SECURE TWO-DIMENSIONAL TORI ARE FLAT

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Abstract. A riemannian manifold $M$ is secure if the geodesics between any pair of points in $M$ can be blocked by a finite number of point obstacles. Compact, flat manifolds are secure. A standing conjecture says that these are the only secure, compact riemannian manifolds. The conjecture claims, in particular, that a riemannian torus of any dimension is secure if and only if it is flat. We prove this for two-dimensional tori.

1. Introduction

We begin by describing our setting and establishing the terminology. By a riemannian manifold $(M, g)$ we will always mean a complete, connected, infinitely differentiable riemannian manifold. We will view geodesics in $(M, g)$ as curves, $c : I \to M$, $I \subset \mathbb{R}$, parameterized by arclength. If $I = [a, b]$, we say that $x = c(a), y = c(b)$ are the endpoints of $c$. If $z \in M$ is an interior point of $c$, we say that $c$ passes through $z$.

For any pair $x, y \in M$ (including $y = x$) let $G(x, y)$ be the set of geodesics in $(M, g)$ with endpoints $x, y$. We say that $G(x, y)$ consists of the geodesics joining $x$ with $y$. A finite set $B \subset M \setminus \{x, y\}$ is a blocking set for $x, y$ if every geodesic in $G(x, y)$ passes through a point $b \in B$. We will also say that $B$ blocks $x$ and $y$ away from each other.

A pair $x, y \in M$ is insecure if the points $x, y$ cannot be blocked away from each other. A riemannian manifold is insecure if we cannot block $x$ away from $y$ for some points $x, y \in M$. Thus, $(M, g)$ is secure if any point in it can be blocked away from any point, including itself. See [Gut05] for an explanation of the term “security” and related terminology.

Which compact riemannian manifolds are secure? The only examples so far are the flat manifolds [GS06]. Researchers in the subject believe in the following statement [BG08, LS07].

Conjecture 1.1. A compact riemannian manifold is secure if and only if it is flat.
Slightly restricting the setting, we state a counterpart of Conjecture 1.1 for tori.

**Conjecture 1.2.** A riemannian torus is secure if and only if it is flat.

In this note we establish Conjecture 1.2 for two-dimensional tori.

**Theorem 1.3.** A two-dimensional riemannian torus is secure if and only if it is flat.

Conjecture 1.1 holds for locally symmetric spaces [GS06]. Moreover, let \((M, g)\) be a compact locally symmetric space of noncompact type. Then no points \(x, y \in M\) can be blocked away from each other [GS06]. This is true, in particular, for a compact surface of constant negative curvature. There is a direct geometric argument that shows this. It is outlined on page 193 in [GS06]. The proof of our Proposition 6.2 uses a similar idea. We point out, however, that the two approaches differ considerably in detail. The discussion in [GS06] crucially uses the hyperbolicity of the geodesic flow for a surface of constant negative curvature.

Our proof of Theorem 1.3 uses the fact that there exists a free homotopy class of closed curves such that the periodic geodesics in this class do not foliate the entire non-flat two-torus. Thus, there are cylindrical regions free of geodesics in this homotopy class. We pick such a cylinder and let \(p, q\) be an arbitrary pair of points in it. We choose an infinite strip in the universal covering of our torus, projecting onto the cylinder in question. We construct an infinite sequence of minimal geodesics in the strip; they project into a sequence of geodesics in the cylinder connecting \(p\) and \(q\). We analyze the asymptotic behavior of minimal geodesics in the strip, as they become longer and longer. We prove that long minimal geodesics spend almost all of their time in the vicinity of a boundary component of the strip. See sections 4 and 5, in particular, Lemma 5.1. This allows us to conclude that any point in the torus can block at most a finite number of geodesics in our infinite sequence of connecting geodesics. See section 6.

A riemannian manifold is uniformly secure if there exists a positive integer \(s\) such that any points \(x, y \in M\) can be blocked away from each other by at most \(s\) blocking points. The minimal such \(s\) is the security threshold of \((M, g)\). Flat, compact manifolds are uniformly secure. Moreover, their security thresholds are bounded above in terms of the dimension of the manifold [GS06]. The fundamental group of a compact, uniformly secure manifold is virtually nilpotent, and its topological entropy vanishes [BG08].
no conjugate points, then it is flat [BG08]. On the other hand, if a manifold has positive topological entropy and no conjugate points, then no pairs \(x, y\) can be blocked away from each other [LS07, BG08]. The crucial idea used in the proofs of these statements is to relate the uniform blocking in \((M, g)\) with the growth, as \(T \to \infty\), of the number \(n_T(x, y)\) of geodesics in \((M, g)\) joining \(x\) with \(y\) and having length \(\leq T\). Relationships between \(n_T(x, y)\) and the growth of \(\pi_1(M)\), as well as between \(n_T(x, y)\) and the topological entropy of \((M, g)\) are well known [Me97, Ma79].

