Power Series Representations
for Complex Bosonic Effective Actions

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Abstract. We develop a power series representation and estimates for an effective action of the form

\[ \ln \frac{\int e^{f(\alpha_1, \cdots, \alpha_s; z^*, z)} d\mu(z^*, z)}{\int e^{f(0, \cdots, 0; z^*, z)} d\mu(z^*, z)} \]

Here, \( f(\alpha_1, \cdots, \alpha_s; z^*, z) \) is an analytic function of the complex fields \( \alpha_1(x), \cdots, \alpha_s(x) \), \( z^*(x), z(x) \) indexed by \( x \) in a finite set \( X \), and \( d\mu(z^*, z) \) is a compactly supported product measure. Such effective actions occur in the small field region for a renormalization group analysis. Using methods similar to a polymer expansion, we estimate the power series of the effective action.
I. Introduction

Let $X$ be a finite set and $\Phi^X$ the space of complex valued bosonic fields (i.e. functions) on $X$. Furthermore let $d\mu(z^*,z)$ be a product measure on $\Phi^X$ of the form

$$d\mu(z^*,z) = \prod_{x \in X} d\mu_0(z^*(x), z(x))$$

(I.1)

where $d\mu_0(\zeta^*,\zeta)$ is a normalized measure on $\Phi$ that is supported in $|\zeta| \leq r$ for some constant $r$. That is,

$$\int |\zeta|^k d\mu_0(\zeta^*,\zeta) \leq r^k \quad \text{for all } k \in \mathbb{N}$$

(I.2)

For an analytic function $f(\alpha_1, \cdots, \alpha_s; z^*, z)$ of fields $\alpha_1, \cdots, \alpha_s, z^*, z$, we develop criteria under which

$$g(\alpha_1, \cdots, \alpha_s) = \ln \frac{\int e^{f(\alpha_1, \cdots, \alpha_s; z^*, z)} d\mu(z^*, z)}{\int e^{f(0, \cdots, 0; z^*, z)} d\mu(z^*, z)}$$

(I.3)

exists. This is done using norms for $f$ which are defined in terms of the expansion of $f$ in powers of the fields. The construction also gives estimates on the corresponding norms of $g$.

In the note [BFKT3], we described the analogous construction for real valued fields, which is technically simpler. As pointed out there, expression like (I.3) occur during the course of each iteration step in a Wilson style renormalization group flow. Here $(z^*, z)$ are the fluctuation fields integrated out at the current scale, while $\alpha_1, \cdots, \alpha_s$ can be fields that are to be integrated out in future scales or can be source fields that are used to generate and control correlation functions and are never integrated out. We shall use the methods developed here to control the ultraviolet limit in the coherent state functional integral representation for many boson systems described in [BFKT1, Theorems II.2 and III.7].

To give an example of the norms used to control (I.3), assume for simplicity that $s = 1$ and write $\alpha_1 = \alpha$. Then $f$ has a power series expansion

$$f(\alpha; z^*, z) = \sum_{n_1, n_2, n_3 \geq 0} \sum_{\substack{x \in X^{n_1} \\bar{y} \in X^{n_2} \\bar{y} \in X^{n_3}}} a(\bar{x}; \bar{y}^*, \bar{y}) \alpha(x_1) \cdots \alpha(x_{n_1}) z^*(y_{n_1}) \cdots z^*(y_{n_2}) z(y_1) \cdots z(y_{n_3})$$

with coefficients $a(\bar{x}; \bar{y}^*, \bar{y})$ that are symmetric under permutations of the components of the vectors $\bar{x}$, $\bar{y}^*$ and $\bar{y}$, respectively.

Assume that $X$ is a metric space and fix parameters $\kappa$, $m > 0$. An example of a norm that we use is

$$\|f\| = |f(0; 0, 0)| + \sum_{n_1 + n_2 + n_3 \geq 1} \sup_{x \in X} \max_{1 \leq \ell \leq n_1 + n_2 + n_3} \sum_{(x, y^*, \bar{y}) \in X^{n_1} \times X^{n_2} \times X^{n_3}} w(\bar{x}; \bar{y}^*, \bar{y}) |a(\bar{x}; \bar{y}^*, \bar{y})|$$
where the weight system \( w \) is defined as

\[
w(\vec{x}; \vec{y}_*, \vec{y}) = \kappa^{n_1} (4r)^{n_2+n_3} e^{m\tau(\vec{x}, \vec{y}_*, \vec{y})} \quad \text{for} \quad (\vec{x}, \vec{y}_*, \vec{y}) \in X^{n_1} \times X^{n_2} \times X^{n_3}
\]

and \( \tau(\vec{x}, \vec{y}_*, \vec{y}) \) is the minimal length of a tree which contains vertices at the points of the set \( \{x_1, \ldots, x_n, y_1, \ldots, y_{n_2}, y_1, \ldots, y_{n_3}\} \). For this norm, our main result, Theorem III.4, states that

\[
g(\alpha) = \ln \int e^{f(\alpha; z^*, z)} d\mu(z^*, z) \int e^{f(0; z^*, z)} d\mu(z^*, z)
\]

exists, provided \( \|f\| < \frac{1}{16} \), and that, in this case,

\[
\|g\| \leq \frac{\|f\|}{1 - \frac{1}{16}\|f\|}
\]

Theorem III.4 applies to more general norms than those described above. (See Definitions II.3, II.6 and III.1.) To reflect the geometry and scale structure of a “large field/small field” decomposition of \( X \), one can replace the constant \( \kappa \) by a “weight factor” \( \kappa : X \to (0, \infty) \) and the factor \( \kappa^{n_1} \) in (I.4) by \( \kappa(x_1) \cdots \kappa(x_{n_1}) \). See Example II.4.i. Another possible modification (Example II.4.iii) is to multiply (I.4) by a factor \( e^{mD(\{x_1, \ldots, y_{n_3}\}, \Omega^c)} \) where \( \Omega \) is a subset of \( X \) and, for any \( S \subset X \),

\[
D(S, \Omega^c) = \sup_{x \in S} \inf_{y \in \Omega^c} d(x, y)
\]

is the maximum distance from \( S \) to \( \Omega^c \). This is useful when \( \Omega \) is a “small field region” and one wants to control boundary terms (that is, terms that depend on the values of fields at points both inside and outside \( \Omega \)). Still another variation on the norms comes from the fact that one is often led to bound source fields by their sup norm rather than by their \( L^1 \) norm. See Definition II.6.

If the measure \( d\mu_0(\zeta^*, \zeta) \) is rotation invariant and there are no nontrivial monomials of the form \( a(\vec{x}; \vec{y}, \vec{y}) \alpha(x_1) \cdots \alpha(x_{n_1}) (z_*(y_1)z(y_1)) \cdots (z_*(y_{n_2})z(y_{n_2})) \) in the power series of \( f \), then (I.5) can be improved to a quadratic bound. See Corollary III.7. This situation occurs in our analysis of many boson systems. There we also need information on how the \( g \) of (I.3) varies with \( f \). This is provided by Corollary III.8. Also in this situation, \( f \) is decomposed into a small field part, that depends only of the values of the fields inside some small field set \( \Omega \), and a boundary part that is measured using a modified norm as sketched above. One gets a corresponding decomposition of \( g \) which is discussed in Corollary III.9 and Example III.10.

The measures that typically arise in renormalization group steps are rarely product measures. To apply the results of this paper, one must first perform a change of variables so as to diagonalize the (essential part) of the covariance of the measure. Linear changes of variables, as well as substitutions that typically occur in renormalization group steps are controlled in §IV.
II. Norms

To get a general setup for the norms that we shall use, we need a number of definitions.

Definition II.1 \textit{(n–tuples)}

(i) Let \( n \in \mathbb{Z} \) with \( n \geq 0 \) and \( \vec{x} = (x_1, \ldots, x_n) \in X^n \) be an ordered \( n \)-tuple of points of \( X \). We denote by \( n(\vec{x}) = n \) the number of components of \( \vec{x} \). Set

\[
z(\vec{x}) = z(x_1) \cdots z(x_n)
\]

If \( n(\vec{x}) = 0 \), then \( z(\vec{x}) = 1 \). The support of \( \vec{x} \) is defined to be

\[
\text{supp} \, \vec{x} = \{x_1, \ldots, x_n\} \subset X
\]

(ii) For each \( s \in \mathbb{N} \), we denote

\[
X^{(s)} = \bigcup_{n_1, \ldots, n_s \geq 0} X^{n_1} \times \cdots \times X^{n_s}
\]

The support of \( (\vec{x}_1, \ldots, \vec{x}_s) \in X^{(s)} \) is

\[
\text{supp}(\vec{x}_1, \ldots, \vec{x}_s) = \bigcup_{j=1}^{s} \text{supp}(\vec{x}_j)
\]

If \( (\vec{x}_1, \ldots, \vec{x}_{s-1}) \in X^{(s-1)} \) then \( (\vec{x}_1, \ldots, \vec{x}_{s-1}, -) \) denotes the element of \( X^{(s)} \) having \( n(\vec{x}_s) = 0 \).

(iii) We define the concatenation of \( \vec{x} = (x_1, \ldots, x_n) \in X^n \) and \( \vec{y} = (y_1, \ldots, y_m) \in X^m \) to be

\[
\vec{x} \circ \vec{y} = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in X^{n+m}
\]

For \( (\vec{x}_1, \ldots, \vec{x}_s), (\vec{y}_1, \ldots, \vec{y}_s) \in X^{(s)} \)

\[
(\vec{x}_1, \ldots, \vec{x}_s) \circ (\vec{y}_1, \ldots, \vec{y}_s) = (\vec{x}_1 \circ \vec{y}_1, \ldots, \vec{x}_s \circ \vec{y}_s)
\]

Definition II.2 \textit{(Coefficient Systems)}

(i) A coefficient system of length \( s \) is a function \( a(\vec{x}_1, \ldots, \vec{x}_s) \) which assigns a complex number to each \( (\vec{x}_1, \ldots, \vec{x}_s) \in X^{(s)} \). It is called symmetric if, for each \( 1 \leq j \leq s \), \( a(\vec{x}_1, \ldots, \vec{x}_s) \) is invariant under permutations of the components of \( \vec{x}_j \).
(ii) Let $f(\alpha_1, \ldots, \alpha_s)$ be a function which is defined and analytic on a neighbourhood of the origin in $\mathbb{C}^s \setminus X$. Then $f$ has a unique expansion of the form

$$f(\alpha_1, \ldots, \alpha_s) = \sum_{(\vec{x}_1, \ldots, \vec{x}_s) \in X^{(s)}} a(\vec{x}_1, \ldots, \vec{x}_s) \alpha_1(\vec{x}_1) \cdots \alpha_s(\vec{x}_s)$$

with $a(\vec{x}_1, \ldots, \vec{x}_s)$ a symmetric coefficient system. This coefficient system is called the symmetric coefficient system of $f$.

**Definition II.3 (Weight Systems)** A weight system of length $s$ is a function which assigns a positive extended number $w(\vec{x}_1, \ldots, \vec{x}_s) \in (0, \infty]$ to each $(\vec{x}_1, \ldots, \vec{x}_s) \in X^{(s)}$ and satisfies the following conditions:

(a) For each $1 \leq j \leq s$, $w(\vec{x}_1, \ldots, \vec{x}_s)$ is invariant under permutations of the components of $\vec{x}_j$.

(b) 

$$w((\vec{x}_1, \ldots, \vec{x}_s) \circ (\vec{y}_1, \ldots, \vec{y}_s)) \leq w(\vec{x}_1, \ldots, \vec{x}_s)w(\vec{y}_1, \ldots, \vec{y}_s)$$

for all $(\vec{x}_1, \ldots, \vec{x}_s), (\vec{y}_1, \ldots, \vec{y}_s) \in X^{(s)}$ with $\text{supp}(\vec{x}_1, \ldots, \vec{x}_s) \cap \text{supp}(\vec{y}_1, \ldots, \vec{y}_s) \neq \emptyset$.

