A TOPOLOGICAL PROPERTY OF QUASI-REDUCTIVE
GROUP SCHEMES

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Abstract. In a recent paper, Gopal Prasad and Jiu-Kang Yu introduced the
notion of a quasi-reductive group scheme $G$ over a discrete valuation ring $R$,
in the context of Langlands duality. They showed that such a group scheme
$G$ is necessarily of finite type over $R$, with geometrically connected fibres, and
its geometric generic fibre is a reductive algebraic group; however, they found
examples where the special fibre is non-reduced, and the corresponding reduced
subscheme is a reductive group of a different type. In this paper, the formalism
of vanishing cycles in étale cohomology is used to show that the generic fibre
of a quasi-reductive group scheme cannot be a restriction of scalars of a group
scheme in a non-trivial way; this answers a question of Prasad, and implies
that non-reductive quasi-reductive group schemes are essentially those found
by Prasad and Yu.

In a recent paper [9], Gopal Prasad and Jiu-Kang Yu introduced the notion of
a quasi-reductive group scheme over a discrete valuation ring $R$: this is an affine,
flat group scheme $\pi : \mathcal{G} \to \text{Spec } R$, such that

(i) the generic fibre $\mathcal{G}_K$ is a smooth, connected group scheme over the quotient
field $K$ of $R$

(ii) the reduced geometric special fibre $(\mathcal{G}_k)_{\text{red}}$ is of finite type over the algebraic
closure of the residue field $k$ of $R$, and its identity component is a
reductive affine algebraic group

(iii) $\dim \mathcal{G}_K = \dim \mathcal{G}_k$.

They showed that $G$ is necessarily of finite type over $R$, with geometrically con-
nected fibres, and its geometric generic fibre is a reductive algebraic group. Fur-
ther, $G$ is a reductive group scheme over $\text{Spec } R$, except possibly when $R$ has
residue characteristic 2, and the geometric generic fibre $\mathcal{G}_k$ has a non-trivial nor-
mal subgroup of type $SO_{2n+1}$, for some $n \geq 1$. They gave examples to show
that in case $\mathcal{G}_k = SO_{2n+1}$, reductivity can fail to hold, with a non-reduced geo-
metric special fibre, and they gave a classification of such $G$. Their work arose
in response to a question of Vilonen to Prasad, in connection with a Tannakian
construction of Langlands dual groups (see [7]).

In this context, it is natural to ask if there are any other possibilities for non-
reductive, quasi-reductive group schemes $G$, except the examples found by Prasad
and Yu, and others obtained from these by simple modifications (like products
e tc.). From their results, this boils down to the following specific question:

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Does there exist a quasi-reductive group scheme \( \pi : \mathcal{G} \to \text{Spec } R \), where \( R \) is a complete DVR with algebraically closed residue field, such that for some finite, separable (totally ramified) extension field \( L \) of \( K \), of degree \( > 1 \), the generic fibre \( \mathcal{G}_K \) is isomorphic to \( R_{L/K}(SO_{2n+1})_L \), the Weil restriction of scalars of \( (SO_{2n+1})_L \)?

One aim of this paper is to show that there do not exist any such quasi-reductive group schemes, see Corollary 2 below. Gopal Prasad has obtained a stronger conclusion, combining Corollary 2 with arguments based on [9]; at his urging, this is included below (see Theorem 9).

The non-existence proof is based on a topological result, Theorem 1, on the \( \ell \)-adic cohomology of a quasi-reductive group scheme; it says roughly that, though a quasi-reductive group scheme may not be smooth over the base, it is almost so from the point of view of \( \ell \)-adic cohomology. This property of quasi-reductive group schemes (including the non-smooth ones) may also be of interest in potential applications of such group schemes. This topological result was motivated by the well known Serre-Tate criterion (see [11]) for good reduction of abelian varieties, which relies ultimately on the theory of Néron models. In a sense, [9] also relies on aspects of this theory.

