a unique minimum satisfying

$$\left\{ \begin{array}{rcll} d^{*_g} \left[d\theta + \langle \vec{f}_1, d\vec{f}_2 \rangle \right] & = & 0 & \text{in } D^2 \\ \iota^*_{\partial D^2} \left(*_g \left[d\theta + \langle \vec{f}_1, d\vec{f}_2 \rangle \right] \right) & = & 0 & \text{on } \partial D^2, \end{array} \right.$$

where $\iota_{\partial D^2}$ is the canonical inclusion of ∂D^2 into $\overline{D^2}$. Then $\vec{e} := \vec{e}_1 + i\vec{e}_2$ given by

$$\vec{e} = e^{i\theta} \vec{f} \tag{4.7}$$

is Coulomb, as desired:

$$\left\{ \begin{array}{rcll} d^{*_g} \left[\langle \vec{e}_1, d\vec{e}_2 \rangle \right] & = & 0 & \text{in } D^2 \\ \iota^*_{\partial D^2} \left(*_g \left[\langle \vec{e}_1, d\vec{e}_2 \rangle \right] \right) & = & 0 & \text{on } \partial D^2. \end{array} \right. \tag{4.8}$$

4.3 The Chern moving frame method in the weak framework

With a Coulomb frame $(\vec{e}_1, \vec{e}_2)$ satisfying (4.8) at hand, we are now in the position to apply the Chern moving frame method from 2.5. This however has to be done with the additional difficulty of keeping track of the regularity of the different actors at each step of the construction.

By the weak Poincaré Lemma, there exists $\lambda \in W^{1,2}(D^2)$ satisfying

$$\left\{\begin{array}{rcl} d\lambda & = & *_g\langle\vec{e}_1, d\vec{e}_2\rangle \\ \displaystyle\int_{\partial D^2}\lambda & = & 0. \end{array}\right. \tag{4.9}$$

The second equation of (4.8) implies that the restriction to ∂D^2 of the 1-form $d\lambda$ is equal to zero. Hence this last fact combined with the second equation of (4.9) implies that λ is identically equal to zero on ∂D^2. With the notation introduced in Section 2.1, observe that

$$d\langle\vec{e}_1, d\vec{e}_2\rangle = \langle d\vec{e}_1, d\vec{e}_2\rangle = \langle d\vec{f}_1, d\vec{f}_2\rangle, \tag{4.10}$$

by (4.7) and the change of gauge formula (2.61). We have then

$$\left\{\begin{array}{rcll} d *_g d\lambda & = & -\langle d\vec{f}_1, d\vec{f}_2\rangle & \text{on } D^2 \\ \lambda & = & 0 & \text{on } \partial D^2. \end{array}\right. \tag{4.11}$$

In the canonical coordinates of D^2, this reads as

$$\left\{\begin{array}{rll} \partial_{x_i}\left[\sqrt{det\,(g_{ij})}\,g^{ij}\partial_{x_j}\lambda\right] & = \langle\partial_{x_1}\vec{f}_1, \partial_{x_2}\vec{f}_2\rangle - \langle\partial_{x_2}\vec{f}_1, \partial_{x_1}\vec{f}_2\rangle & \\ & & \text{on } D^2 \\ \lambda & = 0 & \text{on } \partial D^2. \end{array}\right.$$

Since

$$\langle \partial_{x_1}\vec{f}_1, \partial_{x_2}\vec{f}_2\rangle - \langle \partial_{x_2}\vec{f}_1, \partial_{x_1}\vec{f}_2\rangle = \sum_{j=1}^{m} \partial_{x_1} f_{1,j}\cdot\partial_{x_2} f_{2,j} - \partial_{x_2} f_{1,j}\cdot\partial_{x_1} f_{2,j}$$

is a sum of Jacobians of $W^{1,2}$-functions, due to (4.6), we are now in the position to make use of Theorem 3.8, a generalization of Wente's theorem on integrability by compensation. It gives us that

$$\lambda \in L^\infty(D^2, \mathbb{R}). \tag{4.12}$$

Let (f_1, f_2) be the frame on D^2 given by

$$d\vec{\Phi}(f_i) = \vec{f}_i, \qquad\qquad i = 1, 2.$$

An easy computation shows that its dual frame satisfies

$$f_i^* = \sum_{k=1}^{2} \langle \vec{e}_i, \partial_{x_k}\vec{\Phi}\rangle \, dx_k \in L^\infty(D^2), \tag{4.13}$$

for $i = 1, 2$. Denote by (e_1, e_2) the frame on D^2 such that

$$d\vec{\Phi}(e_i) = \vec{e}_i.$$

By (4.7), we have

$$e^* = e_1^* + ie_2^* = e^{-i\theta}\,(f_1^* + if_2^*)$$

which is in $L^\infty(D^2)$ due to (4.13). From the proof of Theorem 2.9 follows that

$$d(e^{-\lambda} e_i^*) = 0 \qquad \text{in } \mathcal{D}'(D^2).$$

Applying the weak Poincaré Lemma gives a map $\phi = (\phi_1, \phi_2)$ satisfying

$$d\phi_i := e^{-\lambda} e_i^*.$$

Exactly as in the proof of Theorem 2.9, using Corollary 2.16, one shows that there exists a homeomorphism ψ of D^2, locally bi-Lipschitz, realizing an isothermal chart for $\vec{\Phi}$ (i.e. satisfying (4.1)). $\square$

Lemma 4.1 immediately implies the following theorem.

Theorem 4.3. [Existence of isothermal coordinates]
Let $\vec{\Phi} \in \mathcal{E}_\Sigma$. Then for each $p \in \Sigma$ there exists a neighborhood U and a Lipschitz diffeomorphism ψ from D^2 into U such that ψ is an isothermal chart for $\vec{\Phi}$, i.e. satisfying (4.1)*.*

Proof of Theorem 4.3. Let $p \in \Sigma$ be an arbitrary point. Choose any chart $\varphi\colon D^2 \to U$ around $p \in U$. Due to condition iii), we can assume that

$$\int_{D^2} |d\vec{n}_{\vec{\Phi}\circ\varphi}|^2 dx_1 dx_2 < \frac{8\pi}{3},$$

possibly after restricting φ to a sufficiently small neighborhood of $\varphi^{-1}(p) = 0$. Thus, $\vec{\Phi} \circ \varphi \in \mathcal{E}_{D^2}$ satisfies the assumptions of Lemma 4.1, which then gives us a bi-Lipschitz diffeomorphism ψ of $D^2_{1/2}$ such that $\vec{\Phi} \circ \varphi \circ \psi$ is conformal. $\varphi \circ \psi$ is the desired isothermal chart around p. $\square$

Corollary 4.4. [Existence of a smooth conformal structure] *Let Σ be a closed smooth 2-dimensional manifold. Then any $\vec{\Phi} \in \mathcal{E}_\Sigma$ defines a smooth conformal structure on Σ. In particular there exists a constant curvature metric h of unit volume on Σ and a Lipschitz diffeomorphism Ψ of Σ such that $\vec{\Phi} \circ \Psi$ realizes a conformal immersion of the Riemann surface (Σ, h) and h and $g_{\vec{\Phi}}$ are conformally equivalent.*

Definition 4.5. *If h is a conformal structure on Σ, a map $\vec{\xi}\colon (\Sigma, h) \to \mathbb{R}$ is called* (weakly) conformal *if it satisfies*

$$\begin{cases} \langle \partial_{x_1}\vec{\xi}, \partial_{x_2}\vec{\xi}\rangle = 0 & \textit{a.e. in } \Sigma \\ |\partial_{x_1}\vec{\xi}| = |\partial_{x_2}\vec{\xi}| & \textit{a.e. in } \Sigma. \end{cases}$$

Proof of Corollary 4.4. Σ can be covered by finitely many isothermal charts $(U_i, \psi_i)_{i=1,\ldots,n}$, due to Theorem 4.3 and the fact that Σ is compact. For each $i = 1, \ldots, n$, the map $\vec{\Phi} \circ \psi_i$ realizes a conformal Lipschitz immersion of D^2, denoted by

$$\psi_i^* g_{\vec{\Phi}} = e^{2\lambda_i}(dx^1 + dx^2).$$

If $U_i \cap U_j \neq \emptyset$, the map $\psi_j^{-1} \circ \psi_i \colon \psi_i^{-1}(U_i \cap U_j) \to \mathbb{C}$ is conformal and positive (hence holomorphic) almost everywhere:

$$(\psi_j^{-1} \circ \psi_i)^*(dy^1 + dy^2) = \psi_i^*(e^{-2\lambda_j} g_{\vec{\Phi}}) = e^{2(\lambda_i - \lambda_j)}(dx^1 + dx^2).$$

It follows that $\psi_j^{-1} \circ \psi_i$ is harmonic on $U_i \cap U_j$. From standard distribution theory follows that it is holomorphic everywhere on $U_i \cap U_j$. Thus, the system of charts (U_i, ψ_i) defines a smooth conformal structure c on Σ.

The second statement follows from Theorem 2.18. $\square$

Weak immersions of surfaces with L^2-bounded second fundamental form

Lecture 3

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Tristan Rivière

Lecture Notes written by Nora Loose

5 Sequences of weak immersions

5.1 Compactness question

Eventually we will be interested in finding a minimizer of the Willmore energy. More specifically, we will look at a minimizing sequence of weak immersions with L^2-bounded second fundamental form and ask if it has a limit with respect to some notion of weak convergence and how this limiting object looks like.

Recall that in Section 2.3 we showed that the Willmore energy is invariant under conformal transformations in the target. To get an idea what kind of convergence is the best we can hope for, let us study an example: Assume $\vec{\Phi} \in \mathcal{E}_\Sigma$ immerses a torus in $\mathbb{R}^3$. Figures 1,2 and 3 show that composing $\vec{\Phi}$ with sequences of conformal diffeomorphisms of $\mathbb{R}^3$ of different types produces sequences $\vec{\Phi}_k$, whose limit does not immerse a torus anymore.

Here, for $a \in \mathbb{R}^m$, the map $i_a \in \mathrm{Conf}(\mathbb{R}^3 \cup \{\infty\})$ denotes the inversion at a,

$$i_a \colon x \mapsto \frac{x-a}{|x-a|^2}.$$

Note that i_a is a diffeomorphism from $B_R(a) \setminus B_\varepsilon(a)$ to $B_{1/\varepsilon}(a) \setminus$

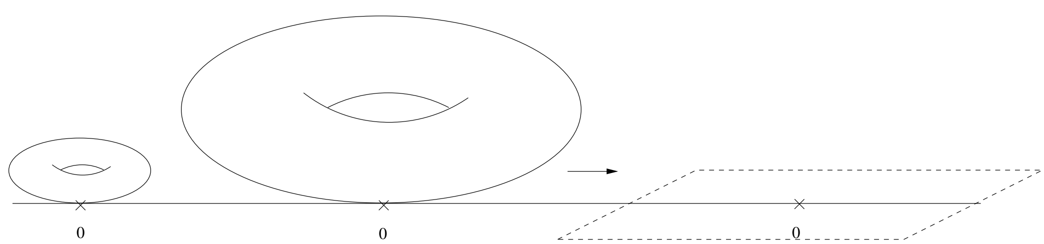

Figure 1: Dilation. $\vec{\Phi}_k := k\vec{\Phi}$. (Loss of energy and topology.)

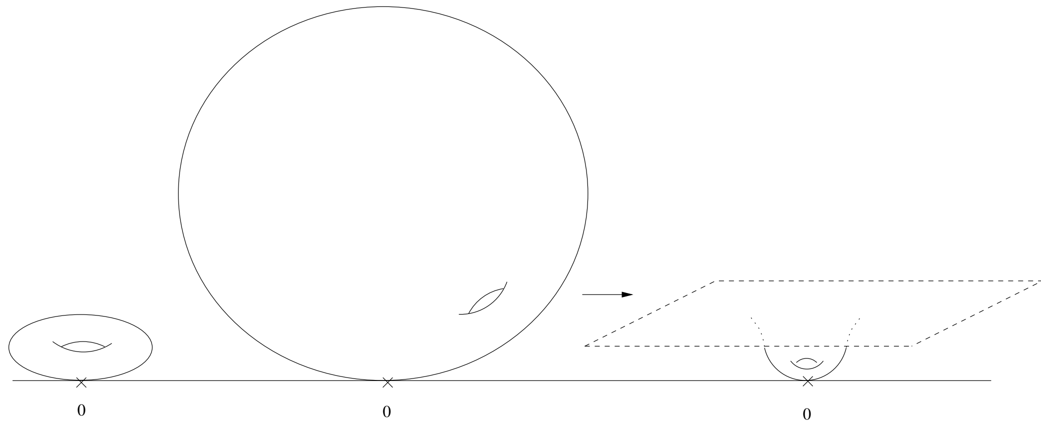

Figure 2: Inversion. $\vec{\Phi}_k := i_{a_k} \circ \vec{\Phi}$, where $a_k \to \vec{\Phi}(\Sigma) \setminus \{0\}$. (Loss of energy.)

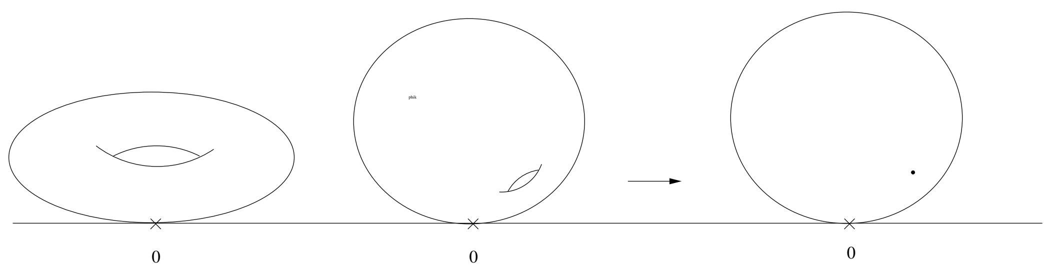

Figure 3: Inversion $\circ$ Dilation. $\vec{\Phi}_k := \operatorname{dist}(a_k, \vec{\Phi}(\Sigma))\ i_{a_k} \circ \vec{\Phi}$, where $a_k \to \vec{\Phi}(\Sigma) \setminus \{0\}$. (Loss of energy and topology.)

$B_{1/R}(a)$ for any $0 < \varepsilon < R < \infty$ and thus, for $a \in \mathbb{R}^m \setminus \vec{\Phi}(\Sigma)$,

$$W(i_a \circ \vec{\Phi}) = W(\vec{\Phi}).$$

These examples illustrate that, given a sequence $\vec{\Phi}_k \in \mathcal{E}_\Sigma$, say with $\sup_k \mathbb{I}(\vec{\Phi}_k) < \infty$, we cannot expect that $\vec{\Phi}_k$ has a limit, which still immerses the surface Σ (in a weak sense). In fact, we need to compose with conformal transformations $\vec{\Xi}_k \in \operatorname{Conf}(\mathbb{R}^m \cup \{\infty\})$ (with the center of inversion of $\vec{\Xi}_k$ not being contained in $\vec{\Phi}_k(\Sigma)$) to avoid degeneracies as shown in Figures 1 - 3.

Furthermore, when passing to the limit there might be energy concentration in single points. Figure 4 provides an example of how such a loss of energy in the limit might occur.

To allow composing with conformal transformations along the sequence in order to obtain a reasonable (weak) limiting object is not enough: We also need to make a compactness assumption on the conformal classes induced by $\vec{\Phi}_k$. Let $\Psi_k\colon (\Sigma, h_k) \to \Sigma$ be the Lipschitz diffeomorphisms from Corollary 4.4 such that

$$\vec{\Psi}_k := \vec{\Phi}_k \circ \Psi_k\colon (\Sigma, h_k) \to \mathbb{R}^m$$

are conformal, where each h_k denotes the reference metric of constant curvature and unit volume of the conformal structure induced by $\vec{\Phi}_k$. Since Ψ_k are elements of the invariance group $\mathrm{Diff}^+(\Sigma)$ of the Willmore functional, we have $W(\vec{\Psi}_k) = W(\vec{\Phi}_k)$. We make the following compactness assumption **(CA)**:

> The conformal classes (Σ, h_k) are contained in a compact subset of $\mathcal{M}_\Sigma$, the moduli space of Σ.

This assumption is necessary for the following reason: If the conformal classes degenerate, we might not only lose energy but also topology in the limit, which is irreversible and has to be avoided. Such a situation is shown in Figure 5. Note that there

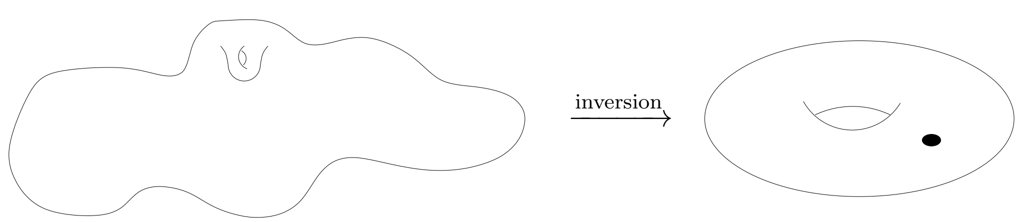

Figure 4: Loss of energy, no loss of topology.

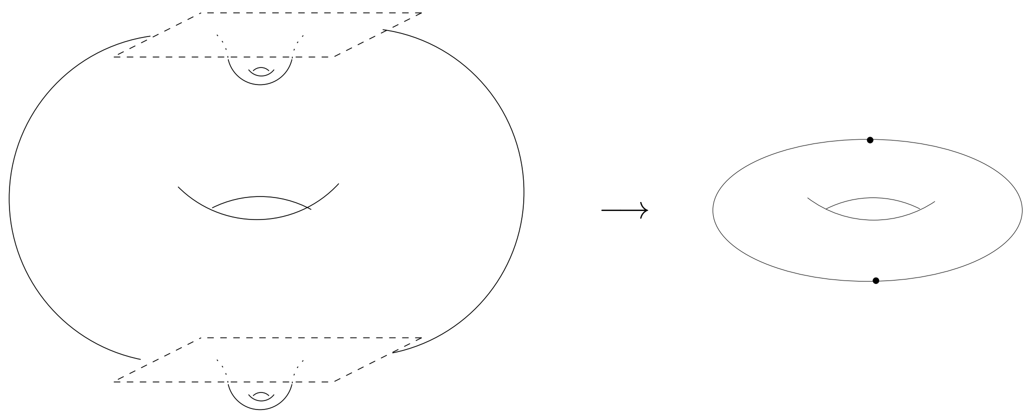

Figure 5: Loss of energy, topology and conformal class.

is no way to preserve the genus in the limit: One can "save" one hole (e.g. by applying dilations and inversions as in Figure 3), but one will lose the other two ones at the same time.

Before using the previous observations to define a notion of *weak convergence*, we note the following fact:

If a sequence of $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ satisfies the compactness assumption (CA), the conformal classes (Σ, h_k) of constant curvature and unit volume, induced by $\vec{\Phi}_k$ resp., satisfy (up to subsequences)

$$h_k \to h_\infty \qquad \text{in } C^l(\Sigma) \quad \forall l \in \mathbb{N},$$

where (Σ, h_∞) is the limiting conformal structure of constant curvature and unit volume.