**Example II.4 (Weight Systems)**

(i) If $\kappa_1, \ldots, \kappa_s$ are functions from $X$ to $(0, \infty]$ (called weight factors) then 

$$w(\vec{x}_1, \ldots, \vec{x}_s) = \prod_{j=1}^s \prod_{\ell=1}^{n(\vec{x}_j)} \kappa_{j, \ell}(x_{j, \ell})$$

is a weight system of length $s$.

(ii) Let $d : X \times X \to \mathbb{R}_{\geq 0}$ be a distance function. By this we mean that $d$ is symmetric and fulfils the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. That is, we impose the conditions of a metric, with the exception(1) that $d(x, y)$ may be zero for different points $x, y \in X$. The length of a tree $T$ with vertices in $X$ is simply the sum of the lengths of all edges of $T$ (where the length of an edge is the distance between its vertices). For a subset $S \subset X$, denote by $\tau(S)$ the length of the shortest tree in $X$ whose set of vertices contains $S$. If $m \geq 0$, then

$$w(\vec{x}_1, \ldots, \vec{x}_s) = e^{m \tau(\text{supp}(\vec{x}_1, \ldots, \vec{x}_s))}$$

is a weight system of length $s$.

(1) The reason for the exception lies in the construction used in the proof of Corollary IV.3.
(iii) Assume again that \(d\) is a distance function on \(X\). By definition, the distance of any point \(x \in X\) to \(\Omega^c\) is \(d(x, \Omega^c) = \inf_{y \in \Omega^c} d(x, y)\) and the distance of any subset \(S\) of \(X\) to \(\Omega^c\) is \(d(S, \Omega^c) = \inf_{x \in S} d(x, y)\). On the other hand, the “maximum distance” of any subset \(S\) of \(X\) to \(\Omega^c\) is \(D(S, \Omega^c) = \sup_{x \in S} d(x, \Omega^c)\). Again let \(m \geq 0\) be a mass. Then

\[
w_{\Omega, m}(\bar{x}_1, \ldots, \bar{x}_s) = e^{md(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s), \Omega^c)}
\]

is a weight system of length \(s\).

If \(\Omega_2 \subseteq \Omega_1\) then, for all \((\bar{x}_1, \ldots, \bar{x}_s)\) with \(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s) \cap \Omega_2 \neq \emptyset\)

\[
w_{\Omega_2, m}(\bar{x}_1, \ldots, \bar{x}_s) \leq e^{-md(\Omega_2, \Omega_1^c)} w_{\Omega_1, m}(\bar{x}_1, \ldots, \bar{x}_s)
\]

(II.1)

To see this, choose \(x \in \text{supp}(\bar{x}_1, \ldots, \bar{x}_s)\) with \(d(x, \Omega_2^c) = D(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s), \Omega_2^c)\) and observe that

\[
D(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s), \Omega_1^c) \geq d(x, \Omega_1^c) \geq d(x, \Omega_2^c) + d(\Omega_2, \Omega_1^c)
\]

This is illustrated in the figure below, with \(D_1 = d(x, \Omega_1^c)\) and \(D_2 = d(x, \Omega_2^c)\).

\[\text{Diagram:} \quad \Omega_1 \supseteq \Omega_2 \supseteq x \supseteq \Omega_2^c \supseteq D_1 \supseteq D_2 \supseteq \Omega_1^c\]

(iv) Generalizing part (ii), assume that \(X\) is a subset of a Riemannian manifold \(X_{\mathbb{R}}\) and \(\mu\) a piecewise continuous non-negative function on \(X_{\mathbb{R}}\) (called the mass density). A tree in \(X_{\mathbb{R}}\) is a finite union of compact curves in \(X_{\mathbb{R}}\) that forms a topological tree. The \(\mu\)-length of a tree is defined to be the integral of \(\mu\) over the tree. For a finite subset \(S\) of \(X_{\mathbb{R}}\) denote by \(\tau_\mu(S)\) the infimum of the \(\mu\)-lengths of trees whose set of vertices contains \(S\). Then

\[
w_\mu(\bar{x}_1, \ldots, \bar{x}_s) = e^{\tau_\mu(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s))}
\]

is a weight system of length \(s\).

(v) Let \(\kappa \geq 1, N \in \mathbb{N}, 1 \leq s_0 \leq s\) and \(Y \subseteq X\). Denote by \(\nu_Y(\bar{x}_1, \ldots, \bar{x}_{s_0})\) the number of components of \(\bar{x}_1, \ldots, \bar{x}_{s_0}\) that are in \(Y\). Then

\[
w(\bar{x}_1, \ldots, \bar{x}_s) = \kappa^{\max\{N - \nu_Y(\bar{x}_1, \ldots, \bar{x}_{s_0}), 0\}}
\]
is a weight system of length $s$. The verification of condition (b) of Definition II.3 follows from
\[
\kappa_{\max \{N-m-n,0\}} \leq \kappa_{\max \{N-m,0\}} \kappa_{\max \{N-n,0\}} \quad \text{for all } m, n \geq 0
\]
(vi) If $w_1(\bar{x}_1, \cdots, \bar{x}_s)$ and $w_2(\bar{x}_1, \cdots, \bar{x}_s)$ are two weight systems of length $s$ then
\[
 w_3(\bar{x}_1, \cdots, \bar{x}_s) = w_1(\bar{x}_1, \cdots, \bar{x}_s)w_2(\bar{x}_1, \cdots, \bar{x}_s)
\]
is also a weight systems of length $s$.

**Definition II.5** Assume that $X$ is a metric space. Given weight factors $\kappa_j : X \to (0, \infty]$ for $j = 1, \cdots, s$, a mass $m \geq 0$ and a subset $\Omega$ of $X$ we call
\[
w(\bar{x}_1, \cdots, \bar{x}_s) = e^{m \tau(\text{supp}(\bar{x}_1, \cdots, \bar{x}_s))}e^{m D(\text{supp}(\bar{x}_1, \cdots, \bar{x}_s), \Omega^c)} \prod_{j=1}^{s} \prod_{\ell=1}^{n(\bar{x}_j)} \kappa_j(x_{j,\ell})
\]
the weight system with mass $m$ and decay set $\Omega$ that associates the weight factor $\kappa_j$ to the field $\alpha_j$. In the case $\Omega = \emptyset$, we call
\[
w(\bar{x}_1, \cdots, \bar{x}_s) = e^{m \tau(\text{supp}(\bar{x}_1, \cdots, \bar{x}_s))} \prod_{j=1}^{s} \prod_{\ell=1}^{n(\bar{x}_j)} \kappa_j(x_{j,\ell})
\]
the weight system with mass $m$ that associates the weight factor $\kappa_j$ to the field $\alpha_j$. It follows from parts (i), (ii), (iii) and (vi) of Example II.4 that these are indeed weight systems.

**Definition II.6 (Norms)**

(i) Let $w$ be a weight system and $a$ a coefficient system of length $s$. Let $0 \leq s' \leq s$. We think of the fields $\alpha_j$ with $1 \leq j \leq s'$ as being sources (that is, we differentiate with respect to these fields to generate correlation functions) and the fields $\alpha_j$ with $s' < j \leq s$ as being internal fields (that is, they will be integrated out). For any $n_1, \cdots, n_s \geq 0$ and any function $b(\bar{x}_1, \cdots, \bar{x}_s)$ on $X^{n_1} \times \cdots \times X^{n_s}$, we define the norm
\[
\|b\|_{n_1, \cdots, n_s} = \begin{cases} 
|b(-, \cdots, -)| & \text{if } \sum_{j=1}^{s} n_j = 0 \\
\max_{x \in X} \max_{s' < j \leq s, \sum_{\ell=1}^{n_j} x_{\ell} = x} \sum_{1 \leq \ell \leq s'} |b(-, \cdots, -x_{s'+1}, \cdots, \bar{x}_s)| & \text{if } \sum_{j=1}^{s} n_j \neq 0, \sum_{j=1}^{s'} n_j = 0 \\
\max_{x_{\ell} \in X^{n_\ell}} \sum_{1 \leq \ell \leq s'} |b(\bar{x}_1, \cdots, \bar{x}_s)| & \text{if } \sum_{j=1}^{s'} n_j \neq 0
\end{cases}
\]
Here \((\bar{x}_j)_i\) is the \(i\)th component of the \(n_j\)-tuple \(\bar{x}_j\). We define the norm of \(a\) with weight \(w\) to be

\[
|a|_w = \sum_{n_1, \ldots, n_s \geq 0} \|w(\bar{x}_1, \ldots, \bar{x}_s) a(\bar{x}_1, \ldots, \bar{x}_s)\|_{n_1, \ldots, n_s}
\]

In some applications, it will be more convenient to turn this norm into a seminorm by ignoring the constant term \(a(-)\). The results of this paper apply equally well to such seminorms.

(ii) Let \(w\) be a weight system and \(f(\alpha_1, \ldots, \alpha_s)\) be a function which is defined and analytic on a neighbourhood of the origin in \(C^s[X]\). The norm, \(\|f\|_w\) of \(f\) with weight \(w\) is defined\(^{(2)}\) to be \(|a|_w\) where \(a\) is the symmetric coefficient system of \(f\).

Remark II.7 Let \(a\) be a (not necessarily symmetric) coefficient system of length \(s\) and

\[
f(\alpha_1, \ldots, \alpha_s) = \sum_{(\bar{x}_1, \ldots, \bar{x}_s) \in X^s} a(\bar{x}_1, \ldots, \bar{x}_s) \alpha_1(\bar{x}_1) \cdots \alpha_s(\bar{x}_s)
\]

Then \(\|f\|_w \leq |a|_w\) for any weight system \(w\). We call \(a\) a not necessarily symmetric coefficient system for \(f\).

\(^{(2)}\) This definition also applies when \(f\) depends only on a subset of the variables \(\alpha_1, \ldots, \alpha_s\).
III. The Main Theorem

In (I.3), we integrate out the last two fields \( z_s, z \) using the measure \( \mu(z^*, z) \) of (I.1). Recall that this measure is supported in \( \|z\|_\infty \leq r \). The weight systems that are adapted to this situation fulfil the Definition III.1, below, with \( \rho = 4r \).

**Definition III.1** A weight system of length \( s + 2 \) has “gives weight at least \( \rho \) to the last two fields” if

\[
w(\bar{x}_1, \cdots, \bar{x}_s; \bar{y}_*, \bar{y}) \geq \rho^{n(\bar{y}_*)+n(\bar{y})} w(\bar{x}_1, \cdots, \bar{x}_s; -,-)
\]

for all \((\bar{x}_1, \cdots, \bar{x}_s) \in X^s\) and \((\bar{y}_*, \bar{y}) \in X^{(2)}\).

**Example III.2** Assume that \( X \) is a metric space and \( \Omega \subset X \). Given weight factors \( \kappa_j : X \to (0,\infty) \) for \( j = 1, \cdots, s \) and a mass \( m \geq 0 \), the weight system with mass \( m \) and decay set \( \Omega \) that associates the weight factor \( \kappa_j \) to the field \( \alpha_j, j = 1, \cdots, s \) and the constant weight factor \( \rho \) to the fields \( \alpha_{s+1} \) and \( \alpha_{s+2} \) gives weight at least \( \rho \) to the last two fields.

We fix, for the rest of this section, a weight system \( w \) of length \( s + 2 \) that gives weight at least \( 4r \) to the last two fields. Furthermore we fix the number \( 0 \leq s' \leq s \) of source fields for the Definition II.6 of \( \| \cdot \|_w \).