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### 2. Rays, corays, and Busemann functions

In this section we recall a few well known and less known facts about rays, corays and the Busemann functions in complete, connected riemannian manifolds of arbitrary dimensions. We will use the notation \((M^n, g)\) for riemannian manifolds, suppressing \(g\) or \(n\) whenever this causes no confusion. We denote by \(d(\cdot, \cdot)\) the riemannian distance on \(M\). We will view geodesics as parameterized curves \(c(t), t \in I\), where \(I \subset \mathbb{R}\) is a nontrivial, possibly infinite interval, and \(t\) is an arclength parameter. We will call the set \(c(I) \subset M\) the trace of \(c\).

**Definition 2.1.** Let \(I \subset \mathbb{R}\) be an interval and let \(c : I \to M\) be a geodesic.

- (a) The geodesic \(c\) is minimal if \(d(c(t), c(s)) = |t - s|\) for all \(s, t \in I\).
- (b) A ray is a minimal geodesic \(c : \mathbb{R}_+ \to M\).
- (c) Let \(c : \mathbb{R}_+ \to M\) be a ray, and let \(C \subset M\) be its trace.

A ray \(\tilde{c}\) is a coray to \(c\) if there exists a sequence of minimal geodesics \(c_n : [0, L_n] \to M\) with \(\lim_{n \to \infty} L_n = \infty\), such that \(\lim_{n \to \infty} \tilde{c}_n(0) = \tilde{c}(0)\) and \(c_n(L_n) \in C\) for all \(n \in \mathbb{N}\).

Taking limits of minimal geodesics of finite length, we obtain the following basic fact.

**Proposition 2.2.** A complete riemannian manifold \((M, g)\) carries a ray if and only if it is not compact. If \(c\) is a ray in \((M, g)\) and \(p \in M\), then there exists a coray \(\tilde{c}\) to \(c\) with \(\tilde{c}(0) = p\).
Definition 2.3. Let \( c : \mathbb{R}_+ \to M \) be a ray. Its Busemann function, \( B_c : M \to \mathbb{R} \), is defined by

\[
B_c(p) = \lim_{t \to \infty} [d(p, c(t)) - t].
\]

By the triangle inequality, the function \( t \to d(p, c(t)) - t \) is monotonically decreasing.\(^1\) Also by the triangle inequality, it satisfies \(-d(c(0), p) \leq d(p, c(t)) - t \). Thus, the limit in equation (1) exists.

Lemma 2.4. Let \( c : \mathbb{R}_+ \to M \) be a ray, and let \( p, q \in M \) be arbitrary points. Then

\[
|B_c(p) - B_c(q)| \leq d(p, q).
\]

Proof. Apply the triangle inequality to the triangle with corners \( p, q, c(t) \), and take the limit \( t \to \infty \). \( \blacksquare \)

By Lemma 2.4, any Busemann function is lipschitz, with the lipschitz constant 1.

Proposition 2.5. Let \( c : \mathbb{R}_+ \to M \) be a ray. A geodesic \( \tilde{c} : \mathbb{R}_+ \to M \) is a coray to \( c \) if and only if for all \( s, t \in \mathbb{R}_+ \) the equation

\[
B_c(\tilde{c}(t)) - B_c(\tilde{c}(s)) = s - t
\]

holds.

Proof. This follows from equations (22.16) and (22.20) in [Bu]. \( \blacksquare \)

We use Proposition 2.5 to relax the requirements in Definition 2.1.

Lemma 2.6. Let \( c : \mathbb{R}_+ \to M \) be a ray, and let \( C \subset M \) be its trace. Let \( L_n \to \infty \) be a positive sequence. Let \( c_n : [0, L_n] \to M \) be minimal geodesics such that \( \lim_{n \to \infty} d(c_n(L_n), C) = 0 \).

If \( \tilde{c} : \mathbb{R}_+ \to M \) is a geodesic such that \( \tilde{c}(0) \) is a point of accumulation of the sequence \( c_n(0) \), then \( \tilde{c} \) is a coray to \( c \).

Proof. We assume without loss of generality that \( \lim_{n \to \infty} \tilde{c}_n(0) = \tilde{c}(0) \). For every \( n \in \mathbb{N} \) there is a number \( t_n \in [0, \infty) \) such that the sequence \( \varepsilon_n = d(c_n(L_n), c(t_n)) \) converges to zero. Since the geodesics \( c_n \) are minimal, the condition \( \lim_{n \to \infty} L_n = \infty \) implies that \( \lim_{n \to \infty} t_n = \infty \). By the triangle inequality, for all \( n \) and any \( t \in [0, L_n] \) we have

\[
L_n - t - \varepsilon_n \leq d(c_n(t), c(t_n)) \leq L_n - t + \varepsilon_n.
\]

\(^1\)In general, not strictly.
Let $s, t > 0$ be arbitrary. Using that $\lim c_n(0) = c(0)$ and equation (2), we have
\[
B_c(\tilde{c}(t)) - B_c(\tilde{c}(s)) = \lim_{n \to \infty} \left[ d(c(t_n), c(t_n)) - d(\tilde{c}(s), c(t_n)) \right]
= \lim_{n \to \infty} \left[ d(c_n(t), c(t_n)) - d(c_n(s), c(t_n)) \right] \geq \lim_{n \to \infty} (s - t - 2\varepsilon_n) = s - t.
\]
Combining this inequality with Lemma 2.4, we obtain
\[
B_c(\tilde{c}(t)) - B_c(\tilde{c}(s)) = s - t.
\]
The claim now follows from Proposition 2.5.