Definition 5.1. *A sequence $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ satisfying assumption (CA) with $h_k \to h_\infty$ is called* weakly convergent *if there exist Lipschitz diffeomorphisms Ψ_k of Σ, conformal transformations $\vec{\Xi}_k$ of $\mathbb{R}^m \cup \{\infty\}$ with*

$$\vec{\Phi}_k(\Sigma) \cap \{\textit{center of inversion of } \vec{\Xi}_k\} = \emptyset$$

and finitely many points $a_1, \ldots, a_N \in \Sigma$, called blow-up points *such that*

$$\vec{\xi}_k := \vec{\Xi}_k \circ \vec{\Phi}_k \circ \Psi_k \colon (\Sigma, h_k) \to \mathbb{R}^m$$

is conformal, and there exists a map $\vec{\xi}_\infty \colon \Sigma \to \mathbb{R}^m$ *such that*

i)

$$\vec{\xi}_\infty \text{ is conformal from } (\Sigma, h_\infty) \text{ into } \mathbb{R}^m;$$

ii)

$$\vec{\xi}_k \rightharpoonup \vec{\xi}_\infty \qquad \text{weakly in } W^{2,2}_{loc}(\Sigma \setminus \{a_1, \ldots, a_N\}), \tag{5.1}$$

iii)

$$\log |d\vec{\xi}_k|^2 \rightharpoonup \log |d\vec{\xi}_\infty|^2 \quad \text{weakly in } (L^\infty)^*_{loc}(\Sigma \setminus \{a_1, \ldots, a_N\}); \tag{5.2}$$

iv)

$$\vec{\xi}_k \rightharpoonup \vec{\xi}_\infty \qquad \text{weakly in } W^{1,2} \cap (L^\infty)^*(\Sigma). \tag{5.3}$$

The following lemma shows that the Willmore functional W and the energy functional $\mathbb{I}$ are lower semicontinuous under weak convergence.

Lemma 5.2. *Let* (Σ, h_k) *be a sequence of conformal structures on* Σ*, where* h_k *denotes the associated metric of constant curvature and unit volume. Assume it satisfies assumption (CA), with* $h_k \to h_\infty$*.*

Let $\vec{\xi}_k \colon (\Sigma, h_k) \to \mathbb{R}^m$ *be a sequence of conformal maps in* $\mathcal{E}_\Sigma$ *with*

$$\sup_k \mathbb{I}(\vec{\xi}_k) < \infty \tag{5.4}$$

and $\vec{\xi}_\infty \colon (\Sigma, h_\infty) \to \mathbb{R}^m$ *a conformal map such that ii) and iii) from Definition 5.1 are satisfied, i.e. there exist* $a_1, \ldots, a_N \in \Sigma$ *such that*

$$\vec{\xi}_k \rightharpoonup \vec{\xi}_\infty \qquad \text{weakly in } W^{2,2}_{loc}(\Sigma \setminus \{a_1, \ldots, a_N\}) \tag{5.5}$$

and

$$\log|d\vec{\xi}_k|^2 \rightharpoonup \log|d\vec{\xi}_\infty|^2 \quad \textit{weakly in } (L^\infty)^*_{loc}(\Sigma\setminus\{a_1,\ldots,a_N\}). \tag{5.6}$$

Then for any $K \Subset \Sigma\setminus\{a_1,\ldots,a_N\}$, we have

$$\int_K |\vec{H}_{\vec{\xi}_\infty}|^2 dvol_{g_\infty} \leq \liminf_k \int_K |\vec{H}_{\vec{\xi}_k}|^2 dvol_{g_k} \tag{5.7}$$

and

$$\int_K |d\vec{n}_{\vec{\xi}_\infty}|^2_{g_\infty} dvol_{g_\infty} \leq \liminf_k \int_K |d\vec{n}_{\vec{\xi}_k}|^2_{g_k} dvol_{g_k}. \tag{5.8}$$

Proof. Since the $\vec{\xi}_k$'s are conformal, the mean curvature vector in isothermal coordinates can be written as

$$\vec{H}_{\vec{\xi}_k} = \frac{1}{2}e^{-2\lambda_k}\Delta\vec{\xi}_k,$$

where $\lambda_k := \log|\partial_{x_1}\vec{\xi}_k|$.

Let $K \Subset \Sigma\setminus\{a_1,\ldots,a_N\}$ be compact. Note that, due to (5.5),

$$\Delta\vec{\xi}_k \rightharpoonup \Delta\vec{\xi}_\infty \qquad \text{weakly in } L^2(K). \tag{5.9}$$

Moreover, using again (5.5) and applying Rellich/Kondrachov, we obtain for all $1 < p < \infty$ a subsequence such that

$$\partial_{x_1}\vec{\xi}_{k'} \to \partial_{x_1}\vec{\xi}_\infty \qquad \text{strongly in } L^p(K).$$

(5.6) ensures that $|\partial_{x_1}\vec{\xi}_k| \geq C$ a.e. for $C > 0$ independent of k. Thus, we have

$$e^{-\lambda_{k'}} = \frac{1}{|\partial_{x_1}\vec{\xi}_{k'}|} \to \frac{1}{|\partial_{x_1}\vec{\xi}_\infty|} = e^{-\lambda_\infty} \qquad \text{strongly in } L^p(K).$$

This, together with (5.9), implies

$$\vec{H}_{\vec{\xi}_{k'}}\sqrt{vol_{g_{k'}}} = \frac{1}{2}e^{-\lambda_{k'}}\Delta\vec{\xi}_{k'} \to \frac{1}{2}e^{-\lambda_\infty}\Delta\vec{\xi}_\infty = \vec{H}_{\vec{\xi}_\infty}\sqrt{vol_{g_\infty}} \quad \text{in } \mathcal{D}'(K). \tag{5.10}$$

But note that $\Delta\vec{\xi}_{k'}$ is uniformly bounded in $L^2(K)$, due to (5.5), and $e^{-\lambda_{k'}}$ is uniformly bounded in $L^\infty(K)$, due to (5.6). It follows that $\vec{H}_{\vec{\xi}_{k'}}\sqrt{vol_{g_{k'}}}$ is uniformly bounded in $L^2(K)$ and consequently, the convergence in (5.10) is a weak convergence in $L^2(K)$. Lower semicontinuity of the L^2-norm under weak L^2-convergence implies the desired result.

To prove (5.8), note that, by $h_k \to h_\infty$, (5.4) implies that

$$\sup_k \int_\Sigma |d\vec{n}_{\vec{\xi}_k}|^2_{h_\infty} dvol_{h_\infty} < \infty.$$

Thus, up to subsequences, there exists $\vec{n}_\infty \in W^{1,2}(\Sigma)$ such that

$$\vec{n}_{\vec{\xi}_{k'}} \rightharpoonup \vec{n}_\infty \qquad \text{weakly in } W^{1,2}(\Sigma).$$

We will show that $\vec{n}_\infty$ equals $\vec{n}_{\vec{\xi}_\infty}$ on any compact set $K \Subset \Sigma \setminus \{a_1, \ldots, a_N\}$, which in turn implies (5.8), by lower semicontinuity of weak $W^{1,2}$-convergence and $h_k \to h_\infty$ again.

Given any $p < \infty$, we have, modulo extraction of a subsequence, strong convergence $\vec{n}_{\vec{\xi}_{k'}} \to \vec{n}_\infty$ in $L^p(\Sigma)$, by Rellich/Kondrachov for compact manifolds (see [Aub82], Theorem 2.34). After passing to a further subsequence, we can assume that

$$\vec{n}_{\vec{\xi}_{k'}} \to \vec{n}_\infty \qquad \text{a.e. in } \Sigma. \tag{5.11}$$

Note that in isothermal coordinates, we have

$$\vec{n}_{\vec{\xi}_{k'}} = e^{-2\lambda_{k'}} \star \left(\partial_{x_1}\vec{\xi}_{k'} \wedge \partial_{x_2}\vec{\xi}_{k'}\right). \tag{5.12}$$

Applying Rellich/ Kondrachov, (5.5) implies that for any $p < \infty$, there is a further subsequence such that

$$d\vec{\xi}_{k'} \to d\vec{\xi}_{\infty} \qquad \text{strongly in } L^p(K).$$

By passing to a further subsequence, we obtain

$$d\vec{\xi}_{k'} \to d\vec{\xi}_{\infty} \qquad \text{a.e. in } K.$$

This implies in particular that

$$\star\left(\partial_{x_1}\vec{\xi}_{k'} \wedge \partial_{x_2}\vec{\xi}_{k'}\right) \to \star\left(\partial_{x_1}\vec{\xi}_{\infty} \wedge \partial_{x_2}\vec{\xi}_{\infty}\right) \quad \text{a.e. in } K. \tag{5.13}$$

Observe that $\sup_{k'} \|\lambda_{k'}\|_{W^{1,2}(K)} \leq C$, due to

$$|\nabla \lambda_{k'}| = \left|\nabla \log\left(\frac{1}{\sqrt{2}}|\nabla \vec{\xi}_{k'}|\right)\right| \leq C + \frac{|\nabla^2 \vec{\xi}_{k'}|}{|\nabla \vec{\xi}_{k'}|},$$

(5.5) and (5.6), which implies $\sup_k |\nabla \vec{\xi}_{k'}| \geq c > 0$. Consequently, we can assume that

$$\lambda_{k'} \to \lambda_{\infty} \qquad \text{in } L^p(K) \text{ and a.e. in } K, \tag{5.14}$$

after extraction of subsequences. (5.12), (5.14) and (5.13) imply that

$$\vec{n}_{\vec{\xi}_{k'}} \to \underbrace{e^{-2\lambda_{\infty}} \star \left(\partial_{x_1}\vec{\xi}_{\infty} \wedge \partial_{x_2}\vec{\xi}_{\infty}\right)}_{=\vec{n}_{\vec{\xi}_{\infty}}} \quad \text{a.e. in } K.$$

(5.11), together with uniqueness of the limit, gives the desired result that $\vec{n}_{\vec{\xi}_{\infty}}$ and $\vec{n}_{\infty}$ coincide on K. $\square$

Theorem 5.3 (Weak almost-closure theorem)**.** *Let $\vec{\Phi}_k \in \mathcal{E}_{\Sigma}$ such that*

$$\sup_k \mathbb{I}(\vec{\Phi}_k) < \infty \tag{5.15}$$

and such that assumption (CA) is satisfied.

Then there exists a weakly converging subsequence of $\vec{\Phi}_k$ (in the sense of Definition 5.1).

In the next two subsections we shall prepare the proof of Theorem 5.3, which will be finally given in Subsection 5.4.

5.2 Control of the conformal factor

Let $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ be a sequence of weak immersions. Corollary 4.4 tells us how to fix a gauge, namely one can find Lipschitz diffeomorphisms $\Psi_k\colon (\Sigma, h_k) \to \Sigma$ such that

$$\vec{\Psi}_k := \vec{\Phi}_k \circ \Psi_k\colon (\Sigma, h_k) \to \mathbb{R}^m$$

are conformal, where each h_k denotes the reference metric of constant curvature and unit volume of the conformal structure induced by $\vec{\Phi}_k$. Denote

$$g_{\vec{\Phi}_k} = e^{2\alpha_k} h_k.$$

A priori the conformal factors $e^{2\alpha_k}$ could go either to $+\infty$ or 0 as $k \to \infty$. In both cases, the limiting map will not be an element of $\mathcal{E}_\Sigma$, in the first case failing the Lipschitz condition and in the second case failing the non-degeneracy of the metric. The question that we want to investigate now is therefore: Can the logarithms of the conformal factors, that are α_k, be controlled in the L^∞-norm by $\sup_k \mathbb{I}(\vec{\Phi}_k)$, when we let $k \to \infty$?

The first result is a global bound for α_k in the $L^{2,\infty}(\Sigma)$-norm.

Theorem 5.4. *Let $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ be a sequence of Lipschitz immersions with L^2-bounded second fundamental form such that*

$$\sup_k \mathbb{I}(\vec{\Phi}_k) < \infty.$$

Let Ψ_k be Lipschitz diffeomorphisms of Σ such that

$$\vec{\Psi}_k := \vec{\Phi}_k \circ \Psi_k \colon (\Sigma, h_k) \to \mathbb{R}^m$$

are conformal, where each h_k denotes the reference metric of constant curvature and unit volume of the conformal structure induced by $\vec{\Phi}_k$. Furthermore, we make assumption (CA), that is the conformal classes (Σ, h_k) are contained in a compact subset of $\mathcal{M}_\Sigma$.

Denote $g_k := \vec{\Phi}_k^* g_{\mathbb{R}^m}$ and

$$g_k = e^{2\alpha_k} h_k.$$

Then

$$\sup_k \|d\alpha_k\|_{L^{2,\infty}(\Sigma)} < \infty.$$

Proof. Since $g_k = e^{2\alpha_k} h_k$, by (2.52), we have for all $k \in \mathbb{N}$

$$-\Delta_{h_k}\alpha_k = e^{2\alpha_k} K_{g_k} - K_{h_k}. \tag{5.16}$$

This identity, together with (2.23), gives us the following estimate for $\Delta_{h_k}\alpha_k$ in $L^1_{h_k}(\Sigma)$:

$$\begin{aligned} &\int_\Sigma |\Delta_{h_k}\alpha_k|\, dvol_{h_k} \le \int_\Sigma e^{2\alpha_k}|K_{g_k}| dvol_{h_k} + \int_\Sigma |K_{h_k}| dvol_{h_k} \\ &= \int_\Sigma |K_{g_k}| dvol_{g_k} + |K_{h_k}| \le \frac{1}{2}\int_\Sigma |\vec{\mathbb{I}}_{\vec{\Phi}_k}|^2_{g_k} dvol_{g_k} + |K_{h_k}| \qquad (5.17) \\ &\le \frac{1}{2}\sup_k \mathbb{I}(\vec{\Phi}_k) + C =: \tilde{C}, \end{aligned}$$

where we used that we chose h_k to be the constant curvature metric of unit volume and $h_k \to h_\infty$. Since $c_k^{-1} g_0 \le h_k \le c_k g_0$ and due to $h_k \to h_\infty$, (5.17) gives also the estimate

$$\|\Delta_{h_k} \alpha_k\|_{L^1_{g_0}(\Sigma)} \le \hat{C}$$

for all $k \in \mathbb{N}$. Applying Theorem 3.6 yields thus the desired uniform bound for $d\alpha_k$ in $L^{2,\infty}(\Sigma)$:

$$\|d\alpha_k\|_{L^{2,\infty}_{g_0}(\Sigma)} \le C_{g_0} \;\|\Delta_{h_k} \alpha_k\|_{L^1_{g_0}(\Sigma)} \le C_{g_0} \cdot \hat{C}. \tag{5.18}$$

□

We now shall investigate the evolvement of the logarithms of the conformal factors in the L^∞-norm. This is done locally, wherever the second fundamental form does not concentrate "too much energy".

Theorem 5.5. *Let $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ be a sequence in $\mathcal{E}_\Sigma$ which satisfies*

$$\sup_k \mathbb{I}(\vec{\Phi}_k) < \infty.$$

Furthermore, we make assumption (CA): the conformal classes (Σ, h_k) are contained in a compact subset of the moduli space of Σ. As before, let Ψ_k be Lipschitz diffeomorphisms of Σ such that

$$\vec{\Psi}_k := \vec{\Phi}_k \circ \Psi_k \colon (\Sigma, h_k) \to \mathbb{R}^m$$

are conformal, where each h_k denotes the reference metric of constant curvature and unit volume of the conformal structure induced by $\vec{\Phi}_k$.

Moreover, let φ_k be a sequence of isothermal charts satisfying

$$\sup_k \int_{D^2} |\nabla \vec{n}_{\vec{\Psi}_k \circ \varphi_k}|^2 dx_1 dx_2 < \frac{8\pi}{3}. \tag{5.19}$$

Denote

$$(\vec{\Psi}_k \circ \varphi_k)^* g_{\mathbb{R}^m} = e^{2\lambda_k}(dx_1^2 + dx_2^2).$$

Then there exist constants $c_k \in \mathbb{R}$ such that

$$\sup_k \|\lambda_k - c_k\|_{L^\infty(\Omega)} \leq C_\Omega, \tag{5.20}$$

for any set $\Omega \Subset D^2$.

The following example shows that in general we cannot control the λ_k in L^∞ when deleting the constants c_k in (5.20): Compose the weak immersions $\vec{\Psi}_k$ with dilations $s_k\cdot$ in $\mathbb{R}^m$, for some $s_k \in \mathbb{R}$. Due to the conformal invariance of the Willmore functional, we have $\mathbb{I}(\vec{\Psi}_k) = \mathbb{I}(s_k\vec{\Psi}_k)$. In contrast, the dilations are reflected in the logarithms of the conformal factors, that is $\lambda_{s_k\vec{\Psi}_k} = \lambda_{\vec{\Psi}_k} + \log s_k$.

Note that this example also shows that the constants c_k are a priori not controlled, when $k \to \infty$.

Proof of Theorem 5.5. Assuming (5.19), for each $k \in \mathbb{N}$, we can apply Hélein's lifting theorem 4.2 to obtain the existence of $\vec{f}_1^k$ and $\vec{f}_2^k$ in $W^{1,2}(D^2, S^{m-1})$ such that

$$\vec{n}_k := \vec{n}_{\vec{\Psi}_k \circ \varphi_k} = \star(\vec{f}_1^k \wedge \vec{f}_2^k),$$

and

$$\int_{D^2} \sum_{i=1}^{2} |\nabla \vec{f}_i^k|^2 \, dx_1 \, dx_2 \leq C \int_{D^2} |\nabla \vec{n}_k|^2 \, dx_1 dx_2,$$

where $\vec{n}_k := \vec{n}_{\vec{\Psi}_k \circ \varphi_k}$.

Using the notation introduced in Section 2.1, the factors λ_k satisfy the following equation:

$$\Delta \lambda_k = -\langle \nabla^\perp \vec{f}_1^k, \nabla \vec{f}_2^k \rangle. \tag{5.21}$$

This can seen by observing that, by (2.45), the latter equation holds for the frame $(\vec{f}_1^k, \vec{f}_2^k)$ being replaced by the frame $(\vec{e}_1^k, \vec{e}_2^k)$ given by

$$\vec{e}_i^k := e^{-\lambda_k} \partial_{x_i}(\vec{\Psi}_k \circ \varphi_k).$$

Furthermore, the change of gauge formula (2.62) leaves the right hand side of (5.21) invariant. Observe that the right hand side of (5.21) is a sum of Jacobians, more specifically

$$\begin{aligned}
-\langle \nabla^\perp \vec{f}_1^k, \nabla \vec{f}_2^k \rangle &= \langle \partial_{x_2} \vec{f}_1^k, \partial_{x_1} \vec{f}_2^k \rangle - \langle \partial_{x_1} \vec{f}_1^k, \partial_{x_2} \vec{f}_2^k \rangle \\
&= \sum_{j=1}^{m} \partial_{x_2} f_{1,j}^k \cdot \partial_{x_1} f_{2,j}^k - \partial_{x_1} f_{1,j}^k \cdot \partial_{x_2} f_{2,j}^k .
\end{aligned}$$

Let μ_k be the solution to

$$\begin{cases} \Delta \mu_k = \sum_{j=1}^{m} \partial_{x_2} f_{1,j}^k \cdot \partial_{x_1} f_{2,j}^k - \partial_{x_1} f_{1,j}^k \cdot \partial_{x_2} f_{2,j}^k & \text{on } D^2 \\ \mu_k = 0 & \text{on } \partial D^2. \end{cases}$$

Wente's Theorem 3.8 gives us the estimate

$$\begin{aligned}
&\|\nabla \mu_k\|_{L^2(D^2)} + \|\mu_k\|_{L^\infty(D^2)} \\
&\leq C \sum_{j=1}^{m} \int_{D^2} \left(|\nabla f_{1,j}^k|^2 + |\nabla f_{2,j}^k|^2 \right) dx_1 dx_2 \qquad (5.22) \\
&\leq C \int_{D^2} |\nabla \vec{n}_k|^2 dx_1 dx_2 \leq C,
\end{aligned}$$

where we used (5.19) in the last step such that the constant C is independent of k.