**Remark III.3**

(i) If \( h \) is an analytic function for which \( h(0, \cdots, 0; z_s, z) \) is constant, then

\[
\left\| \int h(\alpha_1, \cdots, \alpha_s; z^*, z) \, d\mu(z^*, z) \right\|_w \leq \left\| h(\alpha_1, \cdots, \alpha_s; z_s, z) \right\|_w
\]

(ii) Assume that the measure \( d\mu(z^*, z) \) on \( \mathbb{C} \) is rotation invariant. Let \( h_j(\alpha_1, \cdots, \alpha_s; z_s, z), j = 1, 2 \), be analytic functions with \( h_j(0, \cdots, 0; z_s, z) = 0 \). Further assume that the symmetric coefficient system \( a_j(\bar{x}_1, \cdots, \bar{x}_s; \bar{y}_*, \bar{y}) \) of \( h_j \) obeys \( a_j(\bar{x}_1, \cdots, \bar{x}_s; \bar{y}_*, \bar{y}) = 0 \) whenever \( \bar{y} = \bar{y}_* \). Then

\[
\int h_j(\alpha_1, \cdots, \alpha_s; z^*, z) \, d\mu(z^*, z) = 0 \quad \text{for } j = 1, 2
\]

and

\[
\left\| \int h_1(\alpha_1, \cdots, \alpha_s; z^*, z) h_2(\alpha_1, \cdots, \alpha_s; z^*, z) \, d\mu(z^*, z) \right\|_w \leq \left\| h_1 \right\|_w \left\| h_2 \right\|_w
\]
Proof:  (i) follows from the observation that
\[
\left| \int z(\tilde{y}_*)^* z(\tilde{y}) \, d\mu(z^*, z) \right| \leq r^{n(\tilde{y}_*)+n(\tilde{y})}
\]
for all \( \tilde{y}_*, \tilde{y} \in X^{(2)} \).

(ii) Write
\[
\tilde{h}(\alpha_1, \ldots, \alpha_s) = \int h_1(\alpha_1, \ldots, \alpha_s; z^*, z) h_2(\alpha_1, \ldots, \alpha_s; z^*, z) \, d\mu(z^*, z)
= \sum_{(\tilde{x}_1, \ldots, \tilde{x}_s) \in X^{(s)}} \tilde{a}(\tilde{x}_1, \ldots, \tilde{x}_s) \alpha_1(\tilde{x}_1) \cdots \alpha_s(\tilde{x}_s)
\]
with, for each \( \zeta \in X^{(s)} \),
\[
\tilde{a}(\zeta) = \sum_{\xi, \xi' \in X^{(s)}} \sum_{\xi \circ \xi' = \zeta} \sum_{\eta, \eta' \in X^{(2)}} \left[ a_1(\xi; \tilde{y}_*, \tilde{y}) a_2(\xi'; \tilde{y}_*'; \tilde{y}') \right] \int z(\tilde{y}_* \circ \tilde{y}_*')^* z(\tilde{y} \circ \tilde{y}') \, d\mu(z^*, z)
\]
We claim that only terms with \( \text{supp}(\tilde{y}_*, \tilde{y}) \cap \text{supp}(\tilde{y}_*'; \tilde{y}') \neq \emptyset \) can be nonzero. By hypothesis, \( a_1(\xi; \tilde{y}_*, \tilde{y}) = 0 \) if \( \tilde{y}_* = \tilde{y} \). So we may assume that there is a \( y \in \text{supp}(\tilde{y}_*, \tilde{y}) \) with the multiplicities of \( \tilde{y}_* \) and \( \tilde{y} \) at \( y \) being different. Since \( \int z(\tilde{y}_* \circ \tilde{y}_*')^* z(\tilde{y} \circ \tilde{y}') \, d\mu(z^*, z) \) vanishes(1) unless \( \tilde{y}_* \circ \tilde{y}_*' = \tilde{y} \circ \tilde{y}' \), the multiplicities of \( \tilde{y}_* \) and \( \tilde{y}' \) at \( y \) must also be different.

In particular, \( y \in \text{supp}(\tilde{y}_*, \tilde{y}') \).

Consequently, by (1.2),
\[
|\tilde{a}(\zeta)| \leq \sum_{\xi, \xi' \in X^{(s)}} \sum_{\xi \circ \xi' = \zeta} \sum_{\eta, \eta' \in X^{(2)}} a_1(\xi; \eta) \left| a_2(\xi'; \eta') \right| r^n(\eta)+n(\eta')
\]
Since \( w \) gives weight at least \( 2r \) to the last two fields,
\[
w(\zeta) |\tilde{a}(\zeta)| \leq \sum_{\xi, \xi' \in X^{(s)}} \sum_{\xi \circ \xi' = \zeta} \sum_{\eta, \eta' \in X^{(2)}} w(\zeta; \eta \cap \eta') \left| a_1(\xi; \eta) \right| \left| a_2(\xi'; \eta') \right| 2^{-n(\eta)-n(\eta')}
\leq \sum_{\xi, \xi' \in X^{(s)}} \sum_{\xi \circ \xi' = \zeta} \sum_{\eta, \eta' \in X^{(2)}} w(\zeta; \eta) \left| a_1(\xi; \eta) \right| w(\zeta'; \eta') \left| a_2(\xi'; \eta') \right| 2^{-n(\eta)-n(\eta')}
\]

(1) To see this, let \( y \in X \) and suppose that the multiplicity, say \( p_* \), of \( \tilde{y}_* \circ \tilde{y}_*' \) at \( y \) is different from the multiplicity, say \( p \), of \( \tilde{y} \circ \tilde{y}' \) at \( y \). Because \( d\mu \) is invariant under \( \zeta(y) \mapsto \zeta(y) = e^{i\theta} z(y) \), while \( [z_\theta(y)^*]^{p_*} z_\theta(y)^p = e^{i(p-p_\theta)} [z(y)^*]^{p_*} z(y)^p \), the integral \( \int [z(y)^*]^{p_*} z(y)^p \, d\mu(z^*, z) \) must vanish.

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If $\xi = (\tilde{x}_1, \ldots, \tilde{x}_s) \in X^{(s)}$, we write $\xi_i \in X$ for $(\tilde{x}_1 \circ \cdots \circ \tilde{x}_s)_i$. By hypothesis, $\tilde{a}(\xi) = 0$ if $\xi = \varepsilon$. So, if there are no source fields,

$$
\|\tilde{h}\|_w = \sum_{n_1, \ldots, n_s \geq 0} \max_{\mathbf{x} \in X} \max_{1 \leq i \leq n_1 + \cdots + n_s} \sum_{\xi \in X^{n_1} \times \cdots \times X^{n_s}} w(\xi) |\tilde{a}(\xi)|
$$

$$
\leq \sum_{n_1, \ldots, n_s \geq 0} \max_{\mathbf{x} \in X} \max_{1 \leq i \leq \Sigma j (n_j + n'_j)} \sum_{\xi' \in X^{n'_1} \times \cdots \times X^{n'_s}} \sum_{\eta \in X^{m'_1} \times X^{m'_2}} w(\xi; \eta) |a_1(\xi; \eta)| w(\xi'; \eta') |a_2(\xi'; \eta')| 2^{-(m'_1 + m'_2 + m + m')}
$$

Fix, temporarily, $n_1, \ldots, n_s, n'_1, \ldots, n'_s, m_s, m, m'_s, m'$ $\geq 0$. If there are no source fields, also fix, temporarily, $1 \leq i \leq \sum_j (n_j + n'_j)$. There are source fields if $s' > 0$ ($s'$ was specified in Definition II.6) and $\sum_{j=1}^{s'} (n_j + n'_j) \geq 1$. In this case

$$
\max_{\mathbf{x} \in X} \sum_{\xi \in X^{n_1} \times \cdots \times X^{n_s}} \sum_{\xi' \in X^{n'_1} \times \cdots \times X^{n'_s}} w(\xi; \eta) |a_1(\xi; \eta)| w(\xi'; \eta') |a_2(\xi'; \eta')| 2^{-(m'_1 + m'_2 + m + m')}
$$

is replaced by

$$
\max_{\mathbf{x}_j, \xi_j \in X} \max_{1 \leq j \leq s'} \max_{1 \leq i \leq n_j} \sum_{\xi' \in X^{n'_1} \times \cdots \times X^{n'_s}} \sum_{\eta \in X^{m'_1} \times X^{m'_2}} w(\xi; \eta) |a_1(\xi; \eta)| w(\xi'; \eta') |a_2(\xi'; \eta')| 2^{-(m'_1 + m'_2 + m + m')}
$$

If we translate the $(\tilde{x}_j)_i$, $(\tilde{x}'_j)_i$ notation into the corresponding components of $\xi, \xi'$ we may write both of these max/sums in the form

$$
\max_{\mathbf{x}_j \in X} \max_{p \in A} \sum_{\eta \in X^{m'_1} \times X^{m'_2}} w(\xi; \eta) |a_1(\xi; \eta)| w(\xi'; \eta') |a_2(\xi'; \eta')| 2^{-(m'_1 + m'_2 + m + m')}
$$

for some subsets $A \subset \{ p \in \mathbb{N} \mid 1 \leq p \leq \sum_j n_j \}$ and $A' \subset \{ p \in \mathbb{N} \mid 1 \leq p \leq \sum_j n'_j \}$ with $A \cup A'$ nonempty.

Fix $\mathbf{x}_p \in X$ for each $p \in A$ and $\mathbf{x}'_p \in X$ for each $p \in A'$. For notational simplicity, suppose that $p = 1 \in A$. Then

$$
\sum_{\xi \in X^{n'_1} \times \cdots \times X^{n'_s}} \sum_{\eta \in X^{m'_1} \times X^{m'_2}} w(\xi; \eta) |a_1(\xi; \eta)| w(\xi'; \eta') |a_2(\xi'; \eta')| 2^{-(m'_1 + m'_2 + m + m')}
$$
Taking the maximum over $A$. If $A$ is empty (i.e. there are no source fields in $\xi'$), the second large bracket is

$$\left( \max_{1 \leq \ell \leq m + m'} \max_{y \in X} \sum_{\eta' \in X^{m} \times X^{m'}} w(\xi'; \eta') |a_2(\xi'; \eta')| \right)$$

If $A'$ is not empty (i.e. there are source fields in $\xi'$), we bound the second large bracket by

$$\left( \sum_{\xi' \in X^{n} \times X^{n'}} w(\xi'; \eta') |a_2(\xi'; \eta')| \right)$$

Taking the maximum over $x_p$'s and $x_p'$'s, and possibly over $i$, and the remaining sums gives the desired bound.

Recall that we have fixed a weight system $w$ of length $s + 2$ that gives weight at least $4r$ to the last two fields and that we have fixed the number $0 \leq s' \leq s$ of source fields.

**Theorem III.4** If $f(\alpha_1, \cdots, \alpha_s; z_*, z)$ obeys $\|f\|_w < \frac{1}{16}$, then there is an analytic function $g(\alpha_1, \cdots, \alpha_s)$ such that

$$\frac{\int e^{f(\alpha_1, \cdots, \alpha_s; z_*, z)} \, d\mu(z_*, z)}{\int e^{f(0, \cdots, 0; z_*, z)} \, d\mu(z_*, z)} = e^{g(\alpha_1, \cdots, \alpha_s)} \tag{III.1}$$

and

$$\|g\|_w \leq \frac{\|f\|_w}{1 - \frac{1}{16}\|f\|_w}$$
Proof: Let $a(\vec{x}_1, \cdots, \vec{x}_s; \vec{y}, \vec{y})$ be the symmetric coefficient system for $f$. We first introduce some shorthand notation.