3. Outline of the proof

For the benefit of the reader, we will outline the main ideas in the proof of Theorem 1.3. Let $(T^2, \overline{g})$ be a non-flat two-dimensional torus; our goal is to find a pair of points in $(T^2, \overline{g})$ that cannot be blocked away from each other by a finite blocking set.

By a classical theorem of E. Hopf [Ho], a riemannian two-torus is flat if and only if it has no conjugate points. Thus, the torus $(T^2, \overline{g})$ has conjugate points. Then, by a theorem of N. Innami, there exists a nontrivial free homotopy class $\alpha$ of closed curves such that $(T^2, \overline{g})$ cannot be foliated by geodesics in $\alpha$. See [In], Corollary 3.2; see also the proof of Theorem 6.1 in [Ba2].

Let $M_\alpha$ be the set of geodesics of minimal length in the class $\alpha$. By results that go back to M. Morse [Mo] and G. Hedlund [He], these geodesics do not self-intersect and are pairwise disjoint. (Generically, $M_\alpha$ consists of a single geodesic.)

The geodesics in $M_\alpha$ foliate a compact, proper subset, $N \subset T^2$. Let $Z \subset T^2$ be a connected component of $T^2 \setminus N$; let $p, q \in Z$ be any pair of points. We will show that the pair $p, q$ is insecure, i.e., that we cannot block $p$ away from $q$ by a finite blocking set.

We denote by $(\mathbb{R}^2, g)$ the riemannian universal covering; let $\pi : (\mathbb{R}^2, g) \to (T^2, \overline{g})$ be the projection. Let $S \subset \mathbb{R}^2$ be a connected component of $\pi^{-1}(Z)$. Then $S$ is an open strip. The boundary $\partial S$ is a disjoint union of traces of two minimal geodesics, $c_0 : \mathbb{R} \to (\mathbb{R}^2, g)$ and $c_1 : \mathbb{R} \to (\mathbb{R}^2, g)$. Let $C_0, C_1$ be the respective traces; then $\partial S = C_0 \cup C_1$.

Let $P, Q_0 \in S$ be arbitrary points such that $\pi(P) = p$, $\pi(Q_0) = q$. Using the action of the stabilizer of $S$ in $\pi_1(T^2) = \mathbb{Z}^2$, we produce an infinite sequence of points $Q_1, \ldots, Q_n, \ldots \in S$ such that $\pi(Q_n) = q$ and the sequence of distances $L_n = d(P, Q_n)$ goes to infinity. Let now

\[\text{These results provide an important part of the Aubry-Mather theory [Ba1].}\]
Let \( \tilde{c}_n : [0, L_n] \to S \) be a sequence of minimal geodesics such that \( \tilde{c}_n(0) = P \) and \( \tilde{c}_n(L_n) = Q_n \).

Lemma 5.1 in section 5 implies that most of the time the geodesics \( \tilde{c}_n \) are close to \( \partial S \). More precisely, for any \( \varepsilon > 0 \) there exists \( T = T(\varepsilon) > 0 \) such that for all \( t \in [T, L_n - T] \) the points \( \tilde{c}_n(t) \) are \( \varepsilon \)-close to \( \partial S \). Figure 1 illustrates the behavior of this sequence of geodesics.

\[ C_1 \]

\[ \begin{array}{c}
\includegraphics[width=\textwidth]{figure1.png}
\end{array} \]

**Figure 1.** A sequence of minimal geodesics in the universal covering whose projections to the torus cannot be blocked by a finite point set.

Set \( c_n = \pi \circ \tilde{c}_n \). Then the geodesics \( c_n : [0, L_n] \to Z \) join the points \( p, q \). Let \( z \in Z \setminus \{p, q\} \) be an arbitrary point. The preceding discussion implies that at most a finite number of the geodesics \( c_n \) passes through \( z \). On the other hand, if \( z \in T^2 \setminus Z \), no geodesic in our sequence passes through it. Thus, any point \( z \in T^2 \) can block at most a finite number of joining geodesics in the infinite sequence \( c_n \). Hence, we cannot block the points \( p, q \) away from each other by a finite set of blocking points.