Theorem 1. Let \( R \) be a complete DVR with quotient field \( K \) and algebraically closed residue field \( k \). Let

\[ \pi : \mathcal{G} \to \text{Spec } R \]

be a quasi-reductive group scheme. Let \( G \to \text{Spec } K \) be the generic fibre. Let \( \ell \) be a prime number, invertible in \( R \). Then the action of the inertia group \( \text{Gal}(\overline{K}/K) \) on the étale cohomology group \( H^i_{\text{ét}}(G_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \) is trivial, for any \( i, n \geq 0 \). Thus, the inertia action on the \( \ell \)-adic cohomology \( H^i_{\text{ét}}(G_{\overline{K}}, \mathbb{Q}_{\ell}) \) is trivial, for all \( i \geq 0 \).

Corollary 2. Let \( R \) be a complete DVR with quotient field \( K \), and algebraically closed residue field \( k \). Let \( L \) be a finite extension field of \( K \), and let

\[ \pi : \mathcal{G} \to \text{Spec } R \]

be a quasi-reductive group scheme, whose generic fibre \( \mathcal{G}_K \) is isomorphic to the restriction of scalars of a positive dimensional reductive affine algebraic group \( G \) over \( L \). Then we must have \( L = K \).

Proof. We first note that since \( G_{\overline{K}} \) is a positive dimensional reductive algebraic group over an algebraically closed field, it has a non-zero \( \ell \)-adic Betti number in some positive degree; for example, this is a simple consequence of the classification of reductive groups over algebraically closed fields. Let \( i > 0 \) be the smallest such degree.

Next, since the generic fibre of \( \mathcal{G} \to \text{Spec } R \) is a reductive group and is obtained by restriction of scalars from \( L \) to \( K \), the extension field \( L/K \) is necessarily separable. (If \( L/K \) is a purely inseparable finite extension and \( G \) is an algebraic group over \( L \), then the kernel of the natural homomorphism \( R_{L/K}(G)_L \to G \) is unipotent; see [8], A.3.5, for example.)

Now, if \( L/K \) is a separable extension of degree \( n > 1 \), then the geometric generic fibre \( \mathcal{G}_{\overline{K}} \) is isomorphic to a product of \( n \) copies of \( G_{\overline{K}} \), and the inertia
group $\text{Gal}(\overline{K}/K)$ permutes the $n$ factors transitively. From the Kunneth formula, it follows that for the chosen $i > 0$, the étale cohomology group $H^i_{\text{ét}}(G_{\overline{K}}, \mathbb{Q}_l)$ is a direct sum of (a positive number of) copies of a nontrivial permutation Galois module. This contradicts Theorem 1. □

The second author wishes to thank Gopal Prasad for mentioning the above existence problem, along with a guess that such a group scheme does not exist, and also for explaining the context in which this problem arises. He also thanks Brian Conrad for pointing out a mistake in an earlier version of this paper, and thanks the first author for collaborating with him to correct it. The final topological statement obtained is a bit weaker than what was claimed in the earlier manuscript, but suffices for the application.

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1. Some preliminaries

Before proving the Theorem, we discuss some preliminaries.

Recall that, if $k$ is an algebraically closed field, a unipotent isogeny between connected reductive algebraic $k$-groups is a homomorphism, which is a finite surjective morphism, whose kernel does not contain any nontrivial subgroup scheme of multiplicative type (i.e. isomorphic to a subgroup scheme of $\mathbb{G}_m^e$ for some $e \geq 1$).

The following lemma sheds more light on unipotent isogenies (see Corollary 4). We thank Conrad for explaining this argument to us; the reader might compare this with Lemma 2.2 in [9].

**Lemma 3.** Let $H$ be a reduced group scheme over a perfect field $k$, let $G$ be a closed normal subgroup scheme of $H$ and let $G^{\text{red}}$ be the reduced subscheme of $G$. Then $G^{\text{red}}$ is also a normal subgroup scheme of $H$. If $H$ is connected and $G$ is finite then $G^{\text{red}}$ is in the center of $H$.