- For $\eta = (\vec{y}, \vec{y}) \in X^{(2)}$, we write $z(\eta) = z(\vec{y}_s)^* z(\vec{y})$ and

$$a(\eta) = \sum_{(\vec{x}_1, \cdots, \vec{x}_s) \in X^{(s)}} a(\vec{x}_1, \cdots, \vec{x}_s; \vec{y}_s, \vec{y}) \alpha_1(\vec{x}_1) \cdots \alpha_s(\vec{x}_s)$$

With this notation

$$f(\alpha_1, \cdots, \alpha_s; z^*, z) = \sum_{\eta \in X^{(2)}} a(\eta) \ z(\eta)$$

By factoring $e^{f(\alpha_1, \cdots, \alpha_s; z^*, z)}$ out of the integral in the numerator of (III.1), we may assume that $a(-, -) = 0$.

- Let $X_1, \cdots, X_\ell$ be subsets of $X$. The incidence graph $G(X_1, \cdots, X_\ell)$ of $X_1, \cdots, X_\ell$ is the labelled graph with the set of vertices $\{1, \cdots, \ell\}$ and edges between $i \neq j$ whenever $X_i \cap X_j \neq \emptyset$. For $\eta_1, \cdots, \eta_\ell \in X^{(2)}$ we write $G(\eta_1, \cdots, \eta_\ell)$ for $G(\text{supp } \eta_1, \cdots, \text{supp } \eta_\ell)$.

- A collection $\eta_1, \cdots, \eta_n \in X^{(2)}$ is called connected if the incidence graph $G(\eta_1, \cdots, \eta_n)$ is connected. For a subset of $Z \subset X$ we denote by $\mathcal{C}(Z)$ the set of all ordered tuples $(\eta_1, \cdots, \eta_n)$ that are connected and for which $Z = \text{supp } \eta_1 \cup \cdots \cup \text{supp } \eta_n$. We call such a tuple a connected cover of $Z$.

Expanding the exponential

$$e^{f(\alpha_1, \cdots, \alpha_s; z^*, z)} = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} f(\alpha_1, \cdots, \alpha_s; z^*, z)^\ell$$

$$= 1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \sum_{\substack{Z \subset X \\ Z \neq \emptyset}} \sum_{\eta_1, \cdots, \eta_\ell \in X^{(2)}} a(\eta_1) \cdots a(\eta_\ell) \ z(\eta_1) \cdots z(\eta_\ell)$$

Given a subset $Z$ of $X$ and $\eta_1, \cdots, \eta_\ell \in X^{(2)}$ with $Z = \text{supp } \eta_1 \cup \cdots \cup \text{supp } \eta_\ell$, there is a (unique, up to labelling) decomposition of $\{1, \cdots, \ell\}$ into pairwise disjoint subsets $I_1, \cdots, I_n$ and a decomposition of $Z$ into pairwise disjoint subsets $Z_1, \cdots, Z_n$ such that, for each $1 \leq j \leq n$, $(\eta_i, i \in I_j)$ is a connected cover of $Z_j$. This decomposition corresponds to the decomposition of $G(\eta_1, \cdots, \eta_\ell)$ into connected components. Therefore

$$e^f = 1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \sum_{n=1}^{\ell} \frac{1}{n!} \sum_{\substack{Z_1, \cdots, Z_n \subset X \setminus \{\emptyset\} \\ \text{pairwise disjoint}}} \sum_{I_1 \cup \cdots \cup I_n = \{1, \cdots, \ell\}} \sum_{\eta_1, \cdots, \eta_n} a(\eta_1) \cdots a(\eta_\ell) \ z(\eta_1) \cdots z(\eta_\ell)$$

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Fix, for the moment, pairwise disjoint nonempty subsets $Z_1, \ldots, Z_n$ of $X$ and $\ell \geq n$. Then

$$\sum_{I_1, \ldots, I_n \text{ pairwise disjoint}} \sum_{\eta_1, \ldots, \eta_\ell \in C(Z_j)} a(\eta_1) \cdots a(\eta_\ell) \, z(\eta_1) \cdots z(\eta_\ell)$$

$$= \sum_{k_1 + \cdots + k_\ell = \ell} \sum_{I_1, \ldots, I_n \subseteq (1, \ldots, \ell)} \sum_{|I_j| = k_j} a(\eta_1) \cdots a(\eta_\ell) \, z(\eta_1) \cdots z(\eta_\ell)$$

$$= \sum_{k_1 + \cdots + k_\ell = \ell} \frac{\ell!}{k_1! \cdots k_\ell!} \sum_{(\eta_1, \ldots, \eta_\ell)} a(\eta_1) \cdots a(\eta_\ell) \, z(\eta_1) \cdots z(\eta_\ell)$$

We define, for $\emptyset \neq Z \subseteq X$, the function $\Phi(Z)(\alpha_1, \ldots, \alpha_s)$ by

$$\Phi(Z) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{(\eta_1, \ldots, \eta_k) \in C(Z)} a(\eta_1) \cdots a(\eta_k) \int z(\eta_1) \cdots z(\eta_k) \, d\mu(z^*, z)$$  \hspace{1cm} (III.2)

and $\Phi(\emptyset) = 0$. As the measure $\mu$ factorizes with each factor normalized, and the different $Z_j$’s are disjoint,

$$\int z(\eta_1) \cdots z(\eta_\ell) \, d\mu(z^*, z) = \prod_{j=1}^{n} \int z(\eta_{p_j-1} + 1) \cdots z(\eta_{p_j}) \, d\mu(z^*, z)$$

(where $p_0 = 0$ and, for $1 \leq j \leq n$, $p_j = k_1 + \cdots + k_j$) and we have

$$\int e^{f(\alpha_1, \ldots, \alpha_s; z^*, z)} \, d\mu(z^*, z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Z_1, \ldots, Z_n \subseteq X \text{ pairwise disjoint}} \prod_{j=1}^{n} \Phi(Z_j)$$

If we define

$$\zeta(Z, Z') = \begin{cases} 0 & \text{if } Z \cap Z' \neq \emptyset \\ 1 & \text{if } Z \text{ and } Z' \text{ are disjoint} \end{cases}$$

and $G_n = \{ \{i, j\} \subseteq \mathbb{N}^2 \mid i, j \leq n, i \neq j \}$ is the complete graph on $\{1, \ldots, n\}$, then

$$\int e^{f(\alpha_1, \ldots, \alpha_s; z^*, z)} \, d\mu(z^*, z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Z_1, \ldots, Z_n \subseteq X} \prod_{\{i, j\} \in G_n} \zeta(Z_i, Z_j) \prod_{j=1}^{n} \Phi(Z_j)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Z_1, \ldots, Z_n \subseteq X} \left( \sum_{g \subseteq G_n} \prod_{\{i, j\} \in g} (\zeta(Z_i, Z_j) - 1) \right) \prod_{j=1}^{n} \Phi(Z_j)$$

by the binomial expansion. Here, whenever a product $\prod_{\{i, j\} \in G_n}$ or $\prod_{\{i, j\} \in g}$ is empty, as is the case for $n = 1$, it is given the value one. We may identify each $g \subseteq G_n$ with the
labelled graph on the set of vertices \( \{1, \cdots, n\} \) that has an edge joining vertex \( i \) and vertex \( j \) if and only if \( \{i, j\} \in g \). Denote by \( G_n \) the set of all graphs (connected or not) on the set of vertices \( \{1, \cdots, n\} \) that have at most one edge joining each pair of distinct vertices and no edges joining a vertex to itself. Define

\[
\rho(Z_1, \cdots, Z_n) = \begin{cases} 
1 & \text{if } n = 1 \\
\sum_{g \in \mathcal{G}_n} \prod_{\{i,j\} \in g} (\zeta(Z_i, Z_j) - 1) & \text{if } n \geq 2
\end{cases}
\]

In this notation

\[
\int e^{f(\alpha_1, \cdots, \alpha_s, z^*, z)} d\mu(z^*, z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Z_1, \cdots, Z_n \subseteq \mathcal{X}} \rho(Z_1, \cdots, Z_n) \prod_{j=1}^{n} \Phi(Z_j)
\]

Now let \( C_n \subset G_n \) be the set of all connected graphs on the set of vertices \( \{1, \cdots, n\} \) that have at most one edge joining each pair of distinct vertices and no edges joining a vertex to itself. Set

\[
\rho^T(Z_1, \cdots, Z_n) = \begin{cases} 
1 & \text{if } n = 1 \\
\sum_{g \in C_n} \prod_{\{i,j\} \in g} (\zeta(Z_i, Z_j) - 1) & \text{if } n \geq 2
\end{cases}
\]

Then, by a standard argument (see, for example [Sa, Theorem 2.17]),

\[
\ln \int e^{f} d\mu = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Z_1, \cdots, Z_n \subseteq \mathcal{X}} \rho^T(Z_1, \cdots, Z_n) \prod_{j=1}^{n} \Phi(Z_j)
\]

(By “\( \ln \)” we just mean that the exponential of the right hand side is \( \int e^{f} d\mu \).)

Let, for any connected graph \( G \in C_n \),

\[
t(G) = \begin{cases} 
1 & \text{if } n = 1 \\
\sum_{g \subseteq G} (-1)^{|g|} & \text{if } n > 1
\end{cases}
\]

The bound

\[
|t(G)| \leq \# \{ \text{ spanning trees in } G \}
\]

is due to Rota [Ro]. For a simple proof see [Si, Theorem V.7.A.6]. Since

\[
\rho^T(Z_1, \cdots, Z_n) = \sum_{g \in C_n} (-1)^{|g|} = t(G(Z_1, \cdots, Z_n))
\]

we have that

\[
|\rho^T(Z_1, \cdots, Z_n)| \leq \# \{ \text{ tree } T \text{ on } \{1, \cdots, n\} \mid |T| = n - 1, T \subseteq G(Z_1, \cdots, Z_n) \}
\]

(III.4)
In particular, $\rho^T(Z_1, \ldots, Z_n) = 0$ if $G(Z_1, \ldots, Z_n)$ is not connected.

To get $a$, a not necessarily symmetric, coefficient system for $\ln \int e^f \, d\mu$ above we first construct a coefficient system for each $\Phi(Z)$. For each $\xi \in X^{(s)}$ and $\eta \in X^{(2)}$, set

$$\tilde{a}(\xi; \eta) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{(\eta_1, \ldots, \eta_k) \in \text{supp} \eta} \sum_{\xi_1 \cdots \xi_k = \xi} a(\xi_1; \eta_1) \cdots a(\xi_k; \eta_k) \int \sum_{z(\eta)} d\mu(z^*, z) \quad (\text{III.5})$$

By (III.2)

$$\Phi(Z)(\alpha_1, \ldots, \alpha_s) = \sum_{(\bar{x}_1, \ldots, \bar{x}_s) \in X^{(s)}} \sum_{\eta \in X^{(2)}} \tilde{a}(\bar{x}_1, \ldots, \bar{x}_s; \eta) \alpha_1(\bar{x}_1) \cdots \alpha_s(\bar{x}_s)$$

Therefore, by (III.3)

$$\ln \int e^f \, d\mu = \sum_{(\bar{x}_1, \ldots, \bar{x}_s) \in X^{(s)}} a'(\bar{x}_1, \ldots, \bar{x}_s) \alpha_1(\bar{x}_1) \cdots \alpha_s(\bar{x}_s)$$

where, for $\xi \in X^{(s)}$,

$$a'(\xi) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\eta_1, \ldots, \eta_n \in X^{(2)}} \sum_{\eta_1 \cdots \eta_n = \xi} \rho^T(\text{supp} \eta_1, \ldots, \text{supp} \eta_n) \prod_{j=1}^{n} \tilde{a}(\xi_j; \eta_j) \quad (\text{III.6})$$