We now consider the harmonic rest $\nu_k := \lambda_k - \mu_k$. Due to Theorem 5.4, which gives a global estimate of $d\alpha_k$ in $L^{2,\infty}(\Sigma)$, and the strong convergence $h_k \to h_\infty$, we have a uniform bound

$$\sup_k \|\nabla \lambda_k\|_{L^{2,\infty}(D^2)} \leq C.$$

Together with (5.22) (and the fact that $L^2 \hookrightarrow L^{2,\infty}$ continuously), this yields $\nu_k \in L^{2,\infty}(D^2)$ and

$$\sup_k \|\nabla \nu_k\|_{L^{2,\infty}(D^2)} \leq C.$$

From (3.5), we know that

$$\|\nabla \nu_k\|_{L^p(D^2)} \leq C_p \ \|\nabla \nu_k\|_{L^{2,\infty}(D^2)}$$

for all $p < 2$. Applying Poincaré's inequality yields

$$\|\nu_k - \bar{\nu}_k\|_{L^p(D^2)} \leq C_p \ \|\nabla \nu_k\|_{L^p(D^2)},$$

where $\bar{\nu}_k$ denotes the average of ν_k on D^2.

Since $W^{1,p}(D^2) \hookrightarrow W^{1-\frac{1}{p},p}(\partial D^2) \hookrightarrow L^1(\partial D^2)$, we obtain

$$\sup_k \|\nu_k - \bar{\nu}_k\|_{L^1(\partial D^2)} \leq C.$$

Using the Poisson representation formula for harmonic functions on D^2 yields

$$\|\nu_k - \bar{\nu}_k\|_{C^1(\Omega)} \leq C_\Omega \ \|\nu_k - \bar{\nu}_k\|_{L^1(\partial D^2)}$$

for any $\Omega \Subset D^2$ and thus

$$\sup_k \|\nu_k - \bar{\nu}_k\|_{L^\infty(\Omega)} \leq \tilde{C}_\Omega.$$

Combining this result with (5.22) gives us

$$\sup_k \|\lambda_k - c_k\|_{L^\infty(\Omega)} \leq \hat{C}_\Omega,$$

for any set $\Omega \Subset D^2$, where $c_k := \bar{\nu}_k$. $\square$

5.3 The monotonicity formula and consequences

As always, let Σ be a smooth closed oriented surface. The following lemma is a corollary of Simon's monotonicity formula.

Lemma 5.6 (Simon's Monotonicity formula). *Let $\vec{\Phi} \in \mathcal{E}_\Sigma$ be a Lipschitz immersion with L^2-bounded second fundamental form. Denote by $M := \vec{\Phi}(\Sigma)$ the immersed surface.*

Then for any point $x_0 \in \mathbb{R}^m$ and any $0 < t < T < \infty$, we have

$$\frac{\mathrm{Area}\,(M \cap B_T(x_0))}{T^2} - \frac{\mathrm{Area}\,(M \cap B_t(x_0))}{t^2}$$

$$\geq -\frac{1}{4} \int_{M \cap (B_T(x_0) \setminus B_t(x_0))} |\vec{H}|^2 \; dvol_g$$

$$-\frac{1}{T^2} \int_{M \cap B_T(x_0)} \langle x - x_0, \vec{H} \rangle \; dvol_g + \frac{1}{t^2} \int_{M \cap B_t(x_0)} \langle x - x_0, \vec{H} \rangle \; dvol_g.$$

For a proof see [Sim84] or [Riv13].

Definition 5.7. *Let $\vec{\Phi} \in \mathcal{E}_\Sigma$. For $x_0 \in \mathbb{R}^m$, define the density $\theta_{x_0} \in \mathbb{N}$ at x_0 as*

$$\theta_{x_0} := \lim_{t \to 0^+} \frac{1}{\pi t^2} \mathrm{Area}\left(\vec{\Phi}(\Sigma) \cap B_t(x_0)\right), \tag{5.23}$$

whenever the limit exists.

To simplify notation in the proofs of the following corollaries, we introduce for fixed $x_0 \in \mathbb{R}^m$ the quantities

$$W(t) := \int_{M \cap B_t(x_0)} |\vec{H}|^2 \; dvol_g, \tag{5.24}$$

$$Y(t) := \frac{1}{t^2}\,\mathrm{Area}\left(\vec{\Phi}(\Sigma) \cap B_t(x_0)\right). \tag{5.25}$$

Corollary 5.8 (Existence of the density)**.** *Let $\vec{\Phi} \in \mathcal{E}_\Sigma$. Then the density θ_{x_0} exists for every point $x \in \mathbb{R}^m$.*

Proof. Let $x_0 \in \mathbb{R}^m$ be an arbitrary point.

Using the Cauchy-Schwarz inequality, we obtain for any $t > 0$:

$$\begin{aligned}
&\frac{1}{t^2}\int_{M\cap B_t(x_0)} \langle x - x_0, \vec{H}\rangle \; dvol_g \\
&\leq \frac{1}{t}\left(\int_{M\cap B_t(x_0)} \frac{|x-x_0|^2}{t^2}\; dvol_g\right)^{1/2}\left(\int_{M\cap B_t(x_0)} |\vec{H}|^2 \; dvol_g\right)^{1/2} \\
&\leq Y(t)^{1/2}\; W(t)^{1/2} \leq \frac{1}{2}\left(Y(t) + W(t)\right),
\end{aligned} \tag{5.26}$$

where we again denote $M := \vec{\Phi}(\Sigma)$. Since $W(\vec{\Phi}) < \infty$, we have $W(t) \to 0$ for $t \to 0^+$ and thus, as a direct consequence of Lemma 5.6, we obtain for $0 < t < T < \infty$,

$$(1 - o_t(1))\; Y(t) \leq Y(T)\;(1 + o(T)) + o_{t,T}(1).$$

Hence,

$$\lim_{t\to 0^+} Y(t) = \theta_{x_0}$$

exists. $\square$

Corollary 5.9 (Li-Yau inequality)**.** *Let $\vec{\Phi} \in \mathcal{E}_\Sigma$. Then for any $x_0 \in \mathbb{R}^m$,*

$$\theta_{x_0} \leq \frac{1}{4\pi}\int_\Sigma |\vec{H}|^2 \; dvol_g. \tag{5.27}$$

Proof. Let $x_0 \in \mathbb{R}^m$ be an arbitrary point. Note that for $T > 0$,

$$\begin{aligned}
&\frac{1}{T^2}\int_{M\cap B_T(x_0)} \langle x - x_0, \vec{H}\rangle \; dvol_g \\
&\leq \frac{1}{T^2}\left(\|\vec{\Phi}\|_{L^\infty(\Sigma)} + |x_0|\right) \mathrm{Area}(\vec{\Phi}(\Sigma))^{1/2}\; W(\vec{\Phi})^{1/2} \\
&\xrightarrow{T\to\infty} 0,
\end{aligned} \tag{5.28}$$

with $M := \vec{\Phi}(\Sigma)$. Moreover,

$$\frac{1}{T^2}\,\mathrm{Area}(\vec{\Phi}(\Sigma)\cap B_T(x_0)) \leq \frac{1}{T^2}\,\mathrm{Area}(\vec{\Phi}(\Sigma)) \xrightarrow{T\to\infty} 0. \tag{5.29}$$

Using Lemma 5.6, (5.26), (5.28) and (5.29) gives

$$\frac{1}{4}\int_\Sigma |\vec{H}|^2 \; dvol_g \geq (1 - o_t(1))Y(t) + o_{1/T}(1).$$

Hence, if we let $T\to\infty$, we obtain

$$\theta_{x_0} = \lim_{t\to 0^+}\frac{1}{\pi}Y(t) \leq \frac{1}{4\pi}\int_\Sigma |\vec{H}|^2 \; dvol_g.$$

□

Corollary 5.10. *Let $\vec{\Phi} \in \mathcal{E}_\Sigma$ and*

$$W(\vec{\Phi}) < 8\pi. \tag{5.30}$$

Then the immersion $\vec{\Phi}$ is in fact an embedding.

Proof. For every point $x_0 \in \mathbb{R}^m$, the density θ_{x_0} exists and is an element of $\mathbb{N}$. If (5.30) holds true, Corollary 5.9 implies that $\theta_{x_0} \in \{0, 1\}$. This means that $\vec{\Phi}$ is an embedding. □

Corollary 5.11. *Let* $\vec{\Phi} \in \mathcal{E}_\Sigma$. *For any* $x_0 \in \Sigma$ *and* $t > 0$,

$$\mathrm{Area}(\vec{\Phi}(\Sigma) \cap B_t(x_0)) \leq \frac{3}{2} W(\vec{\Phi}) t^2. \tag{5.31}$$

Proof. By Lemma 5.6, (5.26), (5.28) and (5.29) we have

$$\frac{1}{2} Y(t) \leq \left(\frac{1}{4} + \frac{1}{2} \right) W(\vec{\Phi}) + o_{1/T}(1).$$

For $T \to \infty$, the result follows. □

Corollary 5.12. *Let* $\vec{\Phi} \in \mathcal{E}_\Sigma$ *and* $x_0 \in \mathbb{R}^m$. *If* $\theta_{x_0} \neq 0$, *then for any* $T > 0$ *we have*

$$\mathrm{Area}(\vec{\Phi}(\Sigma) \cap B_T(x_0)) \geq \frac{2\pi}{3} T^2 - \frac{T^2}{2} \int_{M \cap B_T(x_0)} |\vec{H}|^2 dvol_g. \tag{5.32}$$

Proof. By Lemma 5.6, we obtain for $T > 0$,

$$\begin{aligned}
\pi \theta_{x_0} &= \lim_{t \to 0^+} Y(t) \\
&\leq Y(T) + \frac{1}{4} W(T) + \frac{1}{T^2} \int_{M \cap B_T(x_0)} \langle x - x_0, \vec{H} \rangle \; dvol_g \\
&\leq \frac{3}{2} Y(T) + \frac{3}{4} W(T),
\end{aligned}$$

where we used (5.26) in the last step. Hence, if $\theta_{x_0} \geq 1$, then

$$\mathrm{Area}\left(\vec{\Phi}(\Sigma) \cap B_T(x_0) \right) \geq \frac{2\pi}{3} T^2 - \frac{T^2}{2} W(T).$$

□

5.4 Proof of the almost-weak closure theorem

Proof of Theorem 5.3. Let $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ be a sequence satisfying the assumptions of Theorem 5.3. Let Ψ_k be diffeomorphisms of Σ such that $\vec{\Psi}_k := \vec{\Phi}_k \circ \Psi_k \colon (\Sigma_k, h_k) \to \mathbb{R}^m$ is conformal for any $k \in \mathbb{N}$, where h_k is the metric of constant curvature and unit volume of the conformal structure induced by $\vec{\Phi}_k$.

Define for each $k \in \mathbb{N}$ and $x \in \Sigma$ a number $\rho_{k,x} > 0$ by

$$\rho_{k,x} := \inf \left\{ \rho > 0 \quad \text{s.t.} \quad \int_{B_\rho^{h_k}(x)} |d\vec{n}_{\vec{\Psi}_k}|_{h_k}^2 dvol_{h_k} \geq \frac{8\pi}{3} \right\}. \quad (5.33)$$

and

$$\rho_{\infty,x} := \liminf_{k\to\infty} \rho_{k,x}. \quad (5.34)$$

Step 1: The case when there are no concentration points. We first want to investigate the case in which the second fundamental form does not concentrate anywhere, i.e. we assume that

$$\rho_\infty := \inf_{x\in\Sigma} \rho_{\infty,x} > 0. \quad (5.35)$$

We will prove that under this assumption, no blow-up points occur such that the limit $\vec{\xi}$ is in $\mathcal{E}_\Sigma$.

Step 1a): Translating and dilating $\vec{\Psi}_k$ to control the conformal factors in L^∞ and to bound the image. Assumption (5.35) gives us

$$\int_{B^{h_k}_{\rho_\infty/2}(x)} |d\vec{n}_{\vec{\Psi}_k}|_{h_k}^2 dvol_{h_k} < \frac{8\pi}{3} \quad (5.36)$$

for all $x \in \Sigma$ and large enough $k \geq k_x$. By conformal invariance of the Dirichlet energy, (5.36) implies that assumption (5.19) is satisfied for any sequence of conformal charts $\{\varphi_k^x\}_{k\geq k_x}$ around an arbitrary point $x \in \Sigma$ with $\varphi_k^x(D^2) = B^{h_k}_{\rho_\infty/2}(x)$. Since we further suppose (5.15) and condition (CA), we are in the assumptions of Theorem 5.5. Thus for any $\Omega \Subset D^2$, we have

$$\sup_{x, k \geq k_x} \|\lambda_k^x - c_k^x\|_{L^\infty(\Omega)} \leq C_\Omega, \tag{5.37}$$

where λ_k^x (c_k^x resp.) are the logarithms of the conformal factors (the obtained constants resp.) in the charts φ_k^x.

By compactness, we can extract a finite subcovering $\{B^{h_\infty}_{\rho_\infty/8}(x_i)\}_{i=1,\ldots,n}$ from $\{B^{h_\infty}_{\rho_\infty/8}(x)\}_{x\in\Sigma}$. Then, since $h_k \to h_\infty$, we can assume that the n balls $\{B^{h_k}_{\rho_\infty/4}(x_i)\}_{i=1,\ldots,n}$ also cover Σ, for $k \in \mathbb{N}$ large enough.

We denote $\lambda_k = \alpha_k + \sigma_k$ for $h_k = e^{2\sigma_k}(dx_1^2 + dx_2^2)$ and $\vec{\Psi}_k^* g_{\mathbb{R}^m} = e^{2\alpha_k} h_k$. Again due to the strong convergence $h_k \to h_\infty$, there is a uniform bound

$$\max_{i=1,\ldots,n} \|\sigma_k^{x_i}\|_{L^\infty(B^{h_k}_{\rho_\infty/4}(x_i))} \leq M.$$

This and (5.37) imply for all $k \geq k_0 := \max\{k_{x_1}, \ldots, k_{x_n}\}$ and $i = 1, \ldots, n$,

$$\begin{aligned} &\|\alpha_k - c_k^{x_i}\|_{L^\infty(B^{h_k}_{\rho_\infty/4}(x_i))} \\ &\leq \|\lambda_k^{x_i} - c_k^{x_i}\|_{L^\infty(B^{h_k}_{\rho_\infty/4}(x_i))} + \|\sigma_k^{x_i}\|_{L^\infty(B^{h_k}_{\rho_\infty/4}(x_i))} \\ &\leq M + C_\Omega =: \tilde{C}, \end{aligned} \tag{5.38}$$

where $\Omega \Subset D^2$ is chosen in such a way that $\varphi_k^{x_i}(\Omega) \supset B_{\rho_\infty/4}(x_i)$ for all $i = 1, \ldots, n$.

As a result of (5.38), observe that if $B^{h_k}_{\rho_\infty/4}(x_i) \cap B^{h_k}_{\rho_\infty/4}(x_j) \neq \emptyset$, we have

$$|c_k^{x_i} - c_k^{x_j}| \leq 2\tilde{C}.$$

Since Σ is path-connected, this yields

$$\sup_{k \geq k_0, i, j} |c_k^{x_i} - c_k^{x_j}| \leq n\tilde{C}. \tag{5.39}$$

Next, we compose each $\vec{\Psi}_k$ with a translation and dilation, conformal transformations in $\mathbb{R}^m$, in the following way: Define

$$\tilde{\vec{\Psi}}_k := e^{-c_k} \left(\vec{\Psi}_k - \vec{\Psi}_k(x_0) \right) \tag{5.40}$$

for some $x_0 \in \Sigma$ and $c_k := c_k^{x_1}$.

The Willmore energy does not change, i.e. we have

$$W(\tilde{\vec{\Psi}}_k) = W(\vec{\Psi}_k) = W(\vec{\Phi}_k).$$

What we have achieved by dilating, however, is that $\tilde{\alpha}_k$, the logarithms of the conformal factors of the new conformal immersions $\tilde{\vec{\Psi}}_k$, are uniformly bounded in $L^\infty(D^2)$:

We have $\tilde{\alpha}_k = \alpha_k - c_k^{x_1}$, and thus for $i = 1, \ldots, n$ and all $k \geq k_0$,

$$\begin{aligned}
\|\tilde{\alpha}_k\|_{L^\infty(B^{h_k}_{\rho_\infty/4}(x_i))} &\leq \|\alpha_k - c_k^{x_i}\|_{L^\infty(B^{h_k}_{\rho_\infty/4}(x_i))} + \|c_k^{x_i} - c_k^{x_1}\|_{L^\infty(B^{h_k}_{\rho_\infty/4}(x_i))} \\
&\leq (1+n)\tilde{C},
\end{aligned}$$

by (5.38) and (5.39). Since $\{B^{h_k}_{\rho_\infty/4}(x_i)\}_{i=1,\ldots,n}$ cover Σ, this yields

$$\sup_{k \geq k_0} \|\tilde{\alpha}_k\|_{L^\infty(\Sigma)} \leq (1+n)\tilde{C} =: \hat{C}. \tag{5.41}$$

Further, we performed the translation in order to bound the image of $\vec{\tilde{\Psi}}_k$ uniformly: $\vec{\tilde{\Psi}}_k$ maps $x_0 \in \Sigma$ to $0 \in \mathbb{R}^m$ and for any point $y \in \Sigma$, we have

$$|\vec{\tilde{\Psi}}_k(y) - \vec{\tilde{\Psi}}_k(x_0)| \leq \int_{x_0}^{y} e^{\tilde{\alpha}_k} dl_{h_k} \leq C, \tag{5.42}$$

where the constant is independent of $k \in \mathbb{N}$, since $\sup_k \|\tilde{\alpha}_k\|_{L^\infty(\Sigma)}$ and $\sup_k \operatorname{diam}(\Sigma, h_k)$ are finite. It follows that

$$\vec{\tilde{\Psi}}_k(\Sigma) \subset B_C(0) \quad \text{for all } k \in \mathbb{N}.$$

Step 1b): Weak $W^{2,2}$-convergence of $\vec{\tilde{\Psi}}_k$. Let φ_k be a sequence of conformal charts and denote the logarithms of the conformal factors of $\vec{\tilde{\Psi}}_k \circ \varphi_k$ as $\tilde{\lambda}_k$. Then, by (2.42) the mean curvature vector can be written as

$$\vec{H}_k = \frac{e^{-2\tilde{\lambda}_k}}{2} \Delta \left(\vec{\tilde{\Psi}}_k \circ \varphi_k \right)$$

and for all $k \in \mathbb{N}$,

$$\left\| \Delta \left(\vec{\tilde{\Psi}}_k \circ \varphi_k \right) \right\|_{L^2(D^2)}^2 = \frac{1}{4} \int_{D^2} e^{4\tilde{\lambda}_k} |\vec{H}_k|^2 dx_1 dx_2$$

$$\leq \frac{1}{4} \|e^{2\tilde{\lambda}_k}\|_{L^\infty(D^2)} \int_{\varphi_k(D^2)} |\vec{H}_k|^2 dvol_{g_{\vec{\tilde{\Psi}}_k}} \leq \frac{1}{4} \|e^{2\tilde{\lambda}_k}\|_{L^\infty(D^2)} W(\vec{\Phi}_k) \leq C. \tag{5.43}$$

Moreover, for all $k \in \mathbb{N}$ we have

$$\left\| \nabla \left(\vec{\tilde{\Psi}}_k \circ \varphi_k \right) \right\|_{L^2(D^2)}^2 = 2 \int_{D^2} e^{2\tilde{\lambda}_k} dx_1 dx_2 \leq C. \tag{5.44}$$

(5.43) and (5.44) imply that

$$\sup_k \|\vec{\tilde{\Psi}}_k \circ \varphi_k\|_{W^{2,2}(D^2_{1/2})} < \infty. \tag{5.45}$$

To see this, let μ_k be the solution to

$$\begin{cases} \Delta\mu_k &= \Delta\left(\vec{\tilde{\Psi}}_k \circ \varphi_k\right) \qquad \text{on } D^2 \\ \mu_k &= 0 \qquad\qquad\qquad\quad \text{on } \partial D^2. \end{cases}$$

Then

$$\|\mu_k\|_{W^{2,2}(D^2)} \leq C \left\|\Delta\left(\vec{\tilde{\Psi}}_k \circ \varphi_k\right)\right\|_{L^2(D^2)} \leq C \tag{5.46}$$

for all $k \in \mathbb{N}$. For the harmonic rest $\nu_k := \vec{\tilde{\Psi}}_k \circ \varphi_k - \mu_k$, we get for $l = 1, 2$, all $k \in \mathbb{N}$ and $\Omega \Subset D^2$,

$$\|\nabla^l \nu_k\|_{L^2(\Omega)} \leq C_{l,\Omega} \cdot \|\nu_k\|_{L^1(\partial D^2)} \leq \tilde{C}_{l,\Omega}, \tag{5.47}$$

using $W^{1,2}(D^2) \hookrightarrow L^1(\partial D^2)$, the estimates (5.44) and (5.46) and the Poisson representation formula for harmonic functions on D^2. (5.46) and (5.47) imply the desired result (5.45).