**Proof.** We first recall that since $k$ is perfect, the product of reduced $k$-schemes is reduced; hence $G^{\text{red}}$ is a subgroup scheme of $G$. Since $H$ is reduced, so is $H \times G^{\text{red}}$, and hence $(H \times G)^{\text{red}} = H \times G^{\text{red}}$.

Let $c : H \times G \to G$ be the morphism giving the conjugation action of $H$ on $G$ and let $i : G^{\text{red}} \to G$ be the inclusion. Then there is a unique morphism $c^{\text{red}} : H \times G^{\text{red}} \to G^{\text{red}}$ making the diagram below commute:

$$
\begin{array}{ccc}
H \times G^{\text{red}} & \xrightarrow{c^{\text{red}}} & G^{\text{red}} \\
\downarrow{\text{Id} \times i} & & \downarrow{i} \\
H \times G & \xrightarrow{c} & G
\end{array}
$$

Thus $G^{\text{red}}$ is normal.

Now suppose $G$ is finite and $H$ is connected. Since $H(k)$ is non-empty, so $H$ is geometrically connected over $k$ (by [4], IV$_2$, 4.5.13), we may assume that $k$ is algebraically closed. Then the inclusion $e : \text{Spec}(k) \to H$ given by the identity
induces a bijection of connected components of $G^{\text{red}}$ with those of $H \times G^{\text{red}}$. Since $c^{\text{red}}$ is continuous it follows that

$$c^{\text{red}} = p_{G^{\text{red}}},$$

the projection onto $G^{\text{red}}$. Thus $G^{\text{red}}$ is central.

□

**Corollary 4.** The kernel of a unipotent isogeny between connected reductive algebraic groups over an algebraically closed field $k$ is infinitesimal, so that such an isogeny must be purely inseparable.

**Proof.** If $H$ is a connected reductive algebraic group over $k$, and $G$ is the kernel of a unipotent isogeny with domain $H$, then $G$ is a finite, normal subgroup scheme of $H$. By Lemma 3, $G^{\text{red}}$ is a central subgroup scheme, hence contained in a maximal torus. Since $G$, and hence $G^{\text{red}}$, has no nontrivial subgroup scheme of multiplicative type, this means $G^{\text{red}}$ is trivial, i.e., $G$ is infinitesimal.

□

**Lemma 5.** Let $k$ be an algebraically closed field of characteristic $p > 0$, and $\ell$ a prime distinct from $p$. Let $f : G_1 \to G_2$ be either

(i) a unipotent isogeny between connected reductive algebraic groups over $k$, or

(ii) a closed immersion of $k$-schemes of finite type which induces an isomorphism on underlying reduced schemes.

Then

$$\mathcal{F} \mapsto f_* \mathcal{F}, \quad \mathcal{F}' \mapsto f^* \mathcal{F}'$$

determine an equivalence of categories between étale sheaves on $G_1$ and $G_2$, and there are natural isomorphisms $H^i_{\text{ét}}(G_2, f_* \mathcal{F}) \cong H^i_{\text{ét}}(G_1, \mathcal{F})$ for all $i$.

**Proof.** It is a standard property of étale cohomology that a finite, surjective, radical morphism induces an equivalence of categories on étale sheaves, and hence isomorphisms on étale cohomology – see SGA4 VIII, Théorème 1.1, Cor. 1.2 (or for example, see [6], II, 3.17, or [5], 3.6, 3.7, where in [5], the term “purely inseparable” is used instead of “radical”).

□

The main input in the proof of Theorem 1 is the formalism of vanishing cycles, and in particular, the notion of the complex of nearby cycles, as explained in SGA 7 (II), Exp. XIII (compare [5], III, §3). We briefly review what we need.