Also

$$g(\alpha_1, \ldots, \alpha_s) = \ln \frac{\int e^f(\alpha_1, \ldots, \alpha_s; z^*, z) \, d\mu(z^*, z)}{\int e^f(0, \ldots, 0; z^*, z) \, d\mu(z^*, z)} = \sum_{(\bar{x}_1, \ldots, \bar{x}_s) \in X^{(s)}} a'(\bar{x}_1, \ldots, \bar{x}_s) \alpha_1(\bar{x}_1) \cdots \alpha_s(\bar{x}_s)$$

so that $a'$, excluding the constant term $a'(-)$, is a, not necessarily symmetric, coefficient system for $g$. By Remark II.7,

$$\|g\|_W \leq |a'|_W \quad (\text{III.7})$$

For each nontrivial $\xi \in X^{(s)}$, by (III.6) and (III.4),

$$|a'(\xi)| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\eta_1, \ldots, \eta_n \in X^{(2)}} \sum_{\eta_1 \cdots \eta_n = \xi} \prod_{j=1}^{n} |\tilde{a}(\xi_j; \eta_j)|$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{T \text{ labelled tree with vertices } 1, \ldots, n} \sum_{\eta_1 \cdots \eta_n = \xi} \sum_{T \subset G(\eta_1, \ldots, \eta_n)} |\tilde{a}(\xi_1; \eta_1)| \cdots |\tilde{a}(\xi_n; \eta_n)|$$

$$\quad (\text{III.8})$$
Below we develop two lemmata which enable us to use (III.8) to get the bound

\[ |a'|_w \leq \frac{|\tilde{a}|_{w_2}}{1 - 8|\tilde{a}|_{w_2}} \]  

(III.9)

where, for any \( \sigma > 0 \),

\[ w_\sigma(\xi; \vec{y}^*, \vec{y}) = w(\xi; - , - ) \sigma^{n(\vec{y}^*) + n(\vec{y})} \]

These lemmata also enable us to use (III.5) to get the bound

\[ |\tilde{a}|_{w_2} \leq \frac{|a|_w}{1 - 8|a|_w} \]  

(III.10)

Combining (III.7), (III.9) and (III.10) yields

\[ \|g\|_w \leq \frac{|a|_w}{1 - 16|a|_w} = \frac{\|f\|_w}{1 - 16\|f\|_w} \]

Hence to complete the proof of the theorem, it suffices to prove (III.9) and (III.10).

**Proof of (III.9), assuming Lemmas III.5 and III.6:**

By (III.8)

\[ |a'(\xi)| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{T \text{ labelled tree with vertices } 1, \ldots, n} \sum_{\eta \in X^{(2)}} |\tilde{a}|_T(\xi; \eta) \]

where

\[ |\tilde{a}|_T(\xi; \eta) = \sum_{\eta_1, \ldots, \eta_n \in X^{(2)}} \sum_{\eta = \eta_1 \circ \cdots \circ \eta_n} \sum_{\xi = \xi_1 \circ \cdots \circ \xi_n} |\tilde{a}(\xi_1; \eta_1)| \cdots |\tilde{a}(\xi_n; \eta_n)| \]

Therefore, by Lemma III.5,

\[ |a' - a'(-)|_w \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{T \text{ labelled tree with vertices } 1, \ldots, n} |\tilde{a}|_T |w_1| \]

\[ \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{d_1, \ldots, d_n} \sum_{T \text{ labelled tree with coordination numbers } d_1, \ldots, d_n} |\tilde{a}|_T |w_1| \]

\[ \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{d_1, \ldots, d_n} d_1! \cdots d_n! \sum_{T \text{ labelled tree with coordination numbers } d_1, \ldots, d_n} |\tilde{a}|_{w_2} \]

Now apply Lemma III.6 with \( \varepsilon = |\tilde{a}|_{w_2} = |\tilde{a}|_{w_2} \) and \( \nu = 1 \) to get

\[ |a'|_w \leq \frac{|\tilde{a}|_{w_2}}{1 - 8|\tilde{a}|_{w_2}} \]
Proof of (III.10), assuming Lemmas III.5 and III.6: Recall that \( \tilde{a} \) was defined in (III.5). For every \((\eta_1, \ldots, \eta_k)\) contributing to (III.5), \(G(\eta_1, \ldots, \eta_k)\) is connected and hence contains at least one tree. So
\[
|\tilde{a}(\xi; \eta)| \leq \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{T \text{ labelled tree with vertices } 1, \ldots, k} \sum_{\eta \in X(2)} \sum_{T \subseteq G(\eta_1, \ldots, \eta_k)} |a(\xi_1; \eta_1)| \cdots |a(\xi_k; \eta_k)| r^n(\eta)
\]
\[
= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{T \text{ labelled tree with vertices } 1, \ldots, k} r^n(\eta) |a|_T(\xi; \eta)
\]
where
\[
|a|_T(\xi; \eta) = \sum_{\eta_1, \ldots, \eta_n \in X(2)} \sum_{T \subseteq G(\eta_1, \ldots, \eta_n)} |a(\xi_1; \eta_1)| \cdots |a(\xi_n; \eta_n)|
\]
By construction, \( |r^n(\eta)| a|_T(\xi; \eta) \big|_{w_2} = |a|_T \big|_{w_2r} \). Hence, by Lemma III.5, followed by Lemma III.6,
\[
|\tilde{a}|_{w_2} \leq \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{T \text{ labelled tree with vertices } 1, \ldots, k} |a|_T \big|_{w_2r} \leq \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{d_1, \ldots, d_k \in \mathbb{N}} d_1! \cdots d_k! |a|_{w_4r}^k \leq \frac{|a|_w}{1 - 8/a|_w}
\]
since \( w_{4r}(\xi) \leq w(\xi) \) for all \( \xi \in X^s \). This gives (III.10).

\[\square\]

**Lemma III.5** Let \( \omega \) be an arbitrary weight system of length \( s + 2 \) and define the weight system \( \omega_2 \) by
\[
\omega_2(\bar{x}_1, \ldots, \bar{x}_s; \bar{y}_s, \bar{y}) = 2^{n(\bar{y}_s)} + n(\bar{y}) \omega(\bar{x}_1, \ldots, \bar{x}_s; \bar{y}_s, \bar{y})
\]
Let \( T \) be a labelled tree with vertices \( 1, \ldots, n \) and coordination numbers \( d_1, \ldots, d_n \). Let \( b \) be any (not necessarily symmetric) coefficient system of length \( s + 2 \) with \( b(\cdot; \cdot) = 0 \). We define a new coefficient system \( b_T \) by
\[
b_T(\xi; \eta) = \sum_{\eta_1, \ldots, \eta_n \in X(2)} \sum_{T \subseteq G(\eta_1, \ldots, \eta_n)} b(\xi_1; \eta_1) \cdots b(\xi_n; \eta_n)
\]
Then
\[
|b_T|_\omega \leq d_1! \cdots d_n! |b|_{\omega_2}^n
\]
Proof: The case $n = 1$ is trivial, so assume that $n \geq 2$. For each $\vec{N} \in \mathbb{N}_0^{s+2} = \{ (N_1, \ldots, N_{s+2}) \in \mathbb{Z}^{s+2} \mid N_j \geq 0 \text{ for all } 1 \leq j \leq s+2 \}$, let $b_{\vec{N}}(\xi, \eta)$ denote the restriction of $b(\vec{x}_1, \ldots, \vec{x}_s; \vec{y}_s, \vec{y})$ to $(n(\vec{x}_1), \ldots, n(\vec{x}_s), n(\vec{y}_s), n(\vec{y})) = \vec{N}$. That is,

$$b_{\vec{N}}(\vec{x}_1, \ldots, \vec{x}_s; \vec{y}_s, \vec{y}) = \begin{cases} b(\vec{x}_1, \ldots, \vec{x}_s; \vec{y}_s, \vec{y}) & \text{if } (n(\vec{x}_1), \ldots, n(\vec{x}_s), n(\vec{y}_s), n(\vec{y})) = \vec{N} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$b_T = \sum_{\vec{N}(1), \ldots, \vec{N}(n) \in \mathbb{N}_0^{s+2}} b_{\vec{N}(1), \ldots, \vec{N}(n)}$$

where

$$b_{\vec{N}(1), \ldots, \vec{N}(n)}(\xi, \eta) = \sum_{\eta_1, \ldots, \eta_n \in X^{(2)}} \sum_{T \subset G(\eta_1, \ldots, \eta_n)} b_{\vec{N}(1)}(\xi_1; \eta_1) \cdots b_{\vec{N}(n)}(\xi_n; \eta_n)$$

Since

$$|b_T|_\omega \leq \sum_{\vec{N}(1), \ldots, \vec{N}(n) \in \mathbb{N}_0^{s+2}} |b_{\vec{N}(1), \ldots, \vec{N}(n)}|_\omega$$

and

$$|b|_{\omega_2} = \sum_{\vec{N} \in \mathbb{N}_0^{s+2}} |b_{\vec{N}}|_{\omega_2}$$

it suffices to prove that, for any $\vec{N}(1), \ldots, \vec{N}(n) \in \mathbb{N}_0^{s+2}$,

$$|b_{\vec{N}(1), \ldots, \vec{N}(n)}|_\omega \leq d_1! \cdots d_n! \prod_{j=1}^n \left| b_{\vec{N}(j)} \right|_{\omega_2}$$

Furthermore, since $T$ is connected, part (b) of Definition II.3 ensures that

$$\omega(\xi; \eta) \leq \prod_{j=1}^n \omega(\xi_j; \eta_j)$$

for all $\xi_1, \ldots, \xi_n \in X^{(s)}$ and $\eta_1, \ldots, \eta_n \in X^{(2)}$ such that $\xi = \xi_1 \circ \cdots \circ \xi_n$, $\eta = \eta_1 \circ \cdots \circ \eta_n$ and $T \subset G(\eta_1, \ldots, \eta_n)$. So it suffices to consider $\omega = 1$.

Fix any $\vec{N}(1), \ldots, \vec{N}(n) \in \mathbb{N}_0^{s+2}$. Quickly review the definition (Definition II.6) of $|b_{\vec{N}(1), \ldots, \vec{N}(n)}|_\omega$. At least one component of

$$(\xi, \eta) = (\xi_1 \circ \cdots \circ \xi_n, \eta_1 \circ \cdots \circ \eta_n)$$

is to be maxed over, rather than summed over. We shall say that those components are “anchored”. If there are source fields (that is, if $\sum_{j=1}^{s'} \sum_{\ell=1}^n N_j^{(\ell)} \geq 1$) then all components

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of \( \xi \) that correspond to source fields are anchored. Otherwise exactly one component of 
\((\xi, \eta)\) is anchored. By permuting \( \{1, \ldots, n\} \), we may assume that at least one component of 
\((\xi_1, \eta_1)\) is anchored. For notational simplicity, we consider the case that the first component 
\(\xi_{1,1} \) of \(\xi_1\) is anchored. The other cases are virtually identical. Denote by \(A\) the set of 
all indices \((\ell, m)\) such that \(\xi_{\ell,m}\) is anchored. Certainly \((1, 1) \in A\). But if there are source 
fields, \(A\) has cardinality \(\sum_{s' = 1}^s \sum_{t = 1}^n N_{j}^{(s')}\) which may be larger than one. Fix, for each 
\((\ell, m) \in A\), any \(x_{\ell,m} \in X\). Thus it suffices to prove that

\[
\sum_{\eta_1, \ldots, \eta_m \in X^{(2)} \atop T \subset G(\eta_1, \ldots, \eta_m)} |b_{N(1)}(\xi_1; \eta_1)| \cdots |b_{N(n)}(\xi_n; \eta_n)|
\]

is bounded by \(d_1! \cdots d_n! \prod_{j = 1}^n |b_{N(j)}|_{\omega_j = 1}\).