Due to the strong convergence $h_k \to h_\infty$, (5.45) implies

$$\sup_k \|\vec{\tilde{\Psi}}_k\|_{W^{2,2}(\Sigma)} < \infty. \tag{5.48}$$

Thus, we can extract a subsequence $\vec{\tilde{\Psi}}_{k'}$ such that

$$\vec{\tilde{\Psi}}_{k'} \rightharpoonup \vec{\xi}_\infty \qquad \text{weakly in } W^{2,2}(\Sigma) \tag{5.49}$$

for some $\vec{\xi}_\infty \in W^{2,2}(\Sigma)$.

Step 1c): $\vec{\xi}_\infty$ is conformal and $\log|d\tilde{\vec{\Psi}}_k|^2 \overset{*}{\rightharpoonup} \log|d\vec{\xi}_\infty|^2$ in $(L^\infty)^*(\Sigma)$. Applying Rellich/ Kondrachov, (5.49) implies that for any $p < \infty$, there is a further subsequence, also denoted by $\tilde{\vec{\Psi}}_{k'}$, such that

$$d\tilde{\vec{\Psi}}_{k'} \to d\vec{\xi}_\infty \qquad \text{strongly in } L^p(\Sigma).$$

By passing to a further subsequence, again denoted by $\tilde{\vec{\Psi}}_{k'}$, we obtain

$$d\tilde{\vec{\Psi}}_{k'} \to d\vec{\xi}_\infty \qquad \text{a.e. in } \Sigma. \tag{5.50}$$

This implies that in any sequence of conformal charts, we have for $i, j = 1, 2$,

$$e^{2\tilde{\lambda}_{k'}}\delta_{ij} = \partial_{x_i}\tilde{\vec{\Psi}}_{k'}\,\partial_{x_j}\tilde{\vec{\Psi}}_{k'} \to \partial_{x_i}\vec{\xi}_\infty\,\partial_{x_j}\vec{\xi}_\infty \qquad \text{a.e. in } D^2.$$

This yields

$$\partial_{x_i}\vec{\xi}_\infty\,\partial_{x_j}\vec{\xi}_\infty = e^{2\tilde{\lambda}_\infty}\delta_{ij}, \tag{5.51}$$

for $\tilde{\lambda}_\infty := \frac{1}{2}\log\frac{1}{2}|d\vec{\xi}_\infty|^2$. As a result $\vec{\xi}_\infty$ is conformal.

Denoting $g_{\tilde{\vec{\Psi}}_k} = e^{2\tilde{\alpha}_k}h_k$ and $g_{\vec{\xi}_\infty} = e^{2\tilde{\alpha}}h_\infty$, we can use (5.41), (5.50) and the dominated convergence theorem to conclude that

$$\tilde{\alpha}_{k'} \overset{*}{\rightharpoonup} \tilde{\alpha}_\infty \qquad \text{weakly}^* \text{ in } (L^\infty(\Sigma))^*.$$

Step 2: The general case. We consider the general case and drop assumption (5.35), which means that we allow concentration of energy of the second fundamental form. This will imply the occurrence of blow-up points.

Step 2a): Detecting the concentration points. For given $k \in \mathbb{N}$, the collection $\{B^{h_k}_{\rho_{k,x}}(x)\}_{x\in\Sigma}$ forms a Besicovitch covering of Σ, where $\rho_{k,x}$ was defined in (5.33). The Besicovitch covering theorem ([Mat95]) gives a subcovering $\{B^{h_k}_{\rho_{k,i}}(x_i^k)\}_{i\in I_k}$ such that any point in Σ is covered by at most $c_\Sigma \in \mathbb{N}$ balls, where c_Σ does not depend on $k \in \mathbb{N}$. In fact, I_k is finite and its cardinality uniformly bounded in k since

$$
\begin{aligned}
|i \in I_k| \cdot \frac{8\pi}{3} &\leq \sum_{i \in I_k} \int_{B_{\rho_{k,i}}(x_i^k)} |dn_{\vec{\Psi}_k}|_{h_k}^2 \; dvol_{h_k} \\
&= \int_\Sigma \left| i \in I_k : \; x \in B_{\rho_{k,i}}(x_i^k) \right| |dn_{\vec{\Psi}_k}|_{h_k}^2 \; dvol_{h_k} \\
&\leq c_\Sigma \int_\Sigma |dn_{\vec{\Psi}_k}|_{h_k}^2 \; dvol_{h_k} = c_\Sigma \, \mathbb{I}(\vec{\Phi}_k) \leq C.
\end{aligned}
$$

Thus, we can extract a subsequence such that I is independent of k (and finite) and such that for all $i \in I$,

$$x_k^i \to x_\infty^i, \tag{5.52}$$

$$\rho_{k,i} \to \rho_{\infty,i} \tag{5.53}$$

as $k \to \infty$, for some $x_\infty^i \in \Sigma$ and $\rho_{\infty,i} \geq 0$. Let

$$J := \{ i \in I \quad \text{s.t. } \rho_{\infty,i} = 0 \} \quad \text{and} \quad I_0 := I \setminus J.$$

It is clear that $\bigcup_{i \in I_0} \overline{B_{\rho_{\infty,i}}^{h_\infty}(x_\infty^i)}$ covers Σ. Note that the balls $B_{\rho_{\infty,i}}^{h_\infty}(x_\infty^i)$ are strictly convex: this holds if $K_{h_\infty} \leq 0$ and if $K_{h_\infty} = 1$ we assume w.l.o.g. that $\rho_{\infty,i} < \frac{\pi}{2}$. Consequently, the points in Σ which are not contained in the union of the finitely many open balls cannot accumulate and therefore are isolated and hence finite:

$$\{a_1, \ldots, a_N\} := \Sigma \setminus \left(\bigcup_{i \in I_0} B_{\rho_{\infty,i}}^{h_\infty}(x_\infty^i) \right). \tag{5.54}$$

Step 2b): Applying Step 1a away from the concentration points. For arbitrary $i_0 \in I_0$, choose $s^{i_0} < \rho_{\infty,i_0}$. Note that for k large enough, $B^{h_k}_{s_{i_0}}(x^{i_0}_\infty) \subset B^{h_k}_{\rho_{k,i_0}}(x^{i_0}_k)$ because $\rho_{k,i_0} \to \rho_{\infty,i_0}$ and $x^{i_0}_k \to x^{i_0}_\infty$. Therefore we can assume that for all $k \in \mathbb{N}$,

$$\int_{B^{h_k}_{s_{i_0}}(x^{i_0}_\infty)} |d\vec{n}_{\vec{\Psi}_k}|^2_{h_k} dvol_{h_k} < \frac{8\pi}{3}.$$

Theorem 5.5 gives a constant c_k such that

$$\sup_k \|\alpha_k - c_k\|_{L^\infty(B_r(x^{i_0}_\infty))} \leq C_r \tag{5.55}$$

for any $r < s_{i_0}$. Define

$$\vec{\tilde{\Psi}}_k := e^{-c_k}\left(\vec{\Psi}_k - \vec{\Psi}_k(x_0)\right)$$

for some $x_0 \in \Sigma$. Let $K \Subset \Sigma \setminus \{a_1, \ldots, a_N\}$ be any compact set. Note that, due to (5.54),

$$\rho_\infty := \inf_{x \in K} \rho_{\infty,x} > 0, \tag{5.56}$$

where ρ_x was defined in (5.34). Thus, we can apply Step 1a to any such compact K (and since $K \subset \Sigma \setminus (\cup_{i=1,\ldots,N} B_\delta(a_i))$ for some $\delta > 0$, we restrict to compact sets of the latter form). For $\delta < \inf_{i \in I_0} \rho_{\infty,i}$, we obtain

$$\sup_k \|\tilde{\alpha}_k\|_{L^\infty(\Sigma \setminus (\cup_{i=1,\ldots,n} B_\delta(a_i)))} < C_\delta, \tag{5.57}$$

for a constant C_δ depending on δ. Note that (5.57) implies, similarly as in (5.42), that

$$\vec{\tilde{\Psi}}_k \left(\Sigma \setminus (\cup_{i=1,\ldots,n} B_\delta(a_i))\right) \subset B_{C_\delta}(0) \tag{5.58}$$

for $k \in \mathbb{N}$.

Step 2c): Using inversions to control $\tilde{\vec{\Psi}}_k(\Sigma)$. The previous results do not give estimates on what happens in the balls $B_\delta(a_i)$. In order to get an area control on the limiting map, we would like to improve (5.58) and have the entire image $\tilde{\vec{\Psi}}_k(\Sigma)$ contained in a ball.

Since $\tilde{\vec{\Psi}}_k(B_\delta(a_i))$ could degenerate to infinity as $k \to \infty$, the strategy is to bring this back to a ball around 0 by using inversions. If we can find a ball $B_r(p_0)$ such that for all $k \in \mathbb{N}$, $\tilde{\vec{\Psi}}_k(\Sigma) \cap B_r(p_0) = \emptyset$, then composing the maps $\tilde{\vec{\Psi}}_k$ with the inversion $i\colon x \mapsto \frac{x-p_0}{|x-p_0|^2}$ yields $i \circ \tilde{\vec{\Psi}}_k(\Sigma) \subset B_{1/r}(0)$, as desired.

The existence of such $p_0 \in B_1(0) \subset \mathbb{R}^m$, $r > 0$ is given by the following lemma, which follows from the monotonicity formula.

Lemma 5.13. *Let $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ with*

$$\sup_k W(\vec{\Phi}_k) < \infty. \tag{5.59}$$

Then there exists $p_0 \in \mathbb{R}^m$ and $r < 1 - |p_0|$ such that

$$\vec{\Phi}_k(\Sigma) \cap B_r(p_0) = \emptyset \qquad \textit{for all } k \in \mathbb{N}.$$

Proof of Lemma 5.13. Let $S > 0$ and place disjoint balls $B_S(p_i)$ in the unit ball obtaining a total number of balls proportional to $1/S^m$ (consider for instance a grid of length $2S$ and put a ball $B_S(p_i)$ in each cube).

Fix $k \in \mathbb{N}$. If for a ball we have

$$B_{S/2}(p_i) \cap \vec{\Phi}_k(\Sigma) \neq \emptyset,$$

there exists $q_i \in B_{S/2}(p_i)$ with $\theta_{k,q_i} \geq 1$. Since $B_{S/2}(q_i) \subset B_S(p_i)$, Corollary 5.12 gives

$$\text{Area}\left(\vec{\Phi}_k(\Sigma) \cap B_S(p_i)\right) \geq \text{Area}\left(\vec{\Phi}_k(\Sigma) \cap B_{S/2}(q_i)\right)$$

$$\geq \frac{S^2\pi}{6} - \frac{S^2}{8}\int_{B_{S/2}(q_i)} |\vec{H}|^2 dvol_{g_{\vec{\Phi}_k}}.$$

Since the balls $B_S(p_i)$ are disjoint and all contained in $B_1(0)$,

$$\frac{S^2\pi}{6} \cdot \left|\{i \quad \text{s.t. } B_{S/2}(p_i) \cap \vec{\Phi}_k(\Sigma) \neq \emptyset\}\right|$$

$$\leq \text{Area}\left(\vec{\Phi}_k(\Sigma) \cap B_1(0)\right) + \frac{S^2}{8}\int_{B_1(0)} |\vec{H}|^2 dvol_{g_{\vec{\Phi}_k}} \tag{5.60}$$

$$\leq \left(\frac{3}{2} + \frac{S^2}{8}\right) W(\vec{\Phi}_k),$$

where we applied Corollary 5.11 in the last step.

Consequently, due to assumption (5.59), for $S > 0$ chosen small enough, there exists for each $k \in \mathbb{N}$ a point p_{i_k} such that

$$B_{S/2}(p_{i_k}) \cap \tilde{\vec{\Psi}}_k(\Sigma) = \emptyset.$$

(If $\frac{c}{S^m}$ is the total number of balls $B_{S/2}(p_i)$ in $B_1(0)$, choose $S > 0$ in such a way that $\frac{c\pi}{6S^{m-2}} > \left(\frac{3}{2} + \frac{S^2}{8}\right) \sup_k W(\vec{\Phi}_k)$.) Extract a subsequence such that $p_{i_k} = p_0$ is independent of $k \in \mathbb{N}$. $B_{S/2}(p_0)$ is the ball we have been looking for. $\square$

Proof of Theorem 5.3 continued. Recall (2.30) and apply Lemma 5.13 to the sequence $\tilde{\vec{\Psi}}_k$ and let $B_r(p_0)$ be the obtained ball free of mass. Consider the inversion

$$i_0 \colon x \mapsto \frac{x - p_0}{|x - p_0|^2}, \tag{5.61}$$

which is a conformal transformation of $\mathbb{R}^m \cup \{\infty\}$ such that

$$\tilde{\vec{\Psi}}_k(\Sigma) \cap \{\text{center of inversion of } i_0\} = \emptyset.$$

Note that i_0 is a diffeomorphism from $B_R(p_0)\backslash B_r(p_0)$ into $B_{1/r}(0)\backslash B_{1/R}(0)$, for any $R \in (0,\infty)$. Thus,

$$\|\nabla i_0\|_{L^\infty(B_R(p_0)\backslash B_r(p_0))} + \|\nabla i_0^{-1}\|_{L^\infty\left(B_{1/r}(0)\backslash B_{1/R}(0)\right)} \leq C_R. \quad (5.62)$$

Since i_0 is conformal it satisfies the equation

$$di_0(x) = e^{\nu(x)} R$$

for $R \in O(m)$ being some orthogonal matrix. (5.62) implies that for the conformal factor, we have

$$\|\nu\|_{L^\infty(B_R(p_0)\backslash B_r(p_0))} \leq \tilde{C}_R. \quad (5.63)$$

Define for $k \in \mathbb{N}$,

$$\hat{\vec{\Psi}}_k := i_0 \circ \tilde{\vec{\Psi}}_k,$$

and let $\hat{\alpha}_k = \tilde{\alpha}_k + \nu$ denote its conformal factor satisfying $e^{2\hat{\alpha}_k} h_k = g_{\hat{\vec{\Psi}}_k}$.

From (5.58) and the choice of $B_r(p_0)$, we know that for $\delta > 0$ and all $k \in \mathbb{N}$,

$$\tilde{\vec{\Psi}}_k\left(\Sigma \setminus (\cup_{i=1,\dots,n} B_\delta(a_i))\right) \subset B_{C_\delta}(0) \setminus B_r(p_0).$$

Together with (5.57) and (5.63), this implies that for the conformal factors, we have again

$$\sup_k \|\hat{\alpha}_k\|_{L^\infty(\Sigma\backslash(\cup_{i=1,\dots,n} B_\delta(a_i)))} \leq \tilde{C}_\delta. \quad (5.64)$$

What we have gained by inverting is that

$$\hat{\vec{\Psi}}_k(\Sigma) \subset B_{1/r}(0) \qquad \text{for all } k \in \mathbb{N}. \quad (5.65)$$

Corollary 5.11 implies that for all $k \in \mathbb{N}$,

$$\int_\Sigma e^{2\hat{\alpha}_k} dvol_{h_k} = \mathrm{Area}(\hat{\vec{\Psi}}_k(\Sigma)) \leq \frac{3}{2r^2} \sup_k W(\vec{\Phi}_k) \leq C, \quad (5.66)$$

by (2.30).

Step 2d): Weak convergence of $\vec{\tilde{\Psi}}_k$ to $\vec{\xi}_\infty$. Since (5.64) holds for any $\delta > 0$, and due to (5.66), we can argue exactly as in Step 1b) and extract a subsequence such that

$$\vec{\tilde{\Psi}}_{k'} \rightharpoonup \vec{\xi}_\infty \qquad \text{weakly in } W^{2,2}_{\text{loc}}(\Sigma \setminus \{a_1, \dots, a_N\}). \tag{5.67}$$

Furthermore, Step 1c) shows that $\vec{\xi}_\infty$ is conformal and we have

$$\log |d\vec{\tilde{\Psi}}_k|^2 \overset{*}{\rightharpoonup} \log |d\vec{\xi}_\infty|^2 \qquad \text{in } (L^\infty)^*_{\text{loc}}(\Sigma \setminus \{a_1, \dots, a_N\}).$$

It remains to prove Condition iv) from Definition 5.1. Since $h_k \to h_\infty$, (5.66) implies that

$$\sup_k \int_\Sigma |d\vec{\tilde{\Psi}}_k|^2_{h_\infty} \, dvol_{h_\infty} < \infty.$$

Together with (5.67), this implies that for any $\delta > 0$,

$$\int_{\Sigma \setminus (\cup_{i=1,\dots,n} B_\delta(a_i))} |d\vec{\xi}_\infty|^2_{h_\infty} dvol_{h_\infty} \leq \liminf_k \int_\Sigma |d\vec{\tilde{\Psi}}_k|^2_{h_\infty} \, dvol_{h_\infty} \leq C,$$

where C is independent of δ. Thus, $\vec{\xi}_\infty$ extends to a map in $W^{1,2}(\Sigma)$ and we have

$$\vec{\tilde{\Psi}}_k \rightharpoonup \vec{\xi}_\infty \qquad \text{weakly in } W^{1,2}(\Sigma).$$

By (5.65),

$$\sup_k \|\vec{\tilde{\Psi}}_k\|_{L^\infty(\Sigma)} < \infty$$

which implies in a similar way

$$\vec{\tilde{\Psi}}_k \overset{*}{\rightharpoonup} \vec{\xi}_\infty \qquad \text{weakly in } (L^\infty)^*(\Sigma).$$

This finishes the proof of Theorem 5.3 for $\vec{\xi}_k := \vec{\tilde{\Psi}}_k$. $\square$

6 Weak branched immersions

6.1 Expansion at a blow-up point

Motivated by the compactness result of Theorem 5.3, the purpose of this section is to find out more about the limit object $\vec{\xi}$ of a weakly convergent sequence $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ satisfying $\sup_k \mathbb{I}(\vec{\Phi}_k) < \infty$. The first observation is on the Gauss map $\vec{n}_{\vec{\xi}}$.