Suppose given a morphism of schemes $\pi : X \to T$, where $T$ is (say) the spectrum of a complete discrete valuation ring with algebraically closed residue field. Denote the generic point of $T$ by $\eta$, and fix an algebraic closure of the quotient field of the DVR, giving a geometric generic point $\overline{\eta}$ of $T$. Let $X_0$ be the closed fibre, and let $X_\eta$ be the geometric generic fibre.

If $\mathcal{F}$ is any étale sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on $X$, then one can define the complex of nearby cycles $R\psi_T(\mathcal{F})$ on the closed fibre $X_0$ (strictly speaking, it is well-defined as an object in a suitable derived category), as follows: if $i : X_0 \to X$ is the inclusion, and $j : X_\eta \to X$ the evident morphism, then

$$R\psi_T(\mathcal{F}) = i^* Rj_* j^* \mathcal{F}.$$
From the definition of $R\psi_T$, using the Leray spectral sequence for $j : X_\pi \to X$, it follows that if $F_0 = i^*F$ is the restriction of $F$ to the closed fibre $X_0$, then there are homomorphisms on cohomology

\begin{align}
H^i_{et}(X_\pi, j^*F) &\to H^i_{et}(X_0, R\psi_T(F)), \\
H^i_{et}(X_0, F_0) &\to H^i_{et}(X_0, R\psi_T(F)).
\end{align}

Further, $H^i_{et}(X_0, R\psi_T(F))$ carries an action of the inertia group $\text{Gal}(k(\eta)/k(\eta))$, such that the above two maps on cohomology are equivariant (where the inertia action on $H^i_{et}(X_0, F_0)$ is taken to be trivial). We may of course replace the closed fibre $X_0$ by its reduced subscheme in the above, since the categories of étale sheaves on $X_0$ and $(X_0)_{\text{red}}$ are equivalent. If $T = \text{Spec} \, R$, we may write $\psi_R$ instead of $\psi_T$.

The two maps on cohomology in (1.1), (1.2) are also seen to fit into a commutative diagram

\begin{align}
H^i_{et}(X, F) &\to H^i_{et}(X_\pi, j^*F) \\
\downarrow & \\
H^i_{et}(X_0, F_0) &\to H^i_{et}(X_0, R\psi_T(F)).
\end{align}

Here, the left vertical arrow is an isomorphism if $f$ is proper (proper base change theorem).

On the other hand, the bottom horizontal arrow arises from a morphism of complexes

$F_0 \to R\psi_T(F)$.

**Lemma 6.** If in the above situation, $f : X \to T$ is smooth, and $F$ is a locally constant constructible sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules, with $\ell$ invertible in $\mathcal{O}_T$, then the natural map

$F_0 \to R\psi_T(F)$

is a quasi-isomorphism, and so induces isomorphisms on étale cohomology.

**Proof.** The quasi-isomorphism property follows from the definition of $R\psi_T$, and the smooth base change theorem (see [5], page 97, for example).

\[ \square \]

## 2. Proof of the theorem

We now give the proof of the Theorem.

If we apply the formalism of nearby cycles to our quasi-reductive group scheme $\pi : G \to \text{Spec} \, R$ (which is of course not proper), with geometric generic fibre $G_{\overline{K}}$, special fibre $G_0$, and $F = (\mathbb{Z}/\ell^n\mathbb{Z})_G$, where $\ell$ is invertible in $R$, then from (1.1), (1.2) we obtain homomorphisms

\begin{align}
H^i_{et}(G_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) &\to H^i_{et}(G_0, R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})), \\
H^i_{et}(G_0, \mathbb{Z}/\ell^n\mathbb{Z}) &\to H^i_{et}(G_0, R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})).
\end{align}

These are equivariant for the action of the inertia $\text{Gal}(\overline{K}/K)$, where the action on $H^i_{et}(G_0, \mathbb{Z}/\ell^n\mathbb{Z})$ is trivial.