View 1 as the root of \(T\). Then the set of vertices of \(T\) is endowed with a natural 
partial ordering under which 1 is the smallest vertex. For each vertex \(2 \leq j \leq n\), denote 
by \(\pi(j)\) the predecessor vertex of \(j\) under this partial ordering. For example, if \(T\) is the 
tree in the figure

then \(\pi(7) = \pi(3) = \pi(4) = 2, \pi(2) = \pi(5) = 6\) and \(\pi(6) = 1\). The condition that 
\(T \subset G(\eta_1, \ldots, \eta_n)\) ensures that, for each \(2 \leq j \leq n\), the support of \(\eta_j\) intersects the 
support of \(\eta_{\pi(j)}\), so that at least one of the \(n(\eta_j)\) components of \(\eta_j\) takes the same value 
(in \(X\)) as some component of \(\eta_{\pi(j)}\). Note that \(n(\eta_j) = N_{s+1}^{(j)} + N_{s+2}^{(j)}\) and in particular is 
fixed by \(N^{(j)}\). Denote it \(n_j\). So

\[
(III.11) \leq \prod_{j = 2}^n \left[ n_j n_{\pi(j)} \right] \max_{1 \leq m_j \leq n_j \atop 1 \leq p_j \leq n(\eta_j) \atop for all 2 \leq j \leq n} \sum_{\eta_1, \ldots, \eta_n \in X^{(2)} \atop \eta_j, m_j = \eta_{\pi(j)}, p_j} \sum_{\xi_1, \ldots, \xi_n \in X^{(s)} \atop \xi_{\ell,m} = x_{\ell,m} \atop (\ell, m) \in A} \prod_{j = 1}^n |b_{N^{(j)}}(\xi_j; \eta_j)|
\]

Observe that \(\prod_{j = 2}^n \left[ n_j n_{\pi(j)} \right] = \prod_{j = 1}^n \frac{d_j}{d_j^2}\). To bound \(n_j \zeta\) we use that, for any \(0 < \nu < 1\) and \(d \geq 1\),

\[
t^d e^{-\nu t} = e^{d \ln t - \nu t}
\]

takes its maximum value (for \(t \geq 1\)) at \(t = \frac{d}{d \nu}\), so that

\[
t^d e^{-\nu t} \leq e^{d \ln \frac{d}{\nu} - d} \iff t^d \leq d^d e^{-d \ln \nu} e^{\nu t}
\]
By Stirling, $d^d \leq \frac{1}{\sqrt{2\pi}} d! \sqrt{d} e^{d - \frac{d}{2} + \frac{1}{12d}} \leq \frac{1}{\sqrt{2\pi}} d! e^d$ so that

$$n_j^{d_j} \leq \frac{1}{\sqrt{2\pi}} e^{-d_j \ln \nu} d_j! e^{\nu n_j}$$

and

$$\prod_{j=2}^n [n_j n_{\pi(j)}] \leq \frac{1}{(2\pi)^{n}} e^{(d_1 + \cdots + d_n) \ln \frac{1}{\sqrt{2\pi}}} d_1! \cdots d_n! \prod_{j=1}^n e^{\nu n_j} \leq \left(\frac{e^{2\ln \frac{1}{\nu}}}{\sqrt{2\pi}}\right)^n d_1! \cdots d_n! \prod_{j=1}^n e^{\nu n_j}$$

Consequently (III.11) is bounded by

$$\left(\frac{e^{2\ln \frac{1}{\nu}}}{\sqrt{2\pi}}\right)^n d_1! \cdots d_n! \max_{1 \leq m_j \leq n_j} \sum_{\eta_j, \eta_n \in X(2)} \sum_{\eta_j, \eta_n \in X(2)} \prod_{j=1}^n 2^{\nu n_j} |b_{\tilde{N}(j)}(\xi_j; \eta_j)| \leq \prod_{j=1}^n |b_{\tilde{N}(j)}|_{\omega=1}$$

Choosing $\nu = \ln 2$, we have $e^{\nu n_j} = 2^{n(\eta_j)}$ and $\frac{e^{2\ln \frac{1}{\nu}}}{\sqrt{2\pi}} < 1$. So it suffices to prove that

$$\sum_{\eta_j, \eta_n \in X(2)} \sum_{\eta_j, \eta_n \in X(2)} \prod_{j=1}^n 2^{n(\eta_j)} |b_{\tilde{N}(j)}(\xi_j; \eta_j)| \leq \prod_{j=1}^n |b_{\tilde{N}(j)}|_{\omega=1}$$

for all choices of $(m_j, p_j)_{2 \leq j \leq n}$, satisfying $1 \leq m_j \leq n(\eta_j)$ and $1 \leq p_j \leq n(\eta_{\pi(j)})$. But this is done easily by iteratively applying

$$\sum_{\eta_j, \eta_n \in X(2)} \sum_{\xi_j, \xi_n \in X(2)} 2^{n(\eta_j)} |b_{\tilde{N}(j)}(\xi_j; \eta_j)| \leq |b_{\tilde{N}(j)}|_{\omega=1}$$

starting with the largest $j$'s, in the partial ordering of $T$, and ending with $j = 1$. (For $j = 1$, the condition $\eta_j, m_j = \eta_{\pi(j)}, p_j$ is absent, but $(1, 1) \in A$.)

**Lemma III.6** Let $0 < \varepsilon < \frac{1}{8}$ and $\nu \in \mathbb{N}$. Then

$$\sum_{n=\nu}^{\infty} \frac{1}{(n-1)!} \sum_{d_1 + \cdots + d_n = 2(n-1)} \sum_{T \text{ labelled tree with coordination numbers } d_1, \cdots, d_n} d_1! \cdots d_n! e^n \leq \frac{1}{8} \left(\frac{8\varepsilon}{1-8\varepsilon}\right)^\nu$$

**Proof:** First suppose that $\nu \geq 2$. By the Cayley formula (see, for example [Ri, Theorem I.4.1]), the number of labelled trees on $n \geq 2$ vertices with specified coordination numbers $(d_1, d_2, \cdots, d_n)$ is

$$\prod_{j=1}^n \frac{(d_j-2)!}{(d_j-1)!}$$

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The number of possible choices of coordination numbers \((d_1, d_2, \ldots, d_n) \in \mathbb{N}^n\) subject to the constraint \(d_1 + d_2 + \cdots + d_n = 2(n - 1)\) is \(\binom{2(n-1)}{n-1} = \binom{2n-3}{n-1} \leq 2^{2n-3}\). Therefore

\[
\sum_{n=\nu}^{\infty} \frac{1}{(n-1)!} \sum_{d_1+\cdots+d_n=2(n-1)} \sum_{\text{\nu labelled tree with coordination numbers } d_1, \ldots, d_n} d_1! \cdots d_n! \varepsilon^n \leq \sum_{n=\nu}^{\infty} \frac{1}{n-1} \sum_{d_1+\cdots+d_n=2(n-1)} d_1 \cdots d_n \varepsilon^n
\]

\[
\leq \sum_{n=\nu}^{\infty} \frac{1}{n-1} 2^{2n-3} 2^n \varepsilon^n \leq \frac{1}{8} (8\varepsilon)^\nu
\]

For \(n = 1, d_1 = 0\) and the number of trees is 1, so the \(n = 1\) term is \(\varepsilon\). So the full sum for \(\nu = 1\) is bounded by \(\varepsilon + \frac{1}{8} (8\varepsilon)^2 = \frac{\varepsilon}{1-8\varepsilon}\).

\section*{Corollary III.7}
Let \(f(\alpha_1, \cdots, \alpha_s; z, z)\) obey \(\|f\|_w < \frac{1}{32}\) and define, for each complex number \(\zeta\) with \(\|\zeta\|_w < \frac{1}{16}\), the function \(G(\zeta) = G(\zeta; \alpha_1, \cdots, \alpha_s)\) by the condition

\[
\frac{\int e^{\zeta f(\alpha_1, \cdots, \alpha_s; z^*, z)} \, d\mu(z^*, z)}{\int e^{\zeta f(0, \cdots, 0; z^*, z)} \, d\mu(z^*, z)} = e^{G(\zeta; \alpha_1, \cdots, \alpha_s)} \tag{III.12}
\]

as in Theorem III.4. Then \(G(\zeta)\) is a (Banach space valued) analytic function of \(\zeta\) and, for each \(n \in \mathbb{N}\), the \(g(\alpha_1, \cdots, \alpha_s) = G(1)\) of Theorem III.4 obeys

\[
\left\| g - \frac{dG}{d\zeta}(0) - \cdots - \frac{1}{n!} \frac{d^nG}{d\zeta^n}(0) \right\|_w \leq \left( \frac{\|f\|_w}{\frac{1}{20} - \|f\|_w} \right)^{n+1}
\]

We have \(G(0) = 0\),

\[
\frac{dG}{d\zeta}(0) = \int [f(\alpha_1, \cdots, \alpha_s; z^*, z) - f(0, \cdots, 0; z^*, z)] \, d\mu(z^*, z)
\]

and

\[
\frac{d^2G}{d\zeta^2}(0) = \int f(\alpha_1, \cdots, \alpha_s; z^*, z)^2 \, d\mu(z^*, z) - \int f(0, \cdots, 0; z^*, z)^2 \, d\mu(z^*, z)
\]

\[-\left[ \int f(\alpha_1, \cdots, \alpha_s; z^*, z) \, d\mu(z^*, z) \right]^2 + \left[ \int f(0, \cdots, 0; z^*, z) \, d\mu(z^*, z) \right]^2
\]

If, in addition, the measure \(d\mu(z^*, z)\) on \(\Psi\) is rotation invariant, \(f(0, \cdots, 0; z^*, z) = 0\) and that the symmetric coefficient system \(a(\vec{x}_1, \cdots, \vec{x}_s; \vec{y}, \vec{y})\) of \(f\) obeys \(a(\vec{x}_1, \cdots, \vec{x}_s; \vec{y}, \vec{y}) = 0\) whenever \(\vec{y} = \vec{y}_*, \) then \(\frac{dG}{d\zeta}(0) = 0\) and

\[
\frac{d^2G}{d\zeta^2}(0) = \int f(\alpha_1, \cdots, \alpha_s; z^*, z)^2 \, d\mu(z^*, z)
\]
Proof: The analyticity of $G(\zeta)$ is obvious since the series (III.5), (III.6) with $a$ replaced by $\zeta$ converge in the norm $\| \cdot \|_w$ uniformly in $\zeta$ for $|\zeta|\|f\|_w$ bounded by any constant strictly less than $\frac{1}{16}$. That $G(0) = 0$ and $G(1) = g$ are also obvious. So by Taylor’s formula with remainder
\[
g - \frac{dG}{d\zeta}(0) - \cdots - \frac{1}{n!} \frac{d^n G}{d\zeta^n}(0) = \frac{1}{(n+1)!} \frac{d^{n+1} G}{d\zeta^{n+1}}(u)
\]
for some $0 < u < 1$. By the Cauchy integral formula
\[
\frac{1}{(n+1)!} \frac{d^{n+1} G}{d\zeta^{n+1}}(u) = \frac{1}{2\pi i} \int_{|\zeta|=U} \frac{G(\zeta)}{(\zeta-w)^{n+2}} \ d\zeta
\]
for any $u < U < \frac{1}{16\|f\|_w}$. Hence
\[
\left\| g - \frac{dG}{d\zeta}(0) - \cdots - \frac{1}{n!} \frac{d^n G}{d\zeta^n}(0) \right\|_w \leq \frac{1}{2\pi} \sup_{|\zeta|=U} \frac{\|G(\zeta)\|_w}{(U-1)^{n+2}}
\]
\[
\leq U \frac{\|f\|_w}{1-16\|f\|_w} \frac{\|G(\zeta)\|_w}{(U-1)^{n+2}}
\]
Choosing $U$ so that $U\|f\|_w = \frac{1}{20}$ gives
\[
\left\| g - \frac{dG}{d\zeta}(0) - \cdots - \frac{1}{n!} \frac{d^n G}{d\zeta^n}(0) \right\|_w \leq \frac{1}{80} \left( \frac{\|f\|_w^{n+1}}{(\frac{1}{20} - \|f\|_w)^{n+2}} \right) \leq \left( \frac{\|f\|_w}{\frac{1}{20} - \|f\|_w} \right)^{n+1}
\]
since $\|f\|_w < \frac{1}{32}$.