Lemma 6.1. *Let $\vec{\xi}$ be the weak limit of a weakly convergent sequence $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ in the sense of Definition 5.1 which satisfies* $\sup_k \mathbb{I}(\vec{\Phi}_k) < \infty$. *Then*

$$\vec{n}_{\vec{\xi}} \in W^{1,2}(\Sigma).$$

Proof. Denote the blow-up points of $\vec{\xi}$ by $a_1, \ldots, a_N$. If follows from Lemma 5.2 that for any $\delta > 0$, we have

$$\begin{aligned}\int_{\Sigma \setminus (\cup_{i=1,\ldots,n} B_\delta(a_i))} |d\vec{n}_{\vec{\xi}}|_g^2 \, dvol_g &\leq \liminf_k \int_\Sigma |d\vec{n}_{\vec{\Phi}_k}|_{g_k}^2 \, dvol_{g_k} \\ &= \liminf_k \mathbb{I}(\vec{\Phi}_k) \leq C.\end{aligned}$$

Hence, $\vec{n}_{\vec{\xi}} \in W^{1,2}_{\mathrm{loc}}(\Sigma \setminus \{a_1, \ldots, a_N\})$ and since C is independent of δ, $\vec{n}_{\vec{\xi}}$ extends to a map in $W^{1,2}(\Sigma)$. □

The following lemma helps to understand the behavior of $\vec{\xi}$ at its blow-up points.

Lemma 6.2. *Let $\vec{\xi} \colon D^2 \to \mathbb{R}^m$ be a weakly conformal map such that* $\log |\nabla \vec{\xi}| \in L^\infty_{loc}(D^2 \setminus \{0\})$ *and $\vec{\xi} \in W^{2,2}_{loc}(D^2 \setminus \{0\})$. Assume $\vec{\xi}$ extends to a map in $W^{1,2}(D^2)$ and that the corresponding Gauss map $\vec{n}_{\vec{\xi}}$ also extends to a map in $W^{1,2}(D^2, Gr_{m-2}(\mathbb{R}^m))$.*

Then $\vec{\xi} \in W^{1,\infty}(D^2)$ and there exists $n \in \mathbb{N} \setminus \{0\}$ and a constant C such that

$$(C - o(1))\ |z|^{n-1} \leq |\partial_z \vec{\xi}| \leq (C + o(1))\ |z|^{n-1}. \tag{6.1}$$

Remark 6.3. (6.1) *tells us that the behavior of $\vec{\xi}$ at its blow-up point is just the one of a holomorphic curve such as*

$$\mathbb{C} \to \mathbb{C}^2, \quad z \mapsto (z^2, z^3).$$

We thus call a blow-up point branch point *if it has positive branching order $n - 1 > 0$, where $n \in \mathbb{N} \setminus \{0\}$ is given by Lemma 6.2. (Note that if $n - 1 = 0$, there is no branching and we can remove the singularity.)*

Proof of Lemma 6.2. We can localize in order to ensure that

$$\int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|^2\ dx\, dy < \frac{8\pi}{3}.$$

Exactly as in Subsection 4.2, using Hélein's lifting theorem, we deduce the existence of a framing

$$\vec{e} := (\vec{e}_1, \vec{e}_2) \in W^{1,2}(D^2, S^{m-1} \times S^{m-1})$$

such that

$$\langle \vec{e}_1, \vec{e}_2 \rangle = 0, \qquad \vec{n}_{\vec{\xi}} = \star(\vec{e}_1 \wedge \vec{e}_2), \tag{6.2}$$

$$\int_{D^2} \left[|\nabla \vec{e}_1|^2 + |\nabla \vec{e}_2|^2 \right]\ dx\, dy \leq C \int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|^2\ dx\, dy \tag{6.3}$$

and satisfying the Coulomb conditon

$$\begin{cases} div(\vec{e}_1, \nabla \vec{e}_2) = 0 & \text{in } D^2 \\[2mm] \left(\vec{e}_1, \dfrac{\partial \vec{e}_2}{\partial \nu} \right) = 0 & \text{on } \partial D^2. \end{cases} \tag{6.4}$$

We introduce $e_i := d\vec{\xi}^{-1}\vec{e}_i$ and e_i^* to be the dual framing. Denoting $|\partial_x\vec{\xi}|^2 = |\partial_y\vec{\xi}|^2 = e^{2\lambda}$ we have that the metric $g := \vec{\xi}^* g_{\mathbb{R}^m}$ is given by $g = e^{2\lambda}\ \left[dx^2 + dy^2\right]$. Hence with respect to the flat metric $g_0 := \left[dx^2 + dy^2\right]$ one has

$$|e_i|^2_{g_0} = g_0(e_i, e_i) = e^{-2\lambda}\ g(e_i, e_i) = e^{-2\lambda}.$$

and since $e_j^*(e_i) = \delta_{ij}$ we have that $|e_i^*|^2_{g_0} = e^{2\lambda}$. Since $\vec{\xi}$ is assumed to be in $W^{1,2}(D^2)$, we deduce that

$$e_i^* \in L^2(D^2).$$

Since $\vec{\xi}$ is in $W^{1,\infty} \cap W^{2,2}_{loc}(D^2 \setminus \{0\}, \mathbb{R}^m)$ and $\log|\nabla\vec{\xi}| \in L^\infty_{loc}(D^2 \setminus \{0\})$ we have that the framing given by $\vec{f}_i := e^{-\lambda}\ \partial_{x_i}\vec{\xi}$ is in $L^\infty_{loc} \cap W^{1,2}_{loc}(D^2 \setminus \{0\}, \mathbb{R}^m)$. Since $\vec{\xi}$ is conformal the unit framing $(\vec{f}_1, \vec{f}_2)$ is Coulomb:

$$div(\vec{f}_1, \nabla\vec{f}_2) = 0 \qquad \text{in } D^2 \setminus \{0\}.$$

Denoting by $e^{i\theta}$ the rotation which passes from $(\vec{f}_1, \vec{f}_2)$ to $(\vec{e}_1, \vec{e}_2)$, the Coulomb condition satisfied by the two framings implies that θ is harmonic on $D^2 \setminus \{0\}$ and hence analytic on this domain. This implies that

$$e_i^* \in L^\infty_{loc} \cap W^{1,2}_{loc}(D^2 \setminus \{0\}).$$

As in Subsection 4.3 we introduce $f \in W^{1,2}(D^2)$ as the solution to

$$\left\{\begin{array}{rcll} df & = & *_g\langle\vec{e}_1, d\vec{e}_2\rangle & \text{on } D^2 \\[3mm] \displaystyle\int_{\partial D^2} f & = & 0. & \end{array}\right. \tag{6.5}$$

Then f satisfies

$$\left\{\begin{array}{rcll} \Delta_{g_0} f & = & (\nabla^\perp\vec{e}_1, \nabla\vec{e}_2) & \text{on } D^2 \\[3mm] f & = & 0 & \text{on } \partial D^2 \end{array}\right.$$

and Theorem 3.7 implies that $f \in C^0(D^2)$. As in Subsection 4.3, we obtain for $i = 1, 2$

$$d[e^{-f} e_i^*] = 0 \qquad \text{in } \mathcal{D}'(D^2 \setminus \{0\}).$$

By the Schwartz Lemma the distribution $d[e^{-f} e_i^*]$ is a finite linear combination of successive derivatives of the Dirac Mass at the origin but since $e^{-f} e_i^* \in L^2(D^2)$, this linear combination can only be 0. Hence we have for $i = 1, 2$

$$d[e^{-f} e_i^*] = 0 \qquad \text{in } \mathcal{D}'(D^2).$$

Hence, by Poincaré's Lemma, there exists $(\sigma_1, \sigma_2) \in W^{1,2}(D^2, \mathbb{R}^2)$ such that

$$d\sigma_i = e^{-f} e_i^*.$$

The dual basis $(\partial/\partial\sigma_1, \partial/\partial\sigma_2) = e^f(e_1, e_2)$ is positive and orthogonal on $D^2 \setminus \{0\}$. Hence $\sigma = \sigma_1 + i\sigma_2$ is an holomorphic function on $D^2 \setminus \{0\}$ which extends to a $W^{1,2}$-map on D^2. The classical point removability theorem for holomorphic maps implies that σ extends to an holomorphic function on D^2. Possibly after modifying σ by a constant, we can assume that $\sigma(0) = 0$. The holomorphicity of σ implies in particular that

$$|d\sigma|_{g_0} = \sqrt{2}\, e^{\lambda - f}$$

is uniformly bounded and, since $f \in L^\infty(D^2)$, we deduce that λ is bounded from above on D^2. This fact implies that $\vec{\xi}$ extends to a Lipschitz map on D^2. Though $|d\sigma|_{g_0}$ has no zero on D^2, σ' might have a zero at the origin: there exists an holomorphic function $h(z)$ on D^2 satisfying $h(0) = 0$, a complex number c_0 and an integer n such that

$$\sigma(z) = c_0\, z^n\, (1 + h(z)). \tag{6.6}$$

We have that locally

$$\partial_\sigma\vec{\xi} = \partial_{\sigma_1}\vec{\xi} - i\partial_{\sigma_2}\vec{\xi} = d\vec{\xi}(e^f e_1) - i d\vec{\xi}(e^f e_2) = e^f[\vec{e}_1 - i\vec{e}_2].$$

Hence, since f is continuous, we have that

$$|\partial_\sigma\vec{\xi}| = \sqrt{2}\ e^{f(0)}\ (1 + o(1)). \tag{6.7}$$

Combining (6.6) and (6.7) gives

$$|\partial_z\vec{\xi}| = |\partial_\sigma\vec{\xi}|\ \ |\partial_z\sigma| = c_0\ n\ \sqrt{2}\ e^{f(0)}\ |z|^{n-1}\ (1 + o(1)). \tag{6.8}$$

This last identity implies (6.1). $\square$

Definition 6.4. *Let (Σ, h) be a conformal structure on Σ, where h denotes the associated metric of constant curvature and unit volume. The space $\mathcal{F}^{conf}_{(\Sigma,h)}$ denotes the set of measurable maps $\vec{\xi}\colon \Sigma \to \mathbb{R}^m$ that satisfy*

i) $\vec{\xi} \in W^{1,\infty}(\Sigma)$*;*

ii) $\vec{\xi}\colon (\Sigma, h) \to \mathbb{R}^m$ *is weakly conformal;*

iii) there exist finitely many blow-up points $a_1, \dots a_N \in \Sigma$ s.t.

$$\log|d\vec{\xi}| \in L^\infty_{\text{loc}}(\Sigma \setminus \{a_1, \dots, a_N\});$$

iv) $\vec{n}_{\vec{\xi}} \in W^{1,2}(\Sigma, Gr_{m-2}(\mathbb{R}^m))$.

Remark 6.5. *Let $\vec{\Phi}_k$ be a sequence in $\mathcal{E}_\Sigma$ with $\sup_k \mathbb{I}(\vec{\Phi}_k) < \infty$. Let h_k denote the respective metrics of constant curvature and unit volume of the induced conformal structures, which are assumed to satisfy condition (CA) with $h_k \to h_\infty$. Suppose $\vec{\Phi}_k$ weakly converges to $\vec{\xi}_\infty$ in the sense of Definition 5.1. Then Lemma 6.1 and Lemma 6.2 imply that $\vec{\xi}_\infty$ is an element of $\mathcal{F}^{conf}_{(\Sigma,h_\infty)}$.*

We are now ready to introduce the space of *weak branched immersions*, which contains the closure of $\mathcal{E}_\Sigma$ under weak convergence.

Definition 6.6. *Define the space $\mathcal{F}_\Sigma$ of* weak branched immersions *as the space of measurable maps $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$ such that there exists a bi-Lipschitz diffeomorphism Ψ of Σ and a conformal structure on Σ, with h being the associated constant curvature metric of unit volume, such that $\vec{\Phi} \circ \Psi \in \mathcal{F}^{conf}_{(\Sigma,h)}$.*

Let $\vec{\xi} \in \mathcal{F}^{\mathrm{conf}}_{(\Sigma,h)}$ be a weak branched conformal immersion with branch points $\{b_j\}$ and respective branching orders $\{n_j - 1\}$, given by Lemma 6.2. Taking isothermal coordinates around b_j, Lemma 6.2 gives us information on the behavior of the conformal factor

$$\lambda = \log|\partial_{x_1}\vec{\xi}| = \log|\partial_{x_2}\vec{\xi}| = \log|\partial_z\vec{\xi}| - \log\sqrt{2}$$

at $0 = \psi^{-1}(b_j)$. More specifically, we have

$$\begin{cases} -\Delta\lambda &=& e^{2\lambda}K & \text{in } D^2 \setminus \{0\} \\ \\ \lambda(z) &=& (n_j - 1)\log|z| + O(1) & \text{in } D^2, \end{cases} \tag{6.9}$$

where we used Lemma 2.5 on the regular part of $\vec{\xi}$.

$$-\Delta\lambda - e^{2\lambda}K \tag{6.10}$$

is a distribution on D^2 and its support is contained in $\{0\}$. The Schwartz Lemma implies that it is a finite linear combination of δ_0 and its derivatives. Since (6.10) is in $\bigcap_{p<\infty} L^p(D^2)$, standard techniques show that no derivatives of δ_0 can occur. The second line in (6.9) yields that

$$-\Delta\lambda = e^{2\lambda}K - 2\pi(n_j - 1)\delta_0. \tag{6.11}$$

In the same way, Lemmas 2.6 and 6.2 imply the following extension of Liouville's equation to maps in $\mathcal{F}_\Sigma^{\mathrm{conf}}$.

Lemma 6.7. *Let $\vec{\xi} \in \mathcal{F}_{(\Sigma,h)}^{conf}$ have the branch points $b_1, \ldots, b_N$ with respective branching orders $n_1 - 1, \ldots, n_N - 1 \in \mathbb{N} \setminus \{0\}$, given by Lemma 6.2 and let*

$$g = e^{2\alpha} h.$$

Then α satisfies the following PDE in $\mathcal{D}'(\Sigma)$:

$$-\Delta_h \alpha = e^{2\alpha} K_g - K_h - 2\pi \sum_{j=1}^{N} (n_j - 1)\delta_{b_j}. \tag{6.12}$$

The following lemma gives a control of the branch points with multiplicity.

Lemma 6.8. *Let $\vec{\xi} \in \mathcal{F}_{(\Sigma,h)}^{conf}$ have the branch points $b_1, \ldots, b_N$ with respective branching orders $n_1 - 1, \ldots, n_N - 1 \in \mathbb{N} \setminus \{0\}$, given by Lemma 6.2 and let*

$$g = e^{2\alpha} h.$$

Then

$$\sum_{j=1}^{N} (n_j - 1) \leq \frac{1}{4\pi} \mathbb{I}(\vec{\xi}) - \chi(\Sigma). \tag{6.13}$$

Proof of Lemma 6.8. Applying the identity (6.12) of distributions to the constant function 1 on Σ yields

$$\begin{aligned} 2\pi \sum_{j=1}^{N} (n_j - 1) &= \int_\Sigma e^{2\alpha} K_g \, dvol_h - \int_\Sigma K_h \, dvol_h \\ &= \int_\Sigma K_g \, dvol_g - 2\pi\chi(\Sigma) \leq \frac{1}{2}\mathbb{I}(\vec{\xi}) - 2\pi\chi(\Sigma), \end{aligned}$$

where we used (2.23) in the last step. □

6.2 Weak sequentially closedness of $\mathcal{F}_\Sigma$

We called Theorem 5.3 a weak "almost-closure theorem" because starting from a sequence $\vec{\Phi}_k$ in $\mathcal{E}_\Sigma$ the weak limit map $\vec{\xi}_\infty$ is in general not contained in the class $\mathcal{E}_\Sigma$, but only in the strictly larger space $\mathcal{F}_\Sigma$. In the following theorem we show that the space $\mathcal{F}_\Sigma$ of weak branched immersions is in fact closed under weak convergence, i.e. we obtain a *weak closure theorem*.

Theorem 6.9 (Weak closure theorem)**.** *Let $\vec{\Phi}_k \in \mathcal{F}_\Sigma$ be a sequence such that*

$$\sup_k \mathbb{I}(\vec{\Phi}_k) < \infty. \tag{6.14}$$

Suppose assumption (CA) is satisfied and thus, up to subsequences, for the constant curvature metrics h_k of unit volume of the conformal structures induced by $\vec{\Phi}_k$, we have

$$h_k \to h_\infty \qquad \text{in } C^l(\Sigma), \quad \text{for all } l \in \mathbb{N},$$

for h_∞ being the constant curvature metric of unit volume of some conformal structure on Σ.

Then there exists a subsequence of $\vec{\Phi}_k$ which, in the sense of Definition 5.1, weakly converges to an element of the space $\mathcal{F}^{conf}_{(\Sigma,h_\infty)} \subset \mathcal{F}_\Sigma$.

Proof of Theorem 6.9. Compose with diffeomorphisms Ψ_k to obtain $\vec{\xi}_k := \vec{\Phi}_k \circ \Psi_k \in \mathcal{F}^{\mathrm{conf}}_{(\Sigma,h_k)}$. Denote the branch points of $\vec{\xi}_k$ by $b_1^k, \dots, b_{N_k}^k$ with respective branching orders $n_1^k - 1, \dots, n_{N_k}^k - 1 \in \mathbb{N} \setminus \{0\}$. (6.12) and (6.13) imply that

$$\|\Delta \alpha_k\|_{\mathcal{M}(\Sigma)} \leq 2\left(\frac{1}{2}\mathbb{I}(\vec{\Phi}_k) - 2\pi\chi(\Sigma)\right) \leq C.$$

Consequently, Theorem 3.6 implies that, as in the unbranched case, we get a global bound

$$\sup_k \|d\alpha_k\|_{L^{2,\infty}(\Sigma)} < \infty.$$

Thus, Theorem 5.5 holds true for a sequence of isothermal charts φ_k satisfying

$$\sup_k \int_{D^2} |\nabla \vec{n}_{\vec{\xi}_k \circ \varphi_k}|^2 dx_1 dx_2 < \frac{8\pi}{3} \tag{6.15}$$

and containing none of the branch points $b_1^k, \ldots, b_{N_k}^k$.

Lemma 6.8 and condition (6.14) imply that the number of branch points is uniformly bounded:

$$N_k \leq \sum_{j=1}^{N_k} (n_j^k - 1) \leq \frac{1}{4\pi} \mathbb{I}(\vec{\Phi}_k) - \chi(\Sigma) \leq C.$$

Hence, we can extract a subsequence of $\vec{\xi}_k$ such that $N_{k'} := N_0$ is independent of k'.

We perform Step 2a) in the proof of Theorem 5.3 with the only difference that when extracting a further subsequence in order to obtain (5.52) and (5.53), we do this in such a way that additionally, for each $j \in \{1, \ldots, N_0\}$ there exists $b_j^\infty \in \Sigma$ with

$$b_j^{k'} \to b_j^\infty, \tag{6.16}$$

as $k' \to \infty$. Define

$$\{d_1, \ldots, d_M\} := \{b_1^\infty, \ldots, b_{N_0}^\infty\} \cup \{a_1, \ldots, a_N\}, \tag{6.17}$$

where the latter set of points is as defined in (5.54).

In Step 2b), if $B_{s_{i_0}}^{h_\infty}(x_\infty^{i_0})$ contains any of the points $b_1^\infty, \ldots, b_{N_0}^\infty$, choose a smaller ball $\hat{B}_t \subset B_{s_{i_0}}^{h_\infty}(x_\infty^{i_0})$ that is free of these points.

Let $0 < \delta < \min\{\min_{i\in I_0} \rho_{i,\infty}, t\}$ be arbitrary small. Step 1a) can now be applied to $K = \Sigma \setminus \cup_{i=1}^{M} B_\delta(d_i)$, with

$$\rho_\infty = \min\left\{\inf_{x\in K} \rho_{x,\infty}, \frac{\delta}{2}\right\}.$$

The choice of ρ_∞ makes sure that Theorem 5.5 can be applied, i.e. we have a cover of balls satisfying (6.15) and *containing no branch points*.