Thus, Theorem 1 follows if we prove that the maps in (2.1), (2.2) are isomorphisms, for any $n$.

We first consider the situation of a smooth reductive group scheme.
Lemma 7. Let $S$ be a complete DVR with algebraically closed residue field, and let $\varphi: \mathcal{H} \to \text{Spec} S$ be a smooth, reductive group scheme. Let $H_0$ be the closed fibre of $\varphi$, and $H_\eta$ the geometric generic fibre. Then for any prime $\ell$ which is invertible in $S$, we have the following.

(i) the canonical map

$$\left(\mathbb{Z}/\ell^n\mathbb{Z}\right)_{H_0} \to R\psi_S \left(\mathbb{Z}/\ell^n\mathbb{Z}\right)_{\mathcal{H}}$$

is a quasi-isomorphism, that is, the complex of nearby cycles reduces to the constant sheaf $\mathbb{Z}/\ell^n\mathbb{Z}$ (in degree 0) on the closed fibre $H_0$

(ii) The canonical maps

$$H^i_{\text{et}}(H_\eta, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i_{\text{et}}(H_0, R\psi_S(\mathbb{Z}/\ell^n\mathbb{Z})),$$

$$H^i_{\text{et}}(H_0, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i_{\text{et}}(H_0, R\psi_S(\mathbb{Z}/\ell^n\mathbb{Z}))$$

are isomorphisms.

Proof. Since $\varphi$ is a smooth morphism, the quasi-isomorphism in (i) holds by lemma 6.

To get the isomorphisms in (ii), consider the square (1.3) constructed with $X = \mathcal{H}$, $f = \varphi$, $\mathcal{F} = \mathbb{Z}/\ell^n\mathbb{Z}$:

$$
\begin{array}{ccc}
H^i_{\text{et}}(\mathcal{H}, \mathbb{Z}/\ell^n\mathbb{Z}) & \to & H^i_{\text{et}}(H_\eta, \mathbb{Z}/\ell^n\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^i_{\text{et}}(H_0, \mathbb{Z}/\ell^n\mathbb{Z}) & \to & H^i_{\text{et}}(H_0, R\psi_S(\mathbb{Z}/\ell^n\mathbb{Z})).
\end{array}
$$

We claim that the left vertical and top horizontal arrows are isomorphisms; this follow at once from [10], Théorème 3.7 (or SGA5, Exp. VII, Proposition 6.2, p.315) From (i), the bottom horizontal arrow is also an isomorphism, and so the right vertical arrow must be one as well. □

We now return to the case of a “general” quasi-reductive group scheme.

Lemma 8. Let $\pi: \mathcal{G} \to \text{Spec} R$ be a quasi-reductive group scheme, where $R$ is a complete DVR with algebraically closed residue field, and $\ell$ a prime invertible in $R$. Let $\mathcal{G}_0$ be the closed fibre of $\pi$. Then the canonical map

$$\left(\mathbb{Z}/\ell^n\mathbb{Z}\right)_{\mathcal{G}_0} \to R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}}$$

is a quasi-isomorphism.

Proof. Combining Propositions 3.4 and 4.3 of [9], it follows that there is a finite extension field $K'$ of $K$ (contained in our chosen algebraic closure $\overline{K}$) such that if $R'$ is the integral closure of $R$ in $K'$, and

$$\hat{\mathcal{G}} = \text{normalization of } \mathcal{G} \times_{\text{Spec} R} \text{Spec } R',$$

then

(i) $R'$ is a complete DVR (with the same residue field as $R$)

(ii) $\hat{\mathcal{G}} \to \text{Spec } R'$ is a smooth, reductive group scheme with connected fibres
(iii) the induced morphism on reduced, geometric special fibres

\[ \hat{G}_0 \to (G_0)_{\text{red}} \]

is a unipotent isogeny between connected, reductive groups of the same dimension.