To get the formulae for $\frac{dG}{d\zeta}(0)$ and $\frac{d^2 G}{d\zeta^2}(0)$, differentiate (III.12) to give
\[
\frac{dG}{d\zeta}(\zeta) e^{G(\zeta; \alpha_1, \ldots, \alpha_s)} = \frac{\int \frac{f(\alpha_1, \ldots, \alpha_s; z^*, z)}{e^{\xi f(0, \ldots, 0; z^*, z)} d\mu(z^*, z)}}{\int e^{\xi f(0, \ldots, 0; z^*, z)} d\mu(z^*, z)} - \frac{\int \frac{f(0, \ldots, 0; z^*, z)}{e^{\xi f(0, \ldots, 0; z^*, z)} d\mu(z^*, z)}}{\int e^{\xi f(0, \ldots, 0; z^*, z)} d\mu(z^*, z)}
\]
and hence
\[
\frac{dG}{d\zeta}(\zeta) = \frac{\int \frac{f(\alpha_1, \ldots, \alpha_s; z^*, z)}{e^{\xi f(0, \ldots, 0; z^*, z)} d\mu(z^*, z)}}{\int e^{\xi f(0, \ldots, 0; z^*, z)} d\mu(z^*, z)} - \frac{\int \frac{f(0, \ldots, 0; z^*, z)}{e^{\xi f(0, \ldots, 0; z^*, z)} d\mu(z^*, z)}}{\int e^{\xi f(0, \ldots, 0; z^*, z)} d\mu(z^*, z)}
\]
Setting $\zeta = 0$ gives the formula for $\frac{dG}{d\zeta}(0)$. Differentiating again with respect to $\zeta$ and then setting $\zeta = 0$ gives the formula for $\frac{d^2 G}{d\zeta^2}(0)$. When the measure $d\mu(z^*, z)$ is rotation invariant and the symmetric coefficient system $a(\tilde{x}_1, \ldots, \tilde{x}_s; \tilde{y}_*, \tilde{y})$ of $f$ obeys $a(\tilde{x}_1, \ldots, \tilde{x}_s; \tilde{y}_*, \tilde{y}) = 0$ whenever $\tilde{y} = \tilde{y}_*$, we have
\[
\int \frac{f(\alpha_1, \ldots, \alpha_s; z^*, z)}{e^{\xi f(0, \ldots, 0; z^*, z)} d\mu(z^*, z)} = 0
\]
as in Remark III.3.ii, and the remaining formulae follow.
Corollary III.8 Denote by $\mathcal{F}$ the Banach space of functions $f(\alpha_1, \cdots, \alpha_s; z^*, z)$ with $\|f\|_w < \infty$. Let $f, f' \in \mathcal{F}$ with

$$
\int f(\alpha_1, \cdots, \alpha_s; z^*, z) d\mu(z^*, z) - \int f'(\alpha_1, \cdots, \alpha_s; z^*, z) d\mu(z^*, z) = 0
$$

and $\|f\|_w + \|f' - f\|_w < \frac{1}{16}$. Define, $g, g' \in \mathcal{F}$ by the conditions

$$
\frac{\int e^{f(\alpha_1, \cdots, \alpha_s; z^*, z)} d\mu(z^*, z)}{\int e^{f(0, \cdots, 0; z^*, z)} d\mu(z^*, z)} = e^g(\alpha_1, \cdots, \alpha_s)
$$

and

$$
\frac{\int e^{f'(\alpha_1, \cdots, \alpha_s; z^*, z)} d\mu(z^*, z)}{\int e^{f'(0, \cdots, 0; z^*, z)} d\mu(z^*, z)} = e^{g'}(\alpha_1, \cdots, \alpha_s)
$$

as in Theorem III.4. Then

$$
\|g' - g\|_w \leq 4 \frac{\|f' - f\|_w (\|f\|_w + \|f' - f\|_w)}{(\frac{1}{16} - \|f\|_w - \|f' - f\|_w)^2}
$$

Proof: Define, for all $\zeta, \zeta' \in \Phi$ with $|\zeta| \leq U^{-1} \equiv \frac{1}{16} - \|f\|_w - \|f' - f\|_w$ and $|\zeta| \leq U'^{-1} \equiv \frac{1}{16} - \|f\|_w - \|f' - f\|_w$, $2\|f\|_w$,

$$
F(\zeta, \zeta'; \alpha_1, \cdots, \alpha_s; z^*, z) = \zeta' f(\alpha_1, \cdots, \alpha_s; z^*, z) + \zeta (f' - f)(\alpha_1, \cdots, \alpha_s; z^*, z)
$$

and define $G(\zeta)$ by

$$
\frac{\int e^{F(\zeta, \zeta'; \alpha_1, \cdots, \alpha_s; z^*, z)} d\mu(z^*, z)}{\int e^{F(\zeta, \zeta'; 0, \cdots, 0; z^*, z)} d\mu(z^*, z)} = e^{G(\zeta, \zeta'; \alpha_1, \cdots, \alpha_s)}
$$

as in Theorem III.4. Then $G(\zeta, \zeta')$ is an $\mathcal{F}$-valued analytic function of $\zeta, \zeta'$, since the series (III.5), (III.6) with $a$ replaced by the appropriate $a(\zeta, \zeta')$ converge in the norm $\| \cdot \|_w$ uniformly in $\zeta, \zeta'$ for $|\zeta| \|f\|_w + |\zeta||f' - f\|_w$ bounded by any constant strictly less than $\frac{1}{16}$. Furthermore $G(0, 1) = g$ and $G(1, 1) = g'$ so that

$$
g' - g = \int_0^1 \frac{\partial G}{\partial \zeta}(s, 1) ds = \int_0^1 \frac{\partial G}{\partial \zeta}(s, 0) ds + \int_0^1 \int_0^1 \frac{\partial^2 G}{\partial \zeta \partial \zeta'}(s, s') ds ds'
$$

By hypothesis,

$$
\frac{\partial G}{\partial \zeta}(s, 0) = \frac{(f - f')(\alpha_1, \cdots, \alpha_s; z^*, z) e^{s(f - f')(\alpha_1, \cdots, \alpha_s; z^*, z)} d\mu(z^*, z)}{\int e^{s(f - f')(\alpha_1, \cdots, \alpha_s; z^*, z)} d\mu(z^*, z)}
$$

and

$$
\frac{\partial^2 G}{\partial \zeta \partial \zeta'}(s, s') = \frac{(f - f')(0, \cdots, 0; z^*, z) e^{s(f - f')(0, \cdots, 0; z^*, z)} d\mu(z^*, z)}{\int e^{s(f - f')(0, \cdots, 0; z^*, z)} d\mu(z^*, z)}
$$

Therefore

$$
\frac{\partial G}{\partial \zeta}(s, 0) = \frac{(f - f')(\alpha_1, \cdots, \alpha_s; z^*, z) e^{s(f - f')(\alpha_1, \cdots, \alpha_s; z^*, z)} d\mu(z^*, z)}{\int e^{s(f - f')(\alpha_1, \cdots, \alpha_s; z^*, z)} d\mu(z^*, z)}
$$

and

$$
\frac{\partial^2 G}{\partial \zeta \partial \zeta'}(s, s') = \frac{(f - f')(0, \cdots, 0; z^*, z) e^{s(f - f')(0, \cdots, 0; z^*, z)} d\mu(z^*, z)}{\int e^{s(f - f')(0, \cdots, 0; z^*, z)} d\mu(z^*, z)}
$$

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By hypothesis, for all $0 \leq s, s' \leq 1$, \( \zeta, \zeta' \) with $|\zeta| \leq \frac{1}{U}$, $|\zeta'| \leq \frac{1}{U'}$

$$
\|F(s+\zeta, s'+\zeta')\|_w = \|(s'+\zeta')f + (s+\zeta)(f'-f)\|_w \leq (1+|\zeta'|)|f\|_w+(1+|\zeta|)|f'-f\|_w \leq \frac{1}{16}
$$

and hence

$$
\|G(s+\zeta, s'+\zeta')\|_w \leq \frac{\|F(s+\zeta, s'+\zeta')\|_w}{1-16\|F(s+\zeta, s'+\zeta')\|_w} \leq \frac{1}{1-\frac{1}{16}} = 1
$$

So

$$
\|g' - g\|_w \leq \frac{1}{2\pi}2\pi U^{-1}\frac{1}{U-1} + \frac{1}{2\pi}2\pi U^{-1}\frac{1}{U-2}2\pi U'^{-1}\frac{1}{U'-2} = U^2 + UU'
$$

**Corollary III.9** Let $w'$ be a weight system of length $s+2$ that also gives weight at least $4r$ to the last two fields, $z_*$ and $z$. Assume that

$$
 w(\bar{x}_1 \circ \bar{x}_1', \cdots, \bar{x}_s \circ \bar{x}_s'; \bar{y}_s \circ \bar{y}_s', \bar{y} \circ \bar{y}') \leq w(\bar{x}_1, \cdots, \bar{x}_s; \bar{y}_*, \bar{y}) \ w'(\bar{x}_1', \cdots, \bar{x}_s'; \bar{y}_s', \bar{y}') \quad (\text{III.13})
$$

for all $(\bar{x}_1, \cdots, \bar{x}_s; \bar{y}_*, \bar{y})$ and $(\bar{x}_1', \cdots, \bar{x}_s'; \bar{y}_s', \bar{y}')$ in $X^{(s+2)}$ with $\text{supp}(\bar{x}_1, \cdots, \bar{x}_s; \bar{y}_*, \bar{y}) \cap \text{supp}(\bar{x}_1', \cdots, \bar{x}_s'; \bar{y}_s', \bar{y}') \neq \emptyset$.

(a) Let $f(\alpha_1, \cdots, \alpha_s; z_*, z)$ and $f'(\alpha_1, \cdots, \alpha_s; z_*, z)$ be analytic functions. Assume that

$$
 f(0, \cdots, 0; z_*, z) = 0 \quad \text{and} \quad \|f\|_w + \|f'\|_w' < \frac{1}{16}
$$

Then there are analytic functions $g(\alpha_1, \cdots, \alpha_s)$ and $g'(\alpha_1, \cdots, \alpha_s)$ such that

$$
\frac{\int e^{f(\alpha_1, \cdots, \alpha_s; z_*, z)} + f'(\alpha_1, \cdots, \alpha_s; z_*, z) \ d\mu(z_*, z)}{\int e^{f(0, \cdots, 0; z_*, z)} \ d\mu(z_*, z)} = e^{g(\alpha_1, \cdots, \alpha_s) + g'(\alpha_1, \cdots, \alpha_s)} \quad (\text{III.14})
$$
and

\[
\frac{\int e^{f'}(\alpha_1, \ldots, \alpha_s; z^*, z) \, d\mu(z^*, z)}{\int e^{f}(0, \ldots, 0; z^*, z) \, d\mu(z^*, z)} = e^{g'(\alpha_1, \ldots, \alpha_s)} \tag{III.15}
\]

They obey the estimates

\[
\|g\|_w \leq \frac{1}{\|f\|_w + \|f'\|_{w'}} \|f\|_w + \|f'\|_{w'} \tag{16}
\]

\[
\|g'\|_{w'} \leq \frac{1}{\|f\|_w + \|f'\|_{w'}} \|f'\|_{w'} \tag{17}
\]