We will now illustrate the preceding discussion with the example of tori of revolution.
Example 3.1. Let $0 < r < R$, and set $C = C(r, R) = \{(x, 0, z) : (x - R)^2 + z^2 = r^2\}$. This is a circle of radius $r$ in the $xz$-plane. The torus of revolution, $T(r, R) \subset \mathbb{R}^3$, is obtained by revolving $C$ about the $z$-axis. The circles in $T(r, R)$ obtained by revolving points in $C$ are the circles of latitude.

Exactly two of the circles of latitude are geodesics: The inner and the outer equators. The inner (resp. outer) equator $E_{\text{in}}$ (resp. $E_{\text{out}}$) is the circle of latitude corresponding to the point $(R - r, 0, 0) \in C$ (resp. $(R + r, 0, 0) \in C$). Their lengths are $2\pi(R - r)$ and $2\pi(R + r)$ respectively. The two equators are freely homotopic; let $\alpha$ be their homotopy class.

Thus, $N = E_{\text{in}}$, and $Z = T(r, R) \setminus E_{\text{in}}$. Note that the set $\mathcal{M}_\alpha$ consists of a single geodesic; although the tori of revolution are very special, this is the generic situation for riemannian tori. By the preceding argument, any points $p, q \in T(r, R) \setminus E_{\text{in}}$ cannot be blocked away from each other by a finite blocking set.

4. Minimal geodesics in the strip $S$

We will use the notation of section 3; in particular, we use the identification $(T^2, \mathcal{g}) = (\mathbb{R}^2, g)/\mathbb{Z}^2$. If $S \subset \mathbb{R}^2$, we denote by

$$\text{Stab}(S) = \{j \in \mathbb{Z}^2 : S + j = S\}$$

the stabilizer of $S$. Recall that a nonzero vector $j \in \mathbb{Z}^2$ is prime if there do not exist $n \in \mathbb{N}$, $n \geq 2$, and $k \in \mathbb{Z}^2$ such that $j = nk$.

Proposition 4.1. Let $(T^2, \mathcal{g})$ be a nonflat riemannian torus. Let $(\mathbb{R}^2, g)$ be its universal covering, and let $\pi : (\mathbb{R}^2, g) \to (T^2, \mathcal{g})$ be the projection.

Then there exists a connected open set $S \subset \mathbb{R}^2$ with totally geodesic boundary, such that the following statements hold.

(a) The group $\text{Stab}(S)$ is generated by a prime vector.
(b) If $j \in \mathbb{Z}^2 \setminus \text{Stab}(S)$, then $(S + j) \cap S = \emptyset$.
(c) The boundary of $S$ has two connected components, say $C_0$ and $C_1$. There are minimal geodesics $c_0, c_1 : \mathbb{R} \to (\mathbb{R}^2, g)$ whose traces are $C_0$ and $C_1$, respectively.
(d) Let $c : \mathbb{R} \to S$ be a geodesic such that $\pi \circ c : \mathbb{R} \to T^2$ is periodic. Then $c$ is not minimal.

Proof. By Corollary 3.2 in [In], there exists a nontrivial free homotopy class, say $\alpha$, of closed curves in $T^2$ having the following property: There
does not exist a family of closed geodesics in the class $\alpha$ whose traces foliate $T^2$. We can assume that $\alpha$ is prime.

Let $L$ be the minimal length of a curve in $\alpha$; we denote by $\mathcal{M}_\alpha$ the set of closed geodesics in the class $\alpha$ having length $L$. Clearly, $\mathcal{M}_\alpha \neq \emptyset$. By Theorem 6.5 and Theorem 6.6 in [Ba1], the trace of every $c \in \mathcal{M}_\alpha$ is an embedded curve in $T^2$. Moreover, if $c, \tilde{c} \in \mathcal{M}_\alpha$, then either their traces are disjoint or $c$ and $\tilde{c}$ coincide up to a translation of the parameter.

Let $N$ be the union of the traces of geodesics in $\mathcal{M}_\alpha$. By our choice of $\alpha$, the set $N \subset T^2$ is a proper, nonempty, closed subset. Let $Z$ be a connected component of $T^2 \setminus N$. Let $\partial Z = Z \setminus Z$ be its boundary. Then either $\partial Z$ is the trace of a geodesic in $\mathcal{M}_\alpha$ or $\partial Z$ is the union of traces of two geodesics in $\mathcal{M}_\alpha$.

Let $S$ be a connected component of $\pi^{-1}(Z) \subset \mathbb{R}^2$. Then the boundary of $S$ is the union of the traces of two geodesics $c_0, c_1 : [0, 1) \to S$ such that $\pi \circ c_0$ and $\pi \circ c_1$ belong to $\mathcal{M}_\alpha$. Let $k \in \mathbb{Z}^2$ correspond to $\alpha$. Then for all $t \in \mathbb{R}$ we have

$$
c_0(t + L) = c_0(t) + k, \quad c_1(t + L) = c_1(t) + k.
$$

Theorem 6.6 in [Ba1] implies that $c_0$ and $c_1$ are minimal geodesics. The remaining statements in (a), (b), and (c) now follow by elementary topological arguments; claim (d) follows from Theorem 6.7 in [Ba1].