The cited Propositions rely on a result due independently to Raynaud and Faltings, whose proof is given in an appendix to [9] by Brian Conrad.

We note that there is a commutative diagram

\[
\begin{array}{ccc}
\hat{G} & \xrightarrow{i_0} & \hat{G}_1 \\
\Spec R' & \xrightarrow{\alpha} & \Spec R' \\
\end{array}
\]

By choice, the geometric point \( \eta \) of \( \Spec R' \) is also a geometric point of \( \Spec R \).

Let \( \hat{G}_1 = \hat{G} \times \varphi G_0 \). We may regard the special fibre \( \hat{G}_0 \) of \( \pi' : \hat{G} \to \Spec R' \) as a closed subscheme of \( \hat{G}_1 \), and in fact it is just the underlying reduced subscheme. Thus the inclusion

\[ i_0 : \hat{G}_0 \to \hat{G}_1 \]

induces an equivalence of categories between finite étale sheaves on the two schemes; under this equivalence, the constant sheaves \( \mathbb{Z}/\ell^n\mathbb{Z} \) on the two schemes correspond. There is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{G}_\pi & \xrightarrow{j'} & \mathcal{G} \\
\hat{G}_0 & \xrightarrow{i_0} & \hat{G}_1 \xrightarrow{i_1} \hat{G} \\
\mathcal{G}_0 & \xrightarrow{i} & \mathcal{G} \\
\end{array}
\]

where the square is a pullback, and the inclusion \( i' : \hat{G}_0 \to \hat{G} \) is the composition \( i' = i_1 \circ i_0 \).

From the definitions, we have that

\[ R\psi R'(\mathbb{Z}/\ell^n\mathbb{Z})_{\hat{G}} = i'^* Rj'_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}_\pi}. \]

From lemma 7, this is quasi-isomorphic to the constant sheaf \( \mathbb{Z}/\ell^n\mathbb{Z} \) on \( \hat{G}_0 \). Hence we obtain quasi-isomorphisms

\[ (\mathbb{Z}/\ell^n\mathbb{Z})_{\hat{G}_0} \cong i'^* Rj'_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}_\pi} \cong i_0^* i_1^* Rj'_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}_\pi}. \]

This implies that

\[ (\mathbb{Z}/\ell^n\mathbb{Z})_{\hat{G}_1} \cong i_1^* Rj'_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}_\pi}. \]

Applying \( R\beta_* = \beta_* \) (since \( \beta \) is a finite morphism), and using the proper base-change theorem, we get quasi-isomorphisms

\[ \alpha_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\hat{G}_1} \cong i^* R(\beta \circ j'_*)_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}_\pi} = R\psi R(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}}. \]

Now the three arrows

\[ i_0 : \hat{G}_0 \to \hat{G}_1, \hat{G}_0 \to (G_0)_{\text{red}}, (G_0)_{\text{red}} \to G_0 \]

induce equivalences of categories between the respective categories of finite étale sheaves (the middle arrow is a unipotent isogeny, the other two are inclusions...
of underlying reduced schemes). Hence \( \alpha \) also induces such an equivalence of categories. Hence we obtain a quasi-isomorphism

\[
(Z/\ell^nZ)_{\hat{G}_0} \to R\psi_R(Z/\ell^nZ)_{\hat{G}};
\]

it is easy to see from the above that this is induced by the canonical map of complexes. □

In particular, we see that the map (2.2) is an isomorphism. It remains to consider the map (2.1):

\[
H^i_{et}(G_K, Z/\ell^nZ) \to H^i_{et}(G_0, R\psi_RZ/\ell^nZ).
\]

This map is constructed as the composition

\[
H^i_{et}(G_K, Z/\ell^nZ) \cong H^i_{et}(\hat{G}, R\psi_RZ/\ell^nZ) \to H^i_{et}(G_0, i^*R\psi_RZ/\ell^nZ) = H^i_{et}(G_0, R\psi_RZ/\ell^nZ)
\]

where the first isomorphism is by what we earlier alluded to as the Leray spectral sequence. Thus it suffices to show that

\[
H^i_{et}(G, R\psi_RZ/\ell^nZ) \to H^i_{et}(G_0, i^*R\psi_RZ/\ell^nZ)
\]

is an isomorphism.