The rest of Step 2b) and Steps 2c) and d) imply weak convergence to some $\vec{\xi}_\infty \in \mathcal{F}^{\mathrm{conf}}_{(\Sigma,h_\infty)}$ with blow-up points $d_1, \ldots, d_M$. $\square$

Weak immersions of surfaces with L^2-bounded second fundamental form

Lecture 4

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Tristan Rivière

Lecture Notes written by Nora Loose

7 The Willmore surface equation

In this section we shall work mostly in the simplest framework of weak immersions into $\mathbb{R}^3$, though most of the arguments below can be extended to the most general framework of weak immersions into arbitrary Riemannian manifolds ([Riv08], [MR13]).

As always, Σ denotes a smooth closed oriented 2-dimensional manifold. In this section we will work locally in conformal charts most of the time.

We first introduce the notion of *weak Willmore immersions.*

Definition 7.1. *Let $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$ be a weak immersion in $\mathcal{E}_\Sigma$. $\vec{\Phi}$ is a* critical point *for W if*

$$\forall \vec{w} \in C^\infty(\Sigma, \mathbb{R}^m) \qquad \frac{d}{dt} W(\vec{\Phi} + t\vec{w})\bigg|_{t=0} = 0. \tag{7.1}$$

Such a weak immersion is called W*illmore.*

Weak Willmore immersions are characterized by an Euler-Lagrange equation. We want to study it for the case $m = 3$.

Theorem 7.2 (Willmore surface equation)**.** *Let $\vec{\Phi}\colon D^2 \to \mathbb{R}^3$ be a conformal weak immersion in $\mathcal{E}_{D^2}$. Φ is Willmore if and only if*

$$div \left[2\nabla \vec{H} - 3H \ \nabla \vec{n} + \vec{H} \times \nabla^\perp \vec{n} \right] = 0 \quad in \ \mathcal{D}'(D^2). \tag{7.2}$$

Since it is a local property for a weak immersion $\vec{\Phi} \in \mathcal{E}_\Sigma$ to be Willmore, one can work locally in a disc-neighborhood of a point and use isothermal coordinates on this disc. Thus Theorem 7.2 implies immediately the following.

Corollary 7.3. *$\vec{\Phi} \in \mathcal{E}_\Sigma$ is a weak Willmore immersion if and only if in any conformal chart, the following holds true:*

$$div\left[2\nabla \vec{H} - 3H\ \nabla\vec{n} + \vec{H} \times \nabla^\perp \vec{n}\right] = 0 \quad in\ \mathcal{D}'(D^2). \tag{7.3}$$

Proof of Theorem 7.2. Let $\vec{\Phi}\colon \Sigma \to \mathbb{R}^3$ be conformal and in $\mathcal{E}_{D^2}$, and let

$$\vec{\Phi}_t := \vec{\Phi} + t\vec{w}$$

be any variation, for $\vec{w} \in C_c^\infty(D^2, \mathbb{R}^3)$. For $|t|$ small enough, $\vec{\Phi}_t$ is still in $\mathcal{E}_{D^2}$ and we consider

$$W(\vec{\Phi}_t) = \int_{D^2} |\vec{H}_t|^2\ dvol_{g_t} = \int_{D^2} H_t^2\ dvol_{g_t},$$

where $\vec{H}_t := H_t\ \vec{n}_t$. We want to compute

$$\frac{d}{dt}W(\vec{\Phi}_t)\bigg|_{t=0} = 2\int_{D^2} H\ \frac{d}{dt}H_t\bigg|_{t=0}\ dvol_g + \int_{D^2} H^2\ \frac{d}{dt}\left(dvol_{g_t}\right)\bigg|_{t=0}. \tag{7.4}$$

Recall from (2.16) that $H_t = \frac{1}{2}\sum_{i,j}(g_t)^{ij}(\mathbb{I}_t)_{ij}$ where we denote $(\mathbb{I}_t)_{ij} = \langle \vec{\mathbb{I}}_t(\partial_{x_i}, \partial_{x_j}), \vec{n}_t\rangle = -\langle \partial_{x_i}\vec{n}_t, \partial_{x_j}\vec{\Phi}\rangle$. Consequently,

$$\begin{aligned}\frac{d}{dt}H_t\bigg|_{t=0} = -\frac{1}{2}\sum_{i,j} &\frac{d}{dt}(g_t)^{ij}\bigg|_{t=0}\ \langle \partial_{x_i}\vec{n}, \partial_{x_j}\vec{\Phi}\rangle \\ &+ g^{ij}\ \left[\left\langle \partial_{x_i}\ \frac{d}{dt}\vec{n}_t\bigg|_{t=0}, \partial_{x_j}\vec{\Phi}\right\rangle + \langle \partial_{x_i}\vec{n}, \partial_{x_j}\vec{w}\rangle\right].\end{aligned} \tag{7.5}$$

We have $(g_t)_{ij} = \langle \partial_{x_i}\vec{\Phi}_t, \partial_{x_j}\vec{\Phi}_t\rangle$, thus

$$\frac{d}{dt}(g_t)_{ij}\bigg|_{t=0} = \langle \partial_{x_i}\vec{w}, \partial_{x_j}\vec{\Phi}\rangle + \langle \partial_{x_i}\vec{\Phi}, \partial_{x_j}\vec{w}\rangle. \tag{7.6}$$

Since $\sum_i (g_t)^{ki}(g_t)_{ij} = \delta_{kj}$ and $g_{ij} = e^{2\lambda}\, I_2$, where I_2 is the (2×2)-identity matrix, we have

$$\frac{d}{dt}(g_t)^{kj}\bigg|_{t=0} e^{2\lambda} + \; e^{-2\lambda}\; \frac{d}{dt}(g_t)_{kj}\bigg|_{t=0} = 0, \tag{7.7}$$

from which we deduce

$$\frac{d}{dt}(g_t)^{kj}\bigg|_{t=0} = -e^{-4\lambda}\; \frac{d}{dt}(g_t)_{kj}\bigg|_{t=0} = -e^{-4\lambda}\; \left(\langle \partial_{x_k}\vec{w}, \partial_{x_j}\vec{\Phi}\rangle + \langle \partial_{x_k}\vec{\Phi}, \partial_{x_j}\vec{w}\rangle\right). \tag{7.8}$$

Note that we can write

$$\frac{d}{dt}\vec{n}_t\bigg|_{t=0} = a\,\vec{e}_1 + b\,\vec{e}_2,$$

for two functions a and b. They can be identified as follows:

$$e^{\lambda}\, a = \left\langle e^{\lambda}\,\vec{e}_1, \frac{d}{dt}\vec{n}_t\bigg|_{t=0}\right\rangle = -\left\langle \frac{d}{dt}\left(\partial_{x_1}\vec{\Phi}_t\right)\bigg|_{t=0}, \vec{n}\right\rangle = -\langle \partial_{x_1}\vec{w}, \vec{n}\rangle,$$

and similarly one obtains $b = -e^{-\lambda}\,\langle \partial_{x_2}\vec{w}, \vec{n}\rangle$. Thus,

$$\begin{aligned}
&\sum_{i,j} g^{ij}\, \left\langle \partial_{x_i}\, \frac{d}{dt}\vec{n}_t\bigg|_{t=0}, \partial_{x_j}\vec{\Phi}\right\rangle \\
&= -e^{-2\lambda}\,\partial_{x_1}\left(e^{-2\lambda}\,\langle\langle \partial_{x_1}\vec{w}, \vec{n}\rangle\; \partial_{x_1}\vec{\Phi} + \langle \partial_{x_2}\vec{w}, \vec{n}\rangle\; \partial_{x_2}\vec{\Phi}, \partial_{x_1}\vec{\Phi}\rangle\right) \\
&\quad - e^{-2\lambda}\,\partial_{x_2}\left(e^{-2\lambda}\,\langle\langle \partial_{x_1}\vec{w}, \vec{n}\rangle\; \partial_{x_1}\vec{\Phi} + \langle \partial_{x_2}\vec{w}, \vec{n}\rangle\; \partial_{x_2}\vec{\Phi}, \partial_{x_2}\vec{\Phi}\rangle\right) \\
&= -e^{-2\lambda}\, \left(\partial_{x_1}\langle \partial_{x_1}\vec{w}, \vec{n}\rangle + \partial_{x_2}\langle \partial_{x_2}\vec{w}, \vec{n}\rangle\right).
\end{aligned} \tag{7.9}$$

Observe

$$
\begin{aligned}
\frac{d}{dt}\left(dvol_{g_t}\right)\Big|_{t=0} &= \frac{d}{dt}\left(det(g_t)_{ij}\right)^{1/2}\Big|_{t=0} dx_1\wedge dx_2 \\
&= \frac{1}{2}e^{-2\lambda}\ \ \frac{d}{dt}\left((g_t)_{11}\ (g_t)_{22}-(g_t)^2_{12}\right)\Big|_{t=0} dx_1\wedge dx_2 \\
&= \frac{1}{2}\left(\frac{d}{dt}(g_t)_{11}\Big|_{t=0}+\frac{d}{dt}(g_t)_{22}\Big|_{t=0}\right)dx_1\wedge dx_2 \\
&= \left(\langle\partial_{x_1}\vec{\Phi},\partial_{x_1}\vec{w}\rangle+\langle\partial_{x_2}\vec{\Phi},\partial_{x_2}\vec{w}\rangle\right)dx_1\wedge dx_2.
\end{aligned}
\tag{7.10}
$$

where the last step is due to (7.6).

Plugging (7.5), (7.7), (7.9) and (7.10) in (7.4) yields

$$
\begin{aligned}
&\frac{d}{dt}W(\vec{\Phi}_t)\Big|_{t=0} \\
&= \int_{D^2} H\, e^{-4\lambda}\sum_{i,j}\left(\langle\partial_{x_i}\vec{w},\partial_{x_j}\vec{\Phi}\rangle+\langle\partial_{x_i}\vec{\Phi},\partial_{x_j}\vec{w}\rangle\right)\langle\partial_{x_i}\vec{n},\partial_{x_j}\vec{\Phi}\rangle\, e^{2\lambda}\, dx_1dx_2 \\
&\quad+\int_{D^2} H\, e^{-2\lambda}\left(\partial_{x_1}\langle\partial_{x_1}\vec{w},\vec{n}\rangle+\partial_{x_2}\langle\partial_{x_2}\vec{w},\vec{n}\rangle\right)\, e^{2\lambda}\, dx_1dx_2 \\
&\quad-\int_{D^2} H\, e^{-2\lambda}\left(\langle\partial_{x_1}\vec{n},\partial_{x_1}\vec{w}\rangle+\langle\partial_{x_2}\vec{n},\partial_{x_2}\vec{w}\rangle\right)\, e^{2\lambda}\, dx_1dx_2 \\
&\quad+\int_{D^2} H^2\ \left(\langle\partial_{x_1}\vec{\Phi},\partial_{x_1}\vec{w}\rangle+\langle\partial_{x_2}\vec{\Phi},\partial_{x_2}\vec{w}\rangle\right)\, dx_1dx_2 \\
&= \int_{D^2} H\, e^{-2\lambda}\sum_{i,j}\langle\partial_{x_i}\vec{n},\partial_{x_j}\vec{\Phi}\rangle\langle\partial_{x_j}\vec{\Phi},\partial_{x_i}\vec{w}\rangle\, dx_1dx_2
\end{aligned}
$$

$$+\int_{D^2} H\ \left(\partial_{x_1}\langle\partial_{x_1}\vec{w},\vec{n}\rangle+\partial_{x_2}\langle\partial_{x_2}\vec{w},\vec{n}\rangle\right)\, dx_1dx_2$$

$$+\int_{D^2} H^2\ \left(\langle\partial_{x_1}\vec{\Phi},\partial_{x_1}\vec{w}\rangle+\langle\partial_{x_2}\vec{\Phi},\partial_{x_2}\vec{w}\rangle\right)\, dx_1dx_2,$$

where we used that

$$\langle\partial_{x_1}\vec{n},\partial_{x_1}\vec{w}\rangle+\langle\partial_{x_2}\vec{n},\partial_{x_2}\vec{w}\rangle=\sum_{i,j}\langle\partial_{x_i}\vec{n},\partial_{x_j}\vec{\Phi}\rangle\langle\partial_{x_j}\vec{\Phi},\partial_{x_i}\vec{w}\rangle.$$

and

$$\langle\partial_{x_i}\vec{n},\partial_{x_j}\vec{\Phi}\rangle=\langle\partial_{x_j}\vec{n},\partial_{x_i}\vec{\Phi}\rangle.$$

Partial integration gives

$$\begin{aligned}
&\left.\frac{d}{dt}W(\vec{\Phi}_t)\right|_{t=0}\\
&=-\int_{D^2}\left\langle\partial_{x_1}\left(He^{-2\lambda}\langle\partial_{x_1}\vec{n},\partial_{x_1}\vec{\Phi}\rangle\,\partial_{x_1}\vec{\Phi}+He^{-2\lambda}\langle\partial_{x_1}\vec{n},\partial_{x_2}\vec{\Phi}\rangle\,\partial_{x_2}\vec{\Phi}\right),\vec{w}\right\rangle dx_1dx_2\\
&\quad-\int_{D^2}\left\langle\partial_{x_2}\left(He^{-2\lambda}\langle\partial_{x_2}\vec{n},\partial_{x_1}\vec{\Phi}\rangle\,\partial_{x_1}\vec{\Phi}+He^{-2\lambda}\langle\partial_{x_2}\vec{n},\partial_{x_2}\vec{\Phi}\rangle\,\partial_{x_2}\vec{\Phi}\right),\vec{w}\right\rangle dx_1dx_2\\
&\quad+\int_{D^2}\langle\partial_{x_1}\left(\partial_{x_1}H\,\vec{n}\right)+\partial_{x_2}\left(\partial_{x_2}H\,\vec{n}\right),\vec{w}\rangle\ dx_1dx_2\\
&\quad-\int_{D^2}\left\langle\partial_{x_1}\left(H^2\,\partial_{x_1}\vec{\Phi}\right)+\partial_{x_2}\left(H^2\,\partial_{x_2}\vec{\Phi}\right),\vec{w}\right\rangle\ dx_1dx_2\\
&=\int_{D^2}\Big\langle\partial_{x_1}\left(-H\,\pi_T(\partial_{x_1}\vec{n})+\partial_{x_1}H\,\vec{n}-H^2\,\partial_{x_1}\vec{\Phi}\right)\\
&\qquad\qquad+\partial_{x_2}\left(-H\,\pi_T(\partial_{x_2}\vec{n})+\partial_{x_2}H\,\vec{n}-H^2\,\partial_{x_2}\vec{\Phi}\right),\vec{w}\Big\rangle\,dx_1dx_2
\end{aligned}$$

$$= \int_{D^2} \left\langle div \left(-H \nabla \vec{n} + \nabla H \, \vec{n} - H^2 \nabla \vec{\Phi} \right), \vec{w} \right\rangle dx_1 dx_2.$$

Thus, (7.1) holds if and only if

$$div \left(-H \nabla \vec{n} + \nabla H \, \vec{n} - H^2 \nabla \vec{\Phi} \right) = 0. \tag{7.11}$$

Using Lemma 2.60 gives the desired result.

□

Note that in the proof of Theorem 7.2, the assumption of $\vec{\Phi}$ being an element of $\mathcal{E}_{D^2}$ was enough to make sense of each line. In particular, the quantity in (7.2) is an "honest" distribution in $\mathcal{D}'(D^2)$. Indeed, $\vec{H} \in L^2(D^2)$ and consequently

$$\nabla \vec{H} \in H^{-1}(D^2), \quad H \, \nabla \vec{n} \in L^1(D^2), \quad \vec{H} \times \nabla^{\perp} \vec{n} \in L^1(D^2). \tag{7.12}$$

Note that first an alternative Euler-Lagrange equation for *smooth* Willmore immersions was discovered: by Shadow-Thomsen ([Tho23]) in dimension 3, for general $m \geq 3$ by Weiner ([Wei78]). In dimension 3, it is

$$\Delta_g H + 2H \, (H^2 - K) = 0. \tag{7.13}$$

In [Riv08] the equivalence of (7.2) and (7.13) for smooth conformal immersions is shown. Note that equation (7.13) contains the nonlinearity $2H \, (H^2 - K)$, which is *cubic* in the second fundamental form. Thus, it has no meaning for weak immersions with second fundamental form bounded in L^2.

In the next subsection we want to address the question whether weak Willmore immersions are actually *smooth*.

Observe that (7.3) is in *conservative-elliptic form* which is *critical in dimension* 2 under the assumption of L^2-bounded second fundamental form. Indeed, we write the equation as follows:

$$\Delta\vec{H} = div\left[\frac{3}{2}H\,\nabla\vec{n} - \frac{1}{2}\vec{H}\times\nabla^{\perp}\vec{n}\right]. \tag{7.14}$$

As observed in (7.12), the second fundamental form being in L^2 implies that

$$\frac{3}{2}\;H\,\nabla\vec{n} - \frac{1}{2}\vec{H}\times\nabla^{\perp}\vec{n}\in L^1(D^2).$$

Theorem 3.5 implies that

$$\frac{1}{2\pi}\log|x| * div\left[\frac{3}{2}H\,\nabla\vec{n} + \frac{1}{2}\nabla^{\perp}\vec{n}\times\vec{H}\right]\in L^{2,\infty}(D^2).$$

Inserting this information back in (7.14), we obtain $\vec{H}\in L^{2,\infty}_{loc}(D^2)$ which is almost the information we started from. This phenomenon characterizes *critical elliptic systems.*

Weak immersions of surfaces with L^2-bounded second fundamental form

Lecture 5

PCMI Graduate Summer School 2013

Tristan Rivière

Lecture Notes written by Nora Loose

7.1 Conservation laws for weak Willmore immersions

The key for studying the regularity of weak Willmore immersions will be to discover conservation laws for them.

Theorem 7.4. *Let $\vec{\Phi} \in \mathcal{E}_{D^2}$ be a conformal weak Willmore immersion. Then there exists $\vec{L} \in L^{2,\infty}_{loc}(D^2, \mathbb{R}^3)$ such that*

$$\nabla^\perp \vec{L} = 2\ \nabla \vec{H} - 3H\ \nabla \vec{n} + \vec{H} \times \nabla^\perp \vec{n}. \tag{7.15}$$

Moreover the following conservation laws are satisfied:

$$\begin{cases} div\left[\langle \vec{L}, \nabla^\perp \vec{\Phi}\rangle\right] = 0 & (7.16a) \\ \\ div\left[\vec{L} \times \nabla^\perp \vec{\Phi} + 2H\ \nabla^\perp \vec{\Phi}\right] = 0. & (7.16b) \end{cases}$$

Proof of Theorem 7.4. Since $\vec{\Phi}$ satisfies (7.3), by the weak Poincaré Lemma there exists $\vec{L} \in \mathcal{D}'(D^2)$ such that

$$\nabla^\perp \vec{L} = 2\ \nabla \vec{H} - 3H\ \nabla \vec{n} + \vec{H} \times \nabla^\perp \vec{n}. \tag{7.17}$$

Assuming $\vec{\Phi} \in \mathcal{E}_{D^2}$, the right hand side of (7.17) is in $H^{-1} \cap L^1(D^2)$. We deduce from Theorem 3.5 that $\vec{L}$ is the sum of a harmonic function and a function in $L^{2,\infty}(D^2)$, thus

$$\vec{L} \in L^{2,\infty}_{\mathrm{loc}}(D^2).$$

For proving (7.16a), note that

$$\begin{aligned} div\left[\langle \vec{L}, \nabla^\perp \vec{\Phi}\rangle\right] &= \langle \nabla \vec{L}, \nabla^\perp \vec{\Phi}\rangle = -\langle \nabla^\perp L, \nabla \vec{\Phi}\rangle \\ &= -\langle -H\ \nabla \vec{n} + \vec{H} \times \nabla^\perp \vec{n}, \nabla \vec{\Phi}\rangle. \end{aligned} \tag{7.18}$$

Using (2.60), we obtain

$$\langle H\ \nabla \vec{n} + \vec{H} \times \nabla^\perp \vec{n}, \nabla \vec{\Phi}\rangle = -4\ e^{2\lambda}\ H^2. \tag{7.19}$$

Finally,

$$\langle -2H\ \nabla\vec{n}, \nabla\vec{\Phi}\rangle = 2H\left(\mathbb{I}(\partial_{x_1},\partial_{x_1}) + \mathbb{I}(\partial_{x_2},\partial_{x_2})\right) = 4\ e^{2\lambda}\ H^2, \tag{7.20}$$

which, together with (7.18) and (7.19) gives the first conservation law (7.16a).