5. The key lemma

We will use the setting and the notation of Proposition 4.1. The following statement is crucial in our proof of Theorem 1.3. We will refer to it as the Key Lemma.

**Lemma 5.1.** For any $\varepsilon > 0$ and any $\delta > 0$ there exists $T = T(\varepsilon, \delta) > 0$ such that the following holds. If $c : [0, L] \to S$ is a minimal geodesic and $d(c(0), \partial S) \geq \delta$ then $d(c(t), \partial S) \leq \varepsilon$ for all $t \in [T, L - T]$.

The proof of Lemma 5.1 is based on the results of M. Morse [Mo] about minimal geodesics in $S$ and on a result from [Ba2] concerning the rays in $S$. We need a few technical lemmas.

**Lemma 5.2.** Let $c : [0, \infty) \to S$ be a ray. Then $\lim_{t \to \infty} d(c(t), \partial S) = 0$.

**Proof.** The claim follows from Theorem 3.7 in [Ba2], interpreted as a statement about minimal geodesics in $(\mathbb{R}^2, g)$. See Example (1) on page 51 in [Ba2] for details.

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3We point out that $Z$ is homeomorphic to the cylinder $S^1 \times (0, 1)$. 

Throughout this section we will use the following notational conventions. With any geodesic $c : \mathbb{R} \to \overline{S}$ we will associate two geodesics $c_{\pm} : \mathbb{R}_+ \to \overline{S}$ as follows. The geodesic $c_+$ is the restriction of $c$ to the positive half-line. We define the geodesic $c_-$ by $c_-(t) = c(-t)$.

We will denote by $C, C_+, C_- \subset \overline{S}$ the respective traces of $c, c_+, c_-$.  

**Definition 5.3.** Let $c_0, c_1 : \mathbb{R} \to \overline{S}$ be two geodesics such that their traces $C_0, C_1$ are the two components of $\partial S$. We say that the geodesics $c_0, c_1$ are coherently oriented if for any time sequence $t_n \to \infty$ the two point sequences $c_0(t_n), c_1(t_n) \in \partial S$ converge to the same end of $\overline{S}$.

Figure 2 illustrates Definition 5.3. We will also say that $c_0, c_1 : \mathbb{R} \to \overline{S}$ are coherent parameterizations of $\partial S$.

![Figure 2](image.png)

**Lemma 5.4.** Let $\partial S = C_0 \cup C_1$, where $c_0, c_1 : \mathbb{R} \to \overline{S}$ are coherently oriented. Let the geodesics $c_{0, \pm} : \mathbb{R}_+ \to \partial S$ and $c_{1, \pm} : \mathbb{R}_+ \to \partial S$ be as above; let $C_{0, \pm} \subset \partial S$ and $C_{1, \pm} \subset \partial S$ be the respective traces.

Let now $c : \mathbb{R} \to S$ be a minimal geodesic. Then, switching $c_0$ with $c_1$ and reversing the orientation of $c$, if need be, we have

$$
\lim_{t \to -\infty} d(c(t), C_{0,-}) = 0, \quad \lim_{t \to +\infty} d(c(t), C_{1,+}) = 0.
$$

**Proof.** By Lemma 5.2, $c(t)$ converges to $\partial S$ as $|t| \to \infty$. By Theorem 15 in [Mo] or by Theorem 6.7 in [Ba1], the equation $\lim_{t \to -\infty} d(c(t), C_{0,-}) = 0$ implies $\lim_{t \to +\infty} d(c(t), C_{1,+}) = 0$. \qed
Remark 5.5. Let $M$ be a riemannian manifold. If $c : I \to M$ is a geodesic, its inverse is the geodesic $c^{-1} : -I \to M$ defined by $c^{-1}(t) = c(-t)$. Lemma 5.4 is equivalent to the following geometric fact.

Let $Z \subset T^2$ be as in section 4. Assume, for simplicity of exposition, that the closure of $Z$ is a proper subset of $T^2$. Let $\overline{c}_0, \overline{c}_1 : \mathbb{R} \to (T^2, \overline{g})$ be the periodic geodesics in the homotopy class $\alpha$ whose respective traces are the two components of the boundary $\partial Z$.

Let now $c : \mathbb{R} \to Z$ be a geodesic whose lift $\tilde{c} : \mathbb{R} \to S$ is minimal. Then $c$ is a heteroclinic connection either between $\overline{c}_0$ and $\overline{c}_1$ or between $\overline{c}_0^{-1}$ and $\overline{c}_1^{-1}$.

Our next lemma says that if a ray in $S$ is a coray to a ray in a boundary component of $S$, then it is asymptotic to this component.