The analogous map for the group scheme \( \pi': \hat{G} \to \Spec R' \) is similarly expressed as a composition

\[
H^i_{et}(\hat{G}_K, Z/\ell^nZ) \cong H^i_{et}(\hat{G}, R\psi_RZ/\ell^nZ) \to H^i_{et}(\hat{G}_0, i'^*R\psi_RZ/\ell^nZ) = H^i_{et}(\hat{G}_0, R\psi_RZ/\ell^nZ).
\]

As seen in lemma 7, this composition is an isomorphism.

Writing \( i' = i_0 \circ i_1 \), where \( i_0 \) is finite, surjective and radical, we thus obtain an isomorphism

\[
H^i_{et}(\hat{G}, R\psi_RZ/\ell^nZ) \to H^i_{et}(\hat{G}_1, i'^*R\psi_RZ/\ell^nZ),
\]

which we may rewrite, on applying \( R\beta_* \), as an isomorphism

\[
H^i_{et}(G, R\psi_RZ/\ell^nZ) \to H^i_{et}(G_0, R\alpha_*i'^*R\psi_RZ/\ell^nZ),
\]

which in turn is expressible as an isomorphism

\[
H^i_{et}(G, R\psi_RZ/\ell^nZ) \to H^i_{et}(G_0, i^*R\psi_RZ/\ell^nZ).
\]

This is the desired isomorphism, and completes the proof of the theorem.

3. An application

Gopal Prasad has given the following application of Corollary 2; we include his proof here.

**Theorem 9.** Let \( R \) be a strictly Henselian DVR with algebraically closed residue field, and \( K \) be its field of fractions. Then the generic fibre \( G = G_K \) of any quasi-reductive \( R \)-scheme \( \mathcal{G} \) splits over \( K \).

**Proof.** In view of Proposition 4.4(i) of [9], we can assume that \( G \) is either a torus or a semi-simple \( K \)-simple group. Now if \( G \) is a torus, then it follows from Théorème 8.8 of [SGA3, Exp. X] that \( \mathcal{G} \) is a \( R \)-torus, which implies that it splits over \( R \) (Proposition 2.1 of [SGA3, Exp. XXII]), and hence the generic fibre \( G \) is a \( K \)-split torus. We assume now that \( G \) is a semi-simple \( K \)-simple group. If \( G \)
does not contain a normal subgroup, defined and isomorphic over the algebraic closure \( \overline{K} \) of \( K \), to \( SO_{2n+1} \) for some \( n \geq 1 \), then according to Theorem 1.2 of [9], \( G \) is smooth and reductive, so again by Proposition 2.1 of [SGA3, Exp. XXII], \( G \) is split, and so its generic fibre \( G \) is \( K \)-split. On the other hand, if \( G \) contains a normal subgroup defined and isomorphic over \( K \) to \( SO_{2n+1} \) for some \( n \geq 1 \), then as \( SO_{2n+1} \) is a group of adjoint type, and \( G \) is \( K \)-simple, there exists a finite separable extension \( L \subset K \) of \( K \), and an absolutely simple \( L \)-group \( H \) such that

(i) \( H \) is \( \overline{K} \)-isomorphic to \( SO_{2n+1} \), and
(ii) \( G \cong R_{L/K}(H) \), see [2], 6.21(ii) and 6.17.