For showing the second one, recall from (2.10) that

$$div\left[\vec{L}\times\nabla^\perp\vec{\Phi}\right] = \nabla\vec{\Phi}\times\nabla^\perp\vec{L}. \tag{7.21}$$

Using again (2.60), we have

$$\nabla\vec{\Phi}\times\left(H\ \nabla\vec{n} + \vec{H}\times\nabla^\perp\vec{n}\right) = -2H^2\ \nabla\vec{\Phi}\times\nabla\vec{\Phi} = 0. \tag{7.22}$$

We compute

$$\begin{aligned}
&\nabla\vec{\Phi}\times(2\ \nabla H\cdot\vec{n} - 2H\ \nabla\vec{n})\\
&= 2\ \nabla H\cdot\left(\nabla\vec{\Phi}\times\vec{n}\right) - 2H\underbrace{\left(\partial_{x_1}\vec{\Phi}\times\partial_{x_1}\vec{n} + \partial_{x_2}\vec{\Phi}\times\partial_{x_2}\vec{n}\right)}_{=0}\\
&= 2\ \nabla H\cdot\nabla^\perp\vec{\Phi}.
\end{aligned} \tag{7.23}$$

Identities (7.21), (7.22) and (7.22) imply the second conservation law (7.16b). □

Having now found two new conserved quantities, as we did for the first one $2\ \nabla\vec{H} - 3H\ \nabla\vec{n} + \vec{H}\times\nabla^\perp\vec{n}$, we can apply the weak Poincaré Lemma in order to obtain "primitives" of these quantities.

Theorem 7.5. *Let $\vec{\Phi}\in\mathcal{E}_{D^2}$ be a conformal weak Willmore immersion and let $\vec{L}\in L^{2,\infty}_{\mathrm{loc}}(D^2,\mathbb{R}^3)$ be as in Theorem 7.4, satisfing the conservation laws* (7.16a) *and* (7.16b).

There exist[14] $S \in W^{1,(2,\infty)}_{loc}(D^2, \mathbb{R})$ *and* $\vec{R} \in W^{1,(2,\infty)}_{loc}(D^2, \mathbb{R}^3)$ *such that*

$$\begin{cases} \nabla^\perp S = \langle \vec{L}, \nabla^\perp \vec{\Phi} \rangle & (7.24a) \\ \\ \nabla^\perp \vec{R} = \vec{L} \times \nabla^\perp \vec{\Phi} + 2H \ \nabla^\perp \vec{\Phi} & (7.24b) \end{cases}$$

and the following equations hold:

$$\begin{cases} \nabla S = -\langle \vec{n}, \nabla^\perp \vec{R} \rangle & (7.25a) \\ \\ \nabla \vec{R} = \vec{n} \times \nabla^\perp \vec{R} + \nabla^\perp S \cdot \vec{n}. & (7.25b) \end{cases}$$

Proof of Theorem 7.5. Due to conservations laws (7.16a) and (7.16b), and the Poincaré Lemma, there exists $S \in \mathcal{D}'(D^2, \mathbb{R})$ and $\vec{R} \in \mathcal{D}'(D^2, \mathbb{R}^3)$ satisfying (7.24a) and (7.24b).

Since $\vec{\Phi} \in W^{1,\infty}(D^2)$ and $\vec{L} \in L^{2,\infty}_{\text{loc}}(D^2)$, ∇S and $\nabla \vec{R}$ are in $L^{2,\infty}_{\text{loc}}(D^2)$.

Next, we want to show (7.25b). Note that, by (7.24b),

$$\vec{n} \times \nabla^\perp \vec{R} = \vec{n} \times \left(\vec{L} \times \nabla^\perp \vec{\Phi} \right) + 2H \ \vec{n} \times \nabla^\perp \vec{\Phi}. \tag{7.26}$$

We have

$$\vec{n} \times \nabla^\perp \vec{\Phi} = \nabla \vec{\Phi} \tag{7.27}$$

and

$$\begin{aligned} &\vec{n} \times \left(\vec{L} \times \nabla^\perp \vec{\Phi} \right) \\ &= -\nabla^\perp \vec{\Phi} \times \left(\vec{n} \times \vec{L} \right) - \vec{L} \times \left(\nabla^\perp \vec{\Phi} \times \vec{n} \right) \\ &= -\left(\langle \vec{L}, \nabla^\perp \vec{\Phi} \rangle \cdot \vec{n} - \langle \nabla^\perp \vec{\Phi}, \vec{n} \rangle \cdot \vec{L} \right) + \vec{L} \times \nabla \vec{\Phi} \\ &= -\langle \vec{L}, \nabla^\perp \vec{\Phi} \rangle \cdot \vec{n} + \vec{L} \times \nabla \vec{\Phi}. \end{aligned} \tag{7.28}$$

[14]We denote by $W^{1,(2,\infty)}$ the space of distributions in L^2 with gradient in $L^{2,\infty}$.

From (7.24a) we know that

$$\langle \vec{L}, \nabla^\perp \vec{\Phi} \rangle = \nabla^\perp S. \tag{7.29}$$

Combining the identities (7.26), (7.27), (7.28) and (7.29) yields

$$\vec{n} \times \nabla^\perp \vec{R} = -\nabla^\perp S \cdot \vec{n} + \vec{L} \times \nabla \vec{\Phi} + 2H \ \nabla \vec{\Phi}.$$

On the other hand, from (7.24b) we know that

$$\nabla \vec{R} = \vec{L} \times \nabla \vec{\Phi} + 2H \ \nabla \vec{\Phi}.$$

The two last identities together imply (7.25b).

The first equation (7.25a) now follows easily from the second one: Using (7.25b), we have

$$\langle \vec{n}, \nabla^\perp \vec{R} \rangle = -\underbrace{\langle \vec{n}, \vec{n} \times \nabla \vec{R} \rangle}_{=0} - \langle \vec{n}, \nabla S \cdot \vec{n} \rangle = -\nabla S.$$

□

Corollary 7.6. *Let $\vec{\Phi} \in \mathcal{E}_{D^2}$ be a conformal weak Willmore immersion. Let $\vec{L} \in L^{2,\infty}_{loc}(D^2, \mathbb{R}^3)$ be as in Theorem 7.4 and $S \in W^{1,(2,\infty)}_{loc}(D^2, \mathbb{R})$ and $\vec{R} \in W^{1,(2,\infty)}_{loc}(D^2, \mathbb{R}^3)$ as in Theorem 7.5.*

Then the triple $(\vec{\Phi}, S, \vec{R})$ satisfies the following system:

$$\begin{cases} \Delta S = -\langle \nabla \vec{n}, \nabla^\perp \vec{R} \rangle & (7.30a) \\ \Delta \vec{R} = \nabla \vec{n} \times \nabla^\perp \vec{R} + \nabla^\perp S \cdot \nabla \vec{n} & (7.30b) \\ \Delta \vec{\Phi} = \dfrac{1}{2} \left(\nabla^\perp S \cdot \nabla \vec{\Phi} + \nabla^\perp \vec{R} \times \nabla \vec{\Phi} \right). & (7.30c) \end{cases}$$

Proof of Corollary 7.6. (7.30a) and (7.30b) are obtained by taking the divergence of (7.25a) and (7.25b) respectively, recalling (2.9) and (2.10).

Furthermore, using (7.24b), we have

$$\begin{aligned}
&\nabla\vec{\Phi} \times \nabla^{\perp}\vec{R} \\
&= \nabla\vec{\Phi} \times \left(\vec{L} \times \nabla^{\perp}\vec{\Phi}\right) + 2H\ \nabla\vec{\Phi} \times \nabla^{\perp}\vec{\Phi} \\
&= \langle \nabla\vec{\Phi}, \nabla^{\perp}\vec{\Phi}\rangle\ \vec{L} - \langle \nabla\vec{\Phi}, \vec{L}\rangle \cdot \nabla^{\perp}\vec{\Phi} - 2H\ \partial_{x_1}\vec{\Phi} \times \partial_{x_2}\vec{\Phi} \\
&= -\langle \nabla\vec{\Phi}, \vec{L}\rangle \cdot \nabla^{\perp}\vec{\Phi} - 4He^{2\lambda}\ \vec{n},
\end{aligned} \tag{7.31}$$

where $e^{\lambda} := |\partial_{x_1}\vec{\Phi}| = |\partial_{x_1}\vec{\Phi}|$. Using (7.24a), we obtain

$$4e^{2\lambda}\ \vec{H} = -\nabla S \cdot \nabla^{\perp}\vec{\Phi} - \nabla\vec{\Phi} \times \nabla^{\perp}\vec{R}.$$

This, together with the representation (2.42) of the mean curvature vector, implies the desired identity (7.30c). □

Remark 7.7. *Recently, Bernard ([Ber]) found that the three conservations laws* (7.3), (7.16a) *and* (7.16b) *are due to Noether's theorem, i.e. they correspond to particular symmetries of the Willmore functional.*

We recall Noether's theorem for a functional of the form

$$L(u) = \int_{D^2} l(u, \nabla u)\, dxdy, \qquad u \in W^{1,2}(D^2, \mathbb{R}^m),$$

for $l(z,p)$ being C^1 wrt z and C^2 wrt p.

A vector field X on $\mathbb{R}^m$ is called infinitesimal symmetry *of l if for all $u \in W^{1,2}(D^2)$*

$$l(u, \nabla u) = l(F(t,u), \nabla(F(t,u))),$$

where $F(t,z)$ is the flow of X at time t started from $z \in \mathbb{R}^m$ at time 0.

Theorem 7.8 (Emmy Noether, 1918)**.** *Let X be an infinitesimal symmetry of l. If u is a critical point of L, then*

$$div\left(\frac{\partial l}{\partial p}\cdot X(u)\right)=0.$$

$J := \frac{\partial l}{\partial p}\cdot X(u)$ is called the Noether Current *associated to the symmetry X.*

In [Ber], Bernard considers variations of a smooth immersion $\vec{\Phi}\colon \Sigma\to\mathbb{R}^m$ of the form

$$\vec{\Phi}_t := \vec{\Phi}+t(A^j\partial_j\vec{\Phi}+\vec{B}), \tag{7.32}$$

for $\vec{B}=B\vec{n}$. (As before, we shall only consider the case $m=3$ in the sequel.) He derives

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\Sigma_0}|\vec{H}_t|^2 dvol_{g_t}\right)\Bigg|_{t=0}\\ &=\int_{\Sigma_0}\Big[\langle\vec{B},\vec{\mathcal{W}}\rangle+div\underbrace{\left(\langle\vec{H},\nabla\vec{B}\rangle-\langle\vec{B},\nabla\vec{H}\rangle+H^2\begin{pmatrix}A^1\\A^2\end{pmatrix}\right)}_{:=J}\Big]dvol_g,\end{aligned} \tag{7.33}$$

where $\Sigma_0\subset\Sigma$ is any smooth subsurface and

$$\vec{\mathcal{W}}=(\Delta_g H+2H(H^2-K))\vec{n}$$

the Willmore operator. Recall from (7.13) *that $\vec{\mathcal{W}}$ vanishes for a Willmore surface, i.e. a critical point of the Willmore functional. Assuming that $\vec{\Phi}$ is a Willmore surface and choosing $A^j\partial_j\vec{\Phi}+\vec{B}$ in* (7.32) *as a translation, dilation and rotation gives then the following associated Noether currents:*

Translation. *For* $\vec{a} \in \mathbb{R}^3$*, let*

$$\vec{\Phi}_t = \vec{\Phi} + t\vec{a}, \qquad i.e.\ A^j = \langle \vec{a}, g^{ij}\partial_i\vec{\Phi}\rangle, \quad B = \langle \vec{n}, \vec{a}\rangle.$$

Noether current:

$$J_{\vec{a}} = \langle \vec{a}, -\nabla H\vec{n} + H\nabla\vec{n} + H^2\nabla\vec{\Phi}\rangle.$$

Note that

$$div(-\nabla H\vec{n} + H\nabla\vec{n} + H^2\nabla\vec{\Phi}) = 0$$

is equivalent to the Willmore equation (7.3) *in conservative form (cf.* (7.11)*).*

Dilation. *For* $\mu \in \mathbb{R} \setminus \{0\}$*, let*

$$\vec{\Phi}_t = \vec{\Phi} + t\mu\vec{\Phi}, \qquad i.e.\ A^j = \mu\langle g^{ij}\partial_i\vec{\Phi}, \vec{\Phi}\rangle, \quad B = \mu\langle \vec{n}, \vec{\Phi}\rangle.$$

Noether current:

$$J_\mu = \frac{\mu}{2}\langle \vec{L}, \nabla^\perp\vec{\Phi}\rangle,$$

where L *is defined as in* (7.15)*.*

$$div\langle \vec{L}, \nabla^\perp\vec{\Phi}\rangle = 0$$

is conservation law (7.16a)*.*

Rotation. *For* $b \in \mathbb{R}^3$*, let*

$$\vec{\Phi}_t = \vec{\Phi} + t\vec{b}\times\vec{\Phi}, \qquad i.e.\ A^j = -\langle \vec{b}, g^{ij}\partial_i\vec{\Phi}\times\vec{\Phi}\rangle, \quad B = -\langle \vec{b}, \vec{n}\times\vec{\Phi}\rangle.$$

Noether current:

$$J_{\vec{b}} = \langle \vec{b}, -\frac{1}{2}\vec{L}\times\nabla^\perp\vec{\Phi} + \vec{H}\times\nabla\vec{\Phi}\rangle,$$

Note that

$$div(-\frac{1}{2}\vec{L}\times\nabla^\perp\vec{\Phi} + \vec{H}\times\nabla\vec{\Phi}\rangle) = -\frac{1}{2}div(\vec{L}\times\nabla^\perp\vec{\Phi} + 2H\nabla^\perp\vec{\Phi}) = 0$$

is conservation law (7.16b)*.*

Inversion. *Finally, considering a variation corresponding to a translation and an inversion of the form*

$$\vec{\Phi}_t = \vec{\Phi} + t(|\vec{\Phi}|^2\vec{a} - 2\langle\vec{\Phi}, \vec{a}\rangle\vec{\Phi}),$$

for $a \in \mathbb{R}^3$, leads to (7.30c)*, i.e. the equation that establishes a connection between the potentials S and $\vec{R}$, obtained as primitives of the two former conservations laws, and the immersion $\vec{\Phi}$.*

7.2 The regularity of weak Willmore immersions.

We are now ready to prove that weak Willmore immersions are C^∞ in conformal parametrization. The starting point is the elliptic system with quadratic non-linearities which are made of linear combinations of Jacobians. It is somehow reminiscent to the CMC (constant mean curvature) equation

$$\Delta u = 2H\ \partial_{x_1} u \times \partial_{x_2} u, \tag{7.34}$$

for $H \in \mathbb{R}$ being a constant.

Wente showed that any $W^{1,2}(D^2, \mathbb{R}^3)$-solution of (7.34) is actually smooth. This makes us hope to get the same fact for weak Willmore immersions. To put ourselves in the same starting position as in the case of the CMC equation, we show that ∇S and $\nabla \vec{R}$ are not only in $L^{2,\infty}$, but in fact in L^2. This is an easy consequence of the previous corollary, together with some result on integrability by compensation.

Corollary 7.9. *Let $\vec{\Phi} \in \mathcal{E}_{D^2}$ be a conformal weak Willmore immersion and $\vec{L} \in L^{2,\infty}_{loc}(D^2, \mathbb{R}^3)$, $S \in W^{1,(2,\infty)}_{loc}(D^2, \mathbb{R})$ and $\vec{R} \in W^{1,(2,\infty)}_{loc}(D^2, \mathbb{R}^3)$ be as in Theorems 7.4 and 7.5.*

Then $\nabla S \in W^{1,2}_{loc}(D^2, \mathbb{R})$ and $\nabla \vec{R} \in W^{1,2}_{loc}(D^2, \mathbb{R}^3)$.

Proof. Appying Theorem 3.10 to the equations (7.30a) and (7.30b) gives the result. □

We can now attack the proof of the smoothness of weak Willmore immersions, just as it works in the case of the CMC equation. We will need the following lemma.

Lemma 7.10. *Let v be a harmonic function on D^2. Then for every point $p \in D^2$, the function*

$$r \mapsto \frac{1}{r^2} \int_{B_r(p)} |\nabla v|^2 \, dx_1 dx_2$$

is increasing.

Proof. See [Riv], Lemma IV.1.

Theorem 7.11 (Weak Willmore immersions are smooth.)**.** *Let $\vec{\Phi} \in \mathcal{E}_\Sigma$ be a weak Willmore immersion. Then $\vec{\Phi}$ is C^∞ in conformal paratmetrization.*

Proof. For any conformal chart, we can apply Theorems 7.4 and 7.5 as well as Corollaries 7.6 and 7.9 and obtain $\nabla S \in W^{1,2}_{loc}(D^2, \mathbb{R})$ and $\nabla \vec{R} \in W^{1,2}_{loc}(D^2, \mathbb{R}^3)$ such that the following system is satisfied:

$$\begin{cases} \Delta S = -\langle \nabla \vec{n}, \nabla^\perp \vec{R} \rangle & (7.35a) \\ \Delta \vec{R} = \nabla \vec{n} \times \nabla^\perp \vec{R} + \nabla^\perp S \cdot \nabla \vec{n} & (7.35b) \\ \Delta \vec{\Phi} = \dfrac{1}{2} \left(\nabla^\perp S \cdot \nabla \vec{\Phi} + \nabla^\perp \vec{R} \times \nabla \vec{\Phi} \right). & (7.35c) \end{cases}$$

Step 1: Morrey decrease. Our first aim is to prove the existence of a positive constant α such that

$$\sup_{r<1/4,\ p\in B_{1/2}(0)} r^{-\alpha} \int_{B_r(p)} \left(|\nabla S|^2 + |\nabla \vec{R}|^2\right)\, dx_1 dx_2 < \infty. \quad (7.36)$$

Once this Morrey decrease for $|\nabla S|^2 + |\nabla \vec{R}|^2$ is established, by (7.35a) and (7.35b), we get one for $|\Delta S| + |\Delta \vec{R}|$ as well:

$$\sup_{r<1/4,\ p\in B_{1/2}(0)} r^{-\alpha/2} \int_{B_r(p)} \left(|\Delta S| + |\Delta \vec{R}|\right)\, dx_1 dx_2 < \infty. \quad (7.37)$$

Then a classical estimate on Riesz potentials ([Ada75]) gives that

$$\nabla S, \nabla \vec{R} \in L^p_{\text{loc}}(B_{1/2}(0)) \qquad \text{for some } p > 2, \qquad (7.38)$$

from which we will start the bootstrapping in Step 2.