Lemma 5.6. Let $c_0 : \mathbb{R} \to \overline{S}$ be a geodesic whose trace is one of the components of $\partial S$.

Let $c : \mathbb{R}_+ \to S$ be a coray to $c_{0,+}$. Then $\lim_{t \to \infty} d(c(t), C_{0,+}) = 0$.

Proof. By Theorem 3.7 in [Ba2], for any $q \in S$ there exists a ray $\tilde{c} : \mathbb{R}_+ \to S$ such that $\tilde{c}(0) = q$ and $\lim_{t \to \infty} d(\tilde{c}(t), C_{0,+}) = 0$. Hence, by Lemma 5.2 and Lemma 2.6, $\tilde{c}$ is a coray to $c_{0,+}$.

Set $q = c(1)$, and let $\tilde{c} : \mathbb{R}_+ \to S$ be as above. Thus, both $c$ and $\tilde{c}$ are corays to $c_{0,+}$; by construction, $\tilde{c}(0) = c(1)$. The geodesic $t \mapsto c(1+t)$ is also a coray to $c_{0,+}$ starting at $q = c(1) = \tilde{c}(0)$. By Theorem 22.19 in [Bu] or, by Corollary 3.8 in [Ba2], there is only one coray to $c_{0,+}$ starting at $q$. Figure 3 shows a hypothetical configuration of the rays $c$ and $\tilde{c}$ which cannot materialize.

Therefore, the ray $\tilde{c}$ satisfies $\tilde{c}(t) = c(1+t)$. Since $\tilde{c}$ is asymptotic to $C_{0,+}$, the claim follows. $\blacksquare$

We will now prove a preliminary variant of the Key Lemma.

Lemma 5.7. For any $\delta > 0$ there exists $\eta = \eta(\delta) > 0$ such that the following holds. Let $0 < L < \infty$, and let $c : [0, L] \to S$ be a minimal geodesic such that $d(c(0), \partial S) \geq \delta$ and $d(c(L), C_1) < \eta$. Then $d(c(t), C_0) \geq \eta$ for all $t \in [0, L]$.

Proof. Suppose that the claim fails. Then there exists $\delta > 0$, a sequence of minimal geodesics $c_n : [0, L_n] \to S$, and a sequence $t_n \in [0, L_n]$ such that $d(c_n(0), \partial S) \geq \delta$, $\lim_{n \to \infty} d(c_n(L_n), C_1) = 0$, and $\lim_{n \to \infty} d(c_n(t_n), C_0) = 0$.

The closed strip $\overline{S}$ is invariant under the group $\text{Stab}(S) \simeq k\mathbb{Z}$ that acts on $\overline{S}$ by isometries. We have denoted this action by $z \mapsto z + rk$. We will use the same notation for the corresponding action of $\text{Stab}(S)$ on
geodesics in $S$. Then, for any integers $r_1, \ldots, r_n, \ldots \in \mathbb{Z}$ the sequence of geodesics $	ilde{c}_n = c_n + r_n k$ satisfies the above conditions. In view of this observation, and the compactness of the quotient $\overline{S}/k\mathbb{Z}$, we assume without loss of generality that the vectors $	ilde{c}_n(0)$ converge to a limit vector, $v \in T^1(S, g)$; let $p \in S$ be its footpoint.

We will now prove that $\lim_{n \to \infty} t_n = \infty$. If this fails, then, by passing to a subsequence, if need be, we have $\lim t_n = \overline{t} < \infty$. Let $\tilde{c} : \mathbb{R} \to \mathbb{R}^2$ be the geodesic with the initial vector $v$. Then $\tilde{c}(0) = p \in S$, and $\tilde{c}(\overline{t}) = q = \lim_{n \to \infty} c_n(t_n) \in C_0 \subset \partial S$. Since $\partial S$ is geodesic, $\tilde{c}$ intersects it transversally at $q$. Thus, for $t > \overline{t}$ and sufficiently close to $\overline{t}$, we have $\tilde{c}(t) \notin \overline{S}$. Figure 4 illustrates the analysis.

On the other hand, $\liminf L_n \geq \overline{t} + d(C_0, C_1)$ implies that $\tilde{c}(t) \in \overline{S}$ for all $t \in [0, \overline{t} + d(C_0, C_1)]$. In view of this contradiction, $\lim t_n = \infty$.

By Lemma 2.6, the relationships $\lim d(c_n(t_n), C_0) = 0$ and $\lim t_n = \infty$ together imply that $\tilde{c}$ is a coray to $c_{0,+}$ or $c_{0,-}$. Similarly, the relationships $\lim d(c_n(L_n), C_1) = 0$ and $\lim L_n = \infty$ imply that $\tilde{c}$ is a coray to $c_{1,+}$ or $c_{1,-}$. By Lemma 5.6, this is impossible.