Now Corollary 2 implies at once that since \( G \) is quasi-reductive, \( L = K \). Thus \( G \) is a \( K \)-group which is isomorphic to \( SO_{2n+1} \) over \( K \). But as \( K \) is a field of cohomological dimension \( \leq 1 \), according to a well known theorem of Steinberg [12] (if \( K \) is imperfect, see also [1], 8.6), \( G \) is quasi-split over \( K \). But as \( G \) is an absolutely simple \( K \)-group of type \( B_n \), if it is quasi-split over \( K \), then it is \( K \)-split. This completes the proof of the above theorem.

\( \square \)

**Remark 10.** Let \( R \) and \( K \) be as in the above theorem. A quasi-reductive group scheme \( G \) is by definition a **good** (see [9], 8.2) quasi-reductive model of its generic fibre \( G \), if \( G(R) \) is a hyperspecial parahoric subgroup of \( G(K) = G(K) \). If \( G \) admits a good quasi-reductive model, then it is \( K \)-split ([9], Lemma 8.1]). Theorems 9.3-9.5 of [9] classify all good quasi-reductive models of \( G \). It is an interesting problem to determine all quasi-reductive models of a connected \( K \)-split reductive group \( G \). For \( G = SO_{2n+1} \), all such models have been determined in §10 of [9].

### 4. Further remarks

We briefly discuss an analogue of quasi–reductive schemes wherein we replace reductive algebraic groups by abelian varieties.

**Definition 11.** We call a group scheme \( \pi : A \to S \) **quasi-abelian** if it is proper and flat over \( S \) and if it is an abelian scheme when restricted to an open dense subset of \( S \).

If all residue fields of \( S \) are of characteristic zero then a quasi-abelian scheme is necessarily an abelian scheme by Cartier’s theorem.

Now suppose \( S \) is the spectrum of a DVR \( R \) with residue characteristic \( p > 0 \) and \( \pi : A \to S \) a quasi–abelian scheme. The following statements are in contrast with the quasi–reductive case.

1. If \( A \) is normal then it is an abelian scheme. This follows from (i) the existence of Néron models and (ii) the fact that for any commutative group scheme \( G \), flat and of finite type over \( S \), the multiplication by \( n \) morphism \([n] : G \to G, n \in \mathbb{Z}, (n,p) = 1\), is étale. (One can use [3], Exposé VI_A, Proposition on p. 316 to prove that \([n]\) is flat; it is unramified because \( n \) is a unit in \( R \).)

Since \( A \) is proper and its geometric special fibre contains no rational curves, it follows that the rational map from \( A' \), the Néron model of \( A \) to \( A \), extending the identity morphism on the generic fibres, is actually a morphism. By examining prime to \( p \) torsion (using (ii)) we deduce that

\( \square \)
the induced morphism on special fibres is dominant, which implies that \( \mathcal{A}' \) is an abelian scheme. We then use Zariski’s main theorem to conclude.

(2) For any prime number \( p \) there exists \( S \) as above and a quasi-abelian scheme over \( S \) which is not an abelian scheme. Such schemes can be constructed as follows: Let \( \mathcal{B}', \mathcal{B} \) be abelian schemes over \( S \) and \( \phi : \mathcal{B}' \to \mathcal{B} \) a flat isogeny with kernel \( \mathcal{K}' \). Suppose there exists an abelian subscheme \( \mathcal{A}' \) of \( \mathcal{B}' \) such that \( \mathcal{K}' \cap \mathcal{A}' \) is not flat over \( S \). Then \( \mathcal{A} := \phi(\mathcal{A}') \) is quasi-abelian but not abelian. For any \( p \) one may easily find such data with \( \mathcal{B}' \) the product of a one dimensional abelian scheme with itself.

One could generalize the definition of quasi-abelian schemes by considering group schemes \( \pi : \mathcal{A} \to S \) which are flat and of finite type over \( S \), abelian over a dense open subset and such that all reduced geometric fibres are semi-abelian. In this generality, we do not know if the analogue of (1) above continues to hold (though it does if the relative dimension is one since there exist canonical regular compactifications in this case).

REFERENCES


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