To prove (7.36), we let $\varepsilon_0 > 0$ fixing its value later. There exists some radius $r_0 > 0$ such that

$$\sup_{p\in B_{1/2}(0)} \int_{B_{r_0}(p)} |\nabla \vec{n}|^2\, dx_1 dx_2 < \varepsilon_0.$$

Let $p \in B_{1/2}(0)$ be arbitrary. Let Ψ_S and $\vec{\Psi}_{\vec{R}}$ be the solutions of

$$\left\{\begin{array}{rcll} \Delta\Psi_S &=& -\langle \nabla\vec{n}, \nabla^{\perp}\vec{R}\rangle & \text{in } B_{r_0}(p) \\ \Psi_S &=& 0 & \text{on } \partial B_{r_0}(p) \end{array}\right. \qquad (7.39)$$

and

$$\left\{\begin{array}{rcll} \Delta\vec{\Psi}_{\vec{R}} &=& \nabla\vec{n} \times \nabla^{\perp}\vec{R} + \nabla^{\perp} S \cdot \nabla\vec{n} & \text{in } B_{r_0}(p) \\ \vec{\Psi}_{\vec{R}} &=& 0 & \text{on } \partial B_{r_0}(p). \end{array}\right. \qquad (7.40)$$

By Lemma 7.10 and the Dirichlet principle, the harmonic rests

$$v_S := S - \Psi_S \qquad \vec{v}_{\vec{R}} := \vec{R} - \vec{\Psi}_{\vec{R}}$$

satisfy

$$\begin{aligned}&\int_{B_{r_0/2}(p)} \left(|\nabla v_S|^2 + |\nabla \vec{v}_{\vec{R}}|^2\right)\ dx_1 dx_2 \\ &\quad \leq \frac{1}{4}\int_{B_{r_0}(p)} \left(|\nabla S|^2 + |\nabla \vec{R}|^2\right)\ dx_1 dx_2.\end{aligned} \tag{7.41}$$

Applying Wente's Theorem 3.7 to (7.39) and (7.40) yields

$$\begin{aligned}&\int_{B_{r_0}(p)} \left(|\nabla \Psi_S|^2 + |\nabla \vec{\Psi}_{\vec{R}}|^2\right)\ dx_1 dx_2 \\ &\leq C \int_{B_{r_0}(p)} |\nabla \vec{n}|^2\ dx_1 dx_2 \int_{B_{r_0}(p)} \left(|\nabla S|^2 + |\nabla \vec{R}|^2\right)\ dx_1 dx_2 \\ &\leq C\varepsilon_0 \ \int_{B_{r_0}(p)} \left(|\nabla S|^2 + |\nabla \vec{R}|^2\right)\ dx_1 dx_2.\end{aligned} \tag{7.42}$$

Putting (7.41) and (7.42) together yields

$$\begin{aligned}
&\int_{B_{r_0/2}(p)} \left(|\nabla S|^2 + |\nabla \vec{R}|^2\right)\ dx_1 dx_2 \\
&\le 2\int_{B_{r_0}(p)} \left(|\nabla \Psi_S|^2 + |\nabla \vec{\Psi}_{\vec{R}}|^2\right)\ dx_1 dx_2 \\
&\qquad\qquad + 2\int_{B_{r_0/2}(p)} \left(|\nabla v_S|^2 + |\nabla \vec{v}_{\vec{R}}|^2\right)\ dx_1 dx_2 \qquad (7.43)\\
&\le \left(2C\varepsilon_0 + \frac{1}{2}\right)\int_{B_{r_0}(p)} \left(|\nabla S|^2 + |\nabla \vec{R}|^2\right)\ dx_1 dx_2.
\end{aligned}$$

We now choose $\varepsilon_0 := 1/(8C)$. Then iterating (7.43) yields

$$\begin{aligned}
&\int_{B_{2^{-j}r_0}(p)} \left(|\nabla S|^2 + |\nabla \vec{R}|^2\right)\ dx_1 dx_2 \\
&\le \left(\frac{3}{4}\right)^j \int_{B_{r_0}(p)} \left(|\nabla S|^2 + |\nabla \vec{R}|^2\right)\ dx_1 dx_2 \qquad (7.44)\\
&\le C_{r_0}\ \left(2^{-j} r_0\right)^{\alpha},
\end{aligned}$$

where we choose

$$\alpha := \log_2\left(\frac{4}{3}\right), \qquad C_{r_0} := r_0^{-\alpha}\int_{B_1(0)} \left(|\nabla S|^2 + |\nabla \vec{R}|^2\right)\ dx_1 dx_2.$$

Since α and r_0 are independent of $p \in B_{1/2}(0)$, this gives (7.36) for the sup taken over all $r \le r_0$. Noting that

$$\sup_{r_0 < r < 1/4,\ p\in B_{1/2}(0)} r^{-\alpha}\ \int_{B_r(p)} \left(|\nabla S|^2 + |\nabla \vec{R}|^2\right)\ dx_1 dx_2$$

$$\leq r_0^{-\alpha} \int_{B_1(0)} \left(|\nabla S|^2 + |\nabla \vec{R}|^2 \right) \, dx_1 dx_2 < \infty$$

yields (7.36).

Bootstrapping. Note that (7.38) from Step 1 and (7.35c) imply that also

$$\nabla \vec{n} \in L^p_{\text{loc}}(B_{1/2}(0)).$$

This and (7.38) we can use to bootstrap in (7.35a) and (7.35b) and obtain that

$$\nabla S,\, \nabla \vec{R} \in L^q_{\text{loc}}(B_{1/2}(0)) \qquad \text{for all } q < \infty.$$

Bootstrapping this information in (7.35c) yields

$$\vec{\Phi} \in W^{2,q}_{\text{loc}}(B_{1/2}(0)) \qquad \text{for all } q < \infty,$$

from which we deduce that

$$\vec{n} \in W^{1,q}_{\text{loc}}(B_{1/2}(0)) \qquad \text{for all } q < \infty.$$

The latter information injected back in (7.35a) and (7.35b) gives

$$\nabla^2 S,\, \nabla^2 \vec{R} \in L^q_{\text{loc}}(B_{1/2}(0)) \qquad \text{for all } q < \infty,$$

and so on and so forth. Iteration gives that

$$\vec{\Phi} \in W^{k,p}_{\text{loc}}(B_{1/2}(0)) \qquad \text{for all } k \in \mathbb{N},\ 1 \leq p \leq \infty.$$

Hence

$$\vec{\Phi} \in C^\infty_{\text{loc}}(B_{1/2}(0)),$$

which finishes the proof. □

Finally, we want to remark that if one performs the steps in the regularity proof more carefully and bootstraps in (7.14), one obtains the following result.

Theorem 7.12 (ε-regularity for Willmore)**.** *Let $\vec{\Phi} \in \mathcal{E}_{D^2}$ be a conformal weak Willmore immersion with $|\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}| = e^\lambda$. Assume that*

$$\|\nabla\lambda\|_{L^{2,\infty}(D^2)} \leq C_0.$$

Then there exists $\varepsilon_0 > 0$ with the following property:
If

$$\int_{D^2} |\nabla\vec{n}|^2 \, dx_1 dx_2 \leq \varepsilon_0,$$

then

$$\begin{aligned} &\|e^\lambda \, |H|\|^2_{L^\infty(D^2_{1/2})} + \|\nabla\vec{n}\|^2_{L^\infty(D^2_{1/2})} \\ &\leq C\left(\varepsilon_0, \|\nabla\lambda\|_{L^{2,\infty}(D^2)}\right) \int_{D^2_{3/4}} |\nabla\vec{n}|^2 \, dx_1 dx_2. \end{aligned} \tag{7.45}$$

From the ε-regularity (and Theorem 5.4), one sees easily that one can pass to the limit in the Willmore surface equation (7.2), if one considers a sequence of weak Willmore immersions with a uniform L^2-bound on the second fundamental form satisfying (CA). This is possible at all points where there is no energy concentration. More precisely, one obtains the following theorem.

Theorem 7.13. *Let $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ be a sequence of weak Willmore immersions, which satisfies the compactness assumption (CA) and*

$$\sup_k \mathbb{I}(\vec{\Phi}_k) < \infty.$$

Let $\vec{\xi}_\infty$ be the weak limit of a subsequence of $\vec{\Phi}_k$ in the sense of Definition 5.1, which exists due to Theorem 5.3.

Then $\vec{\xi}$ is weakly Willmore away from its blow-up points.

8 A minimization procedure for the Willmore energy among weak branched immersions

We will now merge the compactness theorem from Section 6 and the regularity theorem from Section 7.2 in order to present a general approach for minimizing Willmore energy under various constraints.

The first one is the topological constraint that Σ (or in other words the genus) is prescribed. We shall give a new proof of the following classical result which has originally been derived with different techniques ([Sim93],[BK03]).

Theorem 8.1. *Let* Σ *be a closed, orientable* 2*-dimensional smooth manifold. Then*

$$\inf_{\vec{\Phi}\in\mathcal{E}_\Sigma} W(\vec{\Phi})$$

is achieved by a smooth embedding.

Proof. We present the proof for $m = 3$.

Because of the closure theorem 6.9, it is natural to work in $\mathcal{F}_\Sigma$, the class of weak branched immersions, instead of $\mathcal{E}_\Sigma$. We thus take a minimizing sequence $\vec{\Phi}_k \in \mathcal{F}_\Sigma$, i.e. satisfying

$$W(\vec{\Phi}_k) \searrow \inf_{\vec{\Phi}\in\mathcal{F}_\Sigma} W(\vec{\Phi}).$$

In order to be able to apply Theorem 6.9 and extract a weakly convergent subsequence, we need that the sequence $\vec{\Phi}_k$ satisfies the compactness assumption (CA). To see this, we note two facts:

i) For every genus g there exists a smooth immersion $\vec{\Phi}\colon \Sigma_g \to \mathbb{R}^3$ such that $W(\vec{\Phi}) < 8\pi$ (for $m = 3$ see e.g. [Kus89]).

ii) If a sequence $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ satisfies $W(\vec{\Phi}_k) < \min\{8\pi, \omega_g^3\}$, it meets the compactness assumption (CA) (see [Riv13], where also Simon's definition of ω_g^n is recalled). For our purposes, it is sufficient to know that $\inf_{\vec{\Phi}\in\mathcal{F}_\Sigma} W(\vec{\Phi}) < \omega_g^n$. This follows in particular from [BK03].

From i) follows that for sufficiently large k, we have

$$W(\vec{\Phi}_k) < 8\pi.$$

The Li-Yau inequality (5.27) and Lemma 6.1 imply that any $\vec{\Phi} \in \mathcal{F}_\Sigma$ with $W(\vec{\Phi}) < 8\pi$ is in fact a weak embedding, i.e. there are neither branch points nor multiple points. Thus we can assume that $\vec{\Phi}_k \in \mathcal{E}_\Sigma$.

By ii) (and again i)) we can assume that $\vec{\Phi}_k$ satisfy (CA). Now we can apply Theorem 6.9, which gives us a weakly convergent subsequence with limit $\vec{\xi}_\infty \in \mathcal{F}_\Sigma^{\text{conf}}$. By lower semicontinuity of the Willmore functional (Lemma 5.2), we get $W(\vec{\xi}_\infty) = \inf_{\vec{\Phi}\in\mathcal{F}_\Sigma} W(\vec{\Phi})$. As above, by Li-Yau and the branch point lemma, we deduce that $\vec{\xi}_\infty$ is actually in $\mathcal{E}_\Sigma$ and does not have multiple points. Furthermore,

$$W(\vec{\xi}_\infty) = \min_{\vec{\Phi}\in\mathcal{F}_\Sigma} W(\vec{\Phi}) = \min_{\vec{\Phi}\in\mathcal{E}_\Sigma} W(\vec{\Phi}). \tag{8.1}$$

To conclude the proof one uses the following lemma, which is proven in [MR13] for weak immersions from S^2 into a Riemannian manifold.

Lemma 8.2 ([MR13], Lemma IX.5)**.** *Let $\vec{\xi} \in \mathcal{E}_\Sigma$ be a conformal weak immersion. Then W is Fréchet differentiable at $\vec{\xi}$ with respect to variations $\vec{w} \in W^{1,\infty} \cap W^{2,2}(\Sigma, \mathbb{R}^3)$.*

We apply Lemma 8.2 to the conformal weak immersion $\vec{\xi}_\infty \in \mathcal{E}_\Sigma$. Note that for small δ, variations of the form $\vec{\xi}_\infty + \vec{w}$, for $\|\vec{w}\|_{W^{1,\infty}\cap W^{2,2}(\Sigma)} \leq \delta$, are still elements of $\mathcal{E}_\Sigma$. Thus, (8.1) implies that

$$dW_{\vec{\xi}_\infty} = 0.$$

Corollary 7.3 implies that

$$div\left[-2\nabla H_{\vec{\xi}}\,\vec{n}_{\vec{\xi}} + H_{\vec{\xi}}\nabla\vec{n}_{\vec{\xi}} - \vec{H}_{\vec{\xi}} \times \nabla^{\perp}\vec{n}_{\vec{\xi}}\right] = 0.$$

The regularity theorem 7.11 implies that $\vec{\xi} \in C^\infty$. The fact that $\vec{\xi}$ is an embedding was noted before. This finishes the proof. $\square$

References

[Ada75] David R. Adams, *A note on Riesz potentials*, Duke Math. J. **42** (1975), no. 4, 765–778.

[Aub82] Thierry Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 252, Springer-Verlag, New York, 1982.

[Ber] Yann Bernard, *Noether's Theorem and the Willmore Functional*, in prep.

[Bet92] Fabrice Bethuel, *Un résultat de régularité pour les solutions de l'équation de surfaces à courbure moyenne prescrite*, C. R. Acad. Sci. Paris Sér. I Math. **314** (1992), no. 13, 1003–1007.

[BK03] Matthias Bauer and Ernst Kuwert, *Existence of minimizing Willmore surfaces of prescribed genus*, Int. Math. Res. Not. (2003), no. 10, 553–576.

[Bla29] Wilhelm Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. III: Differentialgeometrie der Kreise und Kugeln. Bearbeitet von G. Thomsen.*, X + 474 S. Berlin, J. Springer (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen), 1929 (German).

[Che74] Bang-yen Chen, *Some conformal invariants of submanifolds and their applications*, Boll. Un. Mat. Ital. (4) **10** (1974), 380–385.

[CL92] Sagun Chanillo and Yan Yan Li, *Continuity of solutions of uniformly elliptic equations in* $\mathbf{R}^2$, Manuscripta Math. **77** (1992), no. 4, 415–433.

[dC76] Manfredo P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1976.

[dC92] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992.

[Ge98] Yuxin Ge, *Estimations of the best constant involving the* L^2 *norm in Wente's inequality and compact* H*-surfaces in Euclidean space*, ESAIM Control Optim. Calc. Var. **3** (1998), 263–300 (electronic).

[Ge99] ______, *A remark on generalized harmonic maps into spheres*, Nonlinear Anal. **36** (1999), no. 4, Ser. A: Theory Methods, 495–506.

[GHL04] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine, *Riemannian geometry*, third ed., Universitext, Springer-Verlag, Berlin, 2004.

[Gra08] Loukas Grafakos, *Classical Fourier Analysis*, Graduate Texts in Mathematics, Springer, 2008.

[GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[Hél02] Frédéric Hélein, *Harmonic maps, conservation laws and moving frames*, second ed., Cambridge Tracts in

Mathematics, vol. 150, Cambridge University Press, Cambridge, 2002.

[IT92] Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller spaces*, Springer-Verlag, Tokyo, 1992.

[Jos97] Jürgen Jost, *Compact Riemann surfaces*, Universitext, Springer-Verlag, Berlin, 1997.

[KaRu] Katzman, Dan; Rubinstein, Jacob "Method for the design of multifocal optical elements" United States Patent No.: US006302540B1, Oct. 16, 2001.

[KMR] Keller, Laura; Mondino, Andrea; Rivière, Tristan "Embedded surfaces of arbitrary genus minimizing the Willmore energy under isoperimetric constraint" Arch. Ration. Mech. Anal. 212 (2014), no. 2, 645-682.

[Ki] Kirchhoff, Gustav Über das Gleichgewicht und die Bewegung einer elastischen Scheibe. J. Reine Angew. Math. 40 (1850)

[Kus89] Rob Kusner, *Comparison surfaces for the Willmore problem*, Pacific J. Math. **138** (1989), no. 2, 317–345.

[KS] Kuwert, Ernst; Schätzle, Reiner "Minimizers of the Willmore energy under fixed conformal class." J. Differential Geometry 93 (2013), 471-530

[LaLi] Landau, L. D.; Lifschitz, E. M. "Theory of elasticity" - Course of theoretical physics, volume 7 - third edition revised and enlarged by E.M.Lifshitz, A.M.Kosevich and L.P.Pitaevskii, Butterwoth-Heinemann 1986.

[MN] Marques, Fernando C.; Neves, André Min-max theory and the Willmore conjecture. Ann. of Math. (2) 179 (2014), no. 2, 683782.

[Mat95] Pertti Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995, Fractals and rectifiability.

[MoN] Mondino, Andrea; Nguyen, Huy The "Global Conformal Invariants of Sub-manifolds" arXiv:1501.07527 (2015).

[MR13] Andrea Mondino and Tristan Rivière, *Willmore spheres in compact Riemannian manifolds*, Adv. Math. **232** (2013), 608–676.

[Riv] Tristan Rivière, *Conformally invariant variational problems*, http://www.math.ethz.ch/ riviere/pub.html.

[Riv06] Rivère, Tristan "Analysis aspects of Willmore surfaces" arXiv:math/0612526 (2006).

[Riv08] ______, *Analysis aspects of Willmore surfaces*, Invent. Math. **174** (2008), no. 1, 1–45.

[Riv10] Rivère, Tristan "Variational Principles for immersed Surfaces with L^2-bounded Second Fundamental Form" arXiv:1007.2997 (2010). Published in J. Reine und Angewandte Math. 695, (2014).

[Riv13] ______, *Lipschitz conformal immersions from degenerating Riemann surfaces with L^2-bounded second fundamental forms*, Adv. Calc. Var. **6** (2013), no. 1, 1–31.

[Riv15] Rivière, Tristan "Critical weak immersed surfaces within sub-manifolds of the Teichmüller space." Adv. Math. 283 (2015), 232-274.

[Riv16] Rivière, Tristan "Willmore Minmax Surfaces and the Cost of the Sphere Eversion" arXiv:1512.08918

[Rud87] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987.

[Sim84] Leon Simon, *Lectures on geometric measure theory*, Proc. Centre Math. Anal. Austral. Nat. Univ., Australian National University, 1984.

[Sim86] ______, *Existence of Willmore surfaces*, Miniconference on geometry and partial differential equations (Canberra, 1985), Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 10, Australian National University, Canberra, 1986, pp. 187–216.

[Sim93] Simon, Leon "Existence of surfaces minimizing the Willmore functional", Comm. Anal. Geom., 1, (1993), no 2, 281-325.

[Tho23] G. Thomsen, *Über konforme geometrie 1: Grundlagen der konformen flächentheorie*, Lütcke & Wulff, 1923.

[Top97] Peter Topping, *The optimal constant in Wente's L^∞ estimate*, Comment. Math. Helv. **72** (1997), no. 2, 316–328.

[Wei78] Joel L. Weiner, *On a problem of Chen, Willmore, et al*, Indiana Univ. Math. J. **27** (1978), no. 1, 19–35.

[Wen69] Henry C. Wente, *An existence theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl. **26** (1969), 318–344.

[Wi] Willmore, T. J. Note on embedded surfaces. An. Sti. Univ. "Al. I. Cuza" Iasi Sect. I a Mat. (N.S.) 11B (1965) 493496.