We will now prove the Key Lemma. Recall that we view geodesics in $\overline{S}$ as mappings $c : I \to \overline{S}$ of nontrivial intervals $I \subset \mathbb{R}$. For $t \in I$ the velocity vectors $\dot{c}(t)$ are unit tangent vectors in $\overline{\mathbb{R}^2, g}$. Thus, $\text{length}(c) = |I|$. If $0 \in I$, we will refer to $\dot{c}(0)$ as the initial vector of $c$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Illustration to the proof of Lemma 5.6: A configuration that cannot take place.}
\end{figure}
Proof. (of Lemma 5.1) Assume that the claim fails. Then for some \( \varepsilon > 0 \), \( \delta > 0 \) there exists a sequence of minimal geodesics \( c_n : [0, L_n] \to S \) such that the following conditions are satisfied:

i) For all \( n \in \mathbb{N} \) we have \( d(c_n(0), \partial S) \geq \delta \);

ii) For each \( n \) there is \( t_n \in [n, L_n - n] \) so that \( d(c_n(t_n), \partial S) > \varepsilon \).

As in the proof of Lemma 5.7, we assume without loss of generality that the velocity vectors \( \dot{c}_n(t_n) \) converge to a vector \( v \in T^1(S, g) \). Let \( \tilde{c} : \mathbb{R} \to S \) be the geodesic such that \( v = \dot{\tilde{c}}(0) \). Since all of \( c_n : [0, L_n] \to S \) are minimal, and \( t_n \in [n, L_n - n] \), we conclude that \( \tilde{c} : \mathbb{R} \to S \) is a minimal geodesic. By construction, it satisfies \( d(\tilde{c}(0), \partial S) \geq \delta \).

Let \( \eta = \eta(\delta) > 0 \) be as in Lemma 5.7. By Lemma 5.4, there are \( s_0, s_1 \in \mathbb{R} \) such that \( d(\tilde{c}(s_0), C_0) < \eta \) and \( d(\tilde{c}(s_1), C_1) < \eta \).

Interchanging \( C_0 \) and \( C_1 \), if need be, we may assume that \( s_0 < s_1 \). For any \( t \in \mathbb{R} \) we have \( \lim_{n \to \infty} c_n(t_n + t) = \tilde{c}(t) \). In particular, \( \tilde{c}(s_0) = \lim_{n \to \infty} c_n(t_n + s_0) \) and \( \tilde{c}(s_1) = \lim_{n \to \infty} c_n(t_n + s_1) \). Therefore, for sufficiently large \( n \) the inequalities \( d(c_n(t_n + s_0), C_0) < \eta \) and \( d(c_n(t_n + s_1), C_1) < \eta \) hold. Besides, for sufficiently large \( n \) we have \( 0 < t_n + s_0 < t_n + s_1 < L_n \).

Let \( n \in \mathbb{N} \) be any index such that the above conditions hold. Set \( L = t_n + s_1 \), and let \( c : [0, L] \to S \) be the restriction of \( c_n \) to \([0, t_n + s_1]\).

Then \( d(c(0), \partial S) > \delta \) and \( d(c(L), C_1) < \eta \). But we also have \( d(c(t_n + s_0), C_0) < \eta \), and \( t_n + s_0 \in (0, L) \). By Lemma 5.7, this is impossible. \( \blacksquare \)
6. Nonflat two-tori are insecure

We use the setting of section 4 and the notation of section 5. First, we need a technical lemma.

**Lemma 6.1.** Let \( p, q \in \mathbb{Z} \) be arbitrary points. Let \( P, Q \in S \) be such that \( \pi(P) = p, \pi(Q) = q \). For \( n \in \mathbb{N} \) let \( \tilde{c}_n : [0, L_n] \to S \) be a minimal geodesic such that \( \tilde{c}_n(0) = P \) and \( \tilde{c}_n(L_n) = Q + nk \).

Then in the sequence of unit tangent vectors \( \tilde{c}_n(0) \in T^1_P(S, g) \) every vector occurs at most a finite number of times.

**Proof.** Assume the opposite. Then, by passing to a subsequence of indices, if need be, we find a unit vector \( v \in T^1_P(S, g) \) and a sequence of minimal geodesics \( \tilde{c}_i : [0, T_i] \to S \) such that \( \tilde{c}_i(0) = v, \tilde{c}_i(T_i) = Q + n(i)k \).

We have \( \lim_{i \to \infty} n(i) = \infty \). Thus, \( \lim_{i \to \infty} \text{length}(\tilde{c}_i) = \infty \). Since all of \( \tilde{c}_i \) have the same initial vector, the geodesic \( \tilde{c}_{i+1} \) extends \( \tilde{c}_i \) from \([0, T_i] \) to \([0, T_{i+1}] \). Therefore, the limit geodesic \( \lim_{i \to \infty} \tilde{c}_i = \hat{c} : [0, \infty) \to S \) coincides with \( \tilde{c}_i \) on \([0, T_i] \). Hence, \( \hat{c} : [0, \infty) \to S \) is a ray.

By Lemma 5.2, \( \hat{c} \) is asymptotic to \( \partial S \) at infinity. On the other hand, for \( i \in \mathbb{N} \) we have

\[
\text{d}(\hat{c}(T_i), \partial S) = \text{d}(Q + n(i)k, \partial S) = \text{d}(Q, \partial S) > 0.
\]

We have arrived at a contradiction. \( \blacksquare \)

Theorem 1.3 will follow immediately from the proposition below.

**Proposition 6.2.** Let \( p, q \in \mathbb{Z} \) be arbitrary points. Then they cannot be blocked away from each other by a finite blocking set.

**Proof.** Let \( P, Q \in S \) be such that \( \pi(P) = p, \pi(Q) = q \). For \( n \in \mathbb{N} \) let \( \tilde{c}_n : [0, L_n] \to S \) be a minimal geodesic such that \( \tilde{c}_n(0) = P \) and \( \tilde{c}_n(L_n) = Q + nk \). Set \( c_n = \pi \circ \tilde{c}_n \). We will show that no point belongs to the interior of infinitely many geodesics \( c_n \).

Suppose this is false. Then there is a point \( z \in T^2 \setminus \{p, q\} \), an infinite set \( I \subset \mathbb{N} \), and a function \( i \mapsto t_i \in (0, L_i) \) on \( I \) such that \( c_i(t_i) = z \). We will now analyze all apriori possible behaviors of the sequence \( t_i \), as \( i \to \infty \).

Since all of the geodesics in question belong to \( \mathbb{Z} \), we have \( z \in Z \setminus \{p, q\} \). By construction, \( d(p, z) \leq t_i \) and \( d(z, q) \leq L_i - t_i \). Thus the sequences \( t_i \) and \( L_i - t_i \) are bounded away from zero.

Suppose first that \( \limsup_{i \to \infty} t_i < \infty \). Then, by passing to an appropriate subsequence of \( I \), if need be, we obtain the following situation: \( \lim_{i \to \infty} t_i = T \in (0, \infty) \) and the vectors \( \hat{c}_i(0) \) converge. Let
\[ w = \lim_{i \to \infty} \dot{c}_i(0). \] Let \( c : \mathbb{R}_+ \to Z \) be the geodesic such that \( c(0) = p \) and \( \dot{c}(0) = w. \) Let \( \tilde{c} : \mathbb{R}_+ \to S \) be its lift such that \( \tilde{c}(0) = P. \) Since \( \tilde{c} \) is a limit of minimal geodesics, it is a ray. Thus, \( c \) has no conjugate points. On the other hand, \( c : [0, T] \to Z \) is the limit of the geodesics \( c_i : [0, t_i] \to Z. \) We have \( c_i(0) = p = c(0) \) and \( c_i(t_i) = z = c(T). \) By Lemma 6.1, we can assume that the vectors \( \dot{c}_i(0) \) are distinct. Thus, the points \( p \) and \( z \) are conjugate along \( c; \) we have arrived at a contradiction.

We reduce the case \( \lim \sup_{i \to \infty} (L_i - t_i) < \infty \) to the preceding one by switching the roles of the points \( p \) and \( q \) and simultaneously reversing the directions of the geodesics \( c_i. \) We conclude that \( \lim \sup_{i \to \infty} (L_i - t_i) < \infty \) is impossible as well.

The only remaining possibility is \( \lim_{i \to \infty} t_i = \lim_{i \to \infty} (L_i - t_i) = \infty. \) Then, by Lemma 5.1, \( \lim_{i \to \infty} d(\tilde{c}_i(t_i), \partial S) = 0. \) Since \( z = \pi(\tilde{c}_i(t_i)) \) for all \( i, \) we conclude that \( z \in \pi(\partial S) \cap \pi(S) = \pi(\partial S) \cap Z. \) This contradicts claim (b) in Proposition 4.1.

We have examined all possibilities for the sequence \( t_i \in (0, L_i), i \in I, \) and arrived at a contradiction in each case. Therefore, at most a finite number of the geodesics \( c_n \) pass through any point in \( T^2 \setminus \{p, q\}. \)

**Proof of Theorem 1.3.** By [Gut05] or [GS06], a flat torus is secure. By Proposition 6.2, a nonflat two-torus is insecure.

We point out that the flat tori are distinguished amongst all riemannian two-tori by the security of pairs \( y = x. \)

**Corollary 6.3.** A two-dimensional riemannian torus is flat if and only if all pairs \( x, x \) are secure.

**Proof.** It suffices to show that a nonflat riemannian two-torus contains at least one point that cannot be blocked away from itself. Let \( Z \) be the cylinder from Proposition 6.2. Then any point \( x \in Z \) cannot be blocked away from itself.

**References**