(b) Assume in addition to the hypothesis of part (a), that the measure \(d\mu(z^*, z)\) on \(\mathbb{C}\) is rotation invariant and the symmetric coefficient systems \(a(\tilde{x}_1, \ldots, \tilde{x}_s; \tilde{y}_*, \tilde{y})\) of \(f\) and \(a'(\tilde{x}_1, \ldots, \tilde{x}_s; \tilde{y}_*, \tilde{y})\) of \(f'\) obey

\[
a(\tilde{x}_1, \ldots, \tilde{x}_s; \tilde{y}_*, \tilde{y}) = a'(\tilde{x}_1, \ldots, \tilde{x}_s; \tilde{y}_*, \tilde{y}) = 0
\]

whenever \(\tilde{y} = \tilde{y}_*\). Then the functions \(g\) and \(g'\) of part (i) obey

\[
\|g\|_w \leq \left(\frac{\|f\|_w + \|f'\|_{w'}}{2\|f\|_w + \|f'\|_{w'}}\right)^2 \tag{18}
\]

\[
\|g'\|_{w'} \leq \left(\frac{\|f'\|_{w'}}{2\|f\|_w + \|f'\|_{w'}}\right)^2 \tag{19}
\]

**Proof**: Consider the weight system of length \(2s + 2\)

\[
\omega(\tilde{x}_1, \ldots, \tilde{x}_s; \tilde{x}_1', \ldots, \tilde{x}_s', \tilde{y}_*, \tilde{y}) = \begin{cases} w(\tilde{x}_1 \circ \tilde{x}_1', \ldots, \tilde{x}_s \circ \tilde{x}_s'; \tilde{y}_*, \tilde{y}) \quad \text{if} \quad \sum_{j=1}^s n(\tilde{x}_j) \geq 1 \\ w'(\tilde{x}_1', \ldots, \tilde{x}_s'; \tilde{y}_*, \tilde{y}) \quad \text{if} \quad \tilde{x}_1 = \cdots = \tilde{x}_s = -1
\end{cases}
\]

It is easy to verify, using (III.13), that this is indeed a weight system that gives weight at least \(4r\) to the last two fields. We use it for the fields \(\alpha_1, \ldots, \alpha_s; \alpha_1', \ldots, \alpha_s'; z, z_*\). We designate as source fields the fields \(\alpha_1', \ldots, \alpha_s'\), as well as the original source fields \(\alpha_1, \ldots, \alpha_s\). Clearly

\[
\|f(\alpha_1, \ldots, \alpha_s; z, z_*)\|_\omega = \|f(\alpha_1, \ldots, \alpha_s; z_s, z)\|_w
\]

\[
\|f'(\alpha_1', \ldots, \alpha_s'; z_s, z)\|_\omega = \|f'(\alpha_1, \ldots, \alpha_s; z_s, z)\|_{w'}
\]

By Theorem III.4 there is an analytic function \(\gamma(\alpha_1, \ldots, \alpha_s; \alpha_1', \ldots, \alpha_s')\) such that

\[
\frac{\int e^{s(\alpha_1, \ldots, \alpha_s; z^*, z)} + f'(\alpha_1', \ldots, \alpha_s'; z^*, z) \, d\mu(z^*, z)}{\int e^{s(0, \ldots, 0; z^*, z)} \, d\mu(z^*, z)} = e^{\gamma(\alpha_1, \ldots, \alpha_s; \alpha_1', \ldots, \alpha_s')} \tag{III.16}
\]

and

\[
\|\gamma\|_\omega \leq \frac{1}{\|f\|_w + \|f'\|_{w'}}
\]

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Set

\[ g'(\alpha_1, \cdots, \alpha_s) = \gamma(0, \cdots, 0; \alpha_1, \cdots, \alpha_s) \]
\[ g(\alpha_1, \cdots, \alpha_s) = \gamma(\alpha_1, \cdots, \alpha_s; \alpha_1, \cdots, \alpha_s) - \gamma(0, \cdots, 0; \alpha_1, \cdots, \alpha_s) \]

Setting \[ \alpha'_j = \alpha_j \] in (III.16) gives (III.14). Also,

\[ \left\| g \right\|_w \leq \frac{1}{1 - \frac{\left\| f \right\|_w + \left\| f' \right\|_w}} \]

Setting \[ \alpha_j = 0 \] in (III.16) gives (III.15). The estimate on \[ g' \] follows directly from Theorem III.4. This completes the proof of part (a) of Corollary III.9. The proof of part (b) is similar, but uses Corollary III.7 with \[ n = 1. \]

\[ \square \]

**Example III.10** Assume that \( X \) is a metric space. Let \( \Omega \subset X \) be a decay set, \( m > 0 \) a mass, and \( \kappa_j : X \to (0, \infty) \), for \( j = 1, \cdots, s + 2 \), be weight factors. Let \( w \) be the weight system with mass \( m \) and decay set \( \Omega \) that satisfies the weight factors \( \kappa_j \) to the field \( \alpha_j \), for \( j = 1, \cdots, s \), and the weight factors \( \kappa_{s+1}, \kappa_{s+2} \) to the fields \( z^* \) and \( z \) respectively. (See Definition II.5.) Also, let \( w' \) be the weight system with empty decay set and mass \( 2m \) that again associates the weight factor \( \kappa_j \) to the field \( \alpha_j \), for \( j = 1, \cdots, s \), and the weight factors \( \kappa_{s+1}, \kappa_{s+2} \) to the fields \( z^* \) and \( z \) respectively. In this situation, hypothesis (III.13) of Corollary III.9 is fulfilled.

**Proof:** Let \((\bar{x}_1, \cdots, \bar{x}_{s+2})\) and \((\bar{x}'_1, \cdots, \bar{x}'_{s+2})\) be two elements of \( X^{(s+2)} \) such that \( \text{supp}(\bar{x}_1, \cdots, \bar{x}_{s+2}) \cap \text{supp}(\bar{x}'_1, \cdots, \bar{x}'_{s+2}) \neq \emptyset \). Choose spanning trees \( T \) and \( T' \) for \( \text{supp}(\bar{x}_1, \cdots, \bar{x}_{s+2}) \) and \( \text{supp}(\bar{x}'_1, \cdots, \bar{x}'_{s+2}) \) respectively such that

\[ \tau(\text{supp}(\bar{x}_1, \cdots, \bar{x}_{s+2})) = \text{length}(T), \quad \tau(\text{supp}(\bar{x}'_1, \cdots, \bar{x}'_{s+2})) = \text{length}(T') \]

Since \( \text{supp}(\bar{x}_1, \cdots, \bar{x}_{s+2}) \cap \text{supp}(\bar{x}'_1, \cdots, \bar{x}'_{s+2}) \neq \emptyset \), one can choose a subtree \( \bar{T} \) of \( T \cup T' \) that spans \( \text{supp}(\bar{x}_1 \circ \bar{x}'_1, \cdots, \bar{x}_{s+2} \circ \bar{x}'_{s+2}) \). So

\[ \tau(\text{supp}(\bar{x}_1 \circ \bar{x}'_1, \cdots, \bar{x}_{s+2} \circ \bar{x}'_{s+2})) \leq \text{length}(\bar{T}) \]

\[ \leq \tau(\text{supp}(\bar{x}_1, \cdots, \bar{x}_{s+2})) + \tau(\text{supp}(\bar{x}'_1, \cdots, \bar{x}'_{s+2})) \]

Choose \( x \in \text{supp}(\bar{x}_1 \circ \bar{x}'_1, \cdots, \bar{x}_{s+2} \circ \bar{x}'_{s+2}) \) such that

\[ d(x, \Omega^c) = D(\text{supp}(\bar{x}_1 \circ \bar{x}'_1, \cdots, \bar{x}_{s+2} \circ \bar{x}'_{s+2}), \Omega^c) \]

and choose \( y \in \text{supp}(\bar{x}_1, \cdots, \bar{x}_{s+2}) \cap \text{supp}(\bar{x}'_1, \cdots, \bar{x}'_{s+2}) \). If \( x \in \text{supp}(\bar{x}'_1, \cdots, \bar{x}'_{s+2}) \)

\[ d(x, \Omega^c) \leq d(x, y) + d(y, \Omega^c) \leq \tau(\text{supp}(\bar{x}'_1, \cdots, \bar{x}'_{s+2})) + D(\text{supp}(\bar{x}_1, \cdots, \bar{x}_{s+2}), \Omega^c) \]

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so that

\[
w(\vec{x}_1 \circ \vec{x}_1', \cdots, \vec{x}_{s+2} \circ \vec{x}_{s+2}')
\]

\[
= e^m d(\vec{x}, \Omega^c) e^m \tau(\text{supp}(\vec{x}_1 \circ \vec{x}_1', \cdots, \vec{x}_{s+2} \circ \vec{x}_{s+2}')) \prod_{j=1}^{s+2} \left( \prod_{\ell=1}^{n(\vec{x}_j)} n(\vec{x}_j') \prod_{\ell=1}^{n(\vec{x}_j')} \kappa_j(\vec{x}_j, \ell) \prod_{\ell=1}^{n(\vec{x}_j')} \kappa_j(\vec{x}_j', \ell) \right)
\]

\[
\leq \left[ e^{2m} \tau(\text{supp}(\vec{x}_1', \cdots, \vec{x}_{s+2}')) \prod_{j=1}^{s+2} \prod_{\ell=1}^{n(\vec{x}_j')} \kappa_j(\vec{x}_j', \ell) \right] \left[ e^m \tau(\text{supp}(\vec{x}_1', \cdots, \vec{x}_{s+2}')) e^m D(\text{supp}(\vec{x}_1', \cdots, \vec{x}_{s+2}), \Omega^c) \prod_{j=1}^{s+2} \prod_{\ell=1}^{n(\vec{x}_j)} \kappa_j(\vec{x}_j, \ell) \right]
\]

\[
\leq w'(\vec{x}_1', \cdots, \vec{x}_{s+2}') w(\vec{x}_1, \cdots, \vec{x}_{s+2})
\]

On the other hand, if \( \vec{x} \in \text{supp}(\vec{x}_1, \cdots, \vec{x}_{s+2}) \),

\[
D(\text{supp}(\vec{x}_1 \circ \vec{x}_1', \cdots, \vec{x}_{s+2} \circ \vec{x}_{s+2}'), \Omega^c) = D(\text{supp}(\vec{x}_1, \cdots, \vec{x}_{s+2}), \Omega^c)
\]

and trivially \( w(\vec{x}_1 \circ \vec{x}_1', \cdots, \vec{x}_{s+2} \circ \vec{x}_{s+2}') \leq w(\vec{x}_1, \cdots, \vec{x}_{s+2}) w'(\vec{x}_1', \cdots, \vec{x}_{s+2}') \).
IV. Linear Transformations

In this section, we study properties of the norms of Definition II.6, and in particular, their behaviour under linear changes of variables. So let $X$ be a metric space, $\Omega \subset X$ a decay set, and $m \geq 0$ a mass. Also fix the number $0 \leq s' \leq s$ of source fields.

**Definition IV.1 (Weighted $L^1$--$L^\infty$ operator norms)** Let $\kappa, \kappa' : X \to (0, \infty]$ be weight factors. For a linear map $J$ from $\Phi^X$ to $\Phi^X$, with kernel $J(x, y)$, define

$$
L_m(J; \kappa, \kappa') = \sup_{x \in X} \sum_{y \in X} e^{m d(x, y)} |J(x, y)| \frac{\kappa(y)}{\kappa'(x)}
$$

$$
R_m(J; \kappa, \kappa') = \sup_{y \in X} \sum_{x \in X} e^{m d(x, y)} |J(x, y)| \frac{\kappa(y)}{\kappa'(x)}
$$

$$
N_m(J; \kappa, \kappa') = \max \left\{ L_m(J; \kappa, \kappa'), R_m(J; \kappa, \kappa') \right\}
$$

**Lemma IV.2** Let $J_j, 1 \leq j \leq s$, be operators on $\Phi^X$ with kernels $J_j(x, y)$. Let $f$ be an analytic function on a neighbourhood of the origin in $\Phi^{s|X}$ and define $\tilde{f}$ by

$$
\tilde{f}(\alpha_1, \cdots, \alpha_s) = f(J_1\alpha_1, \cdots, J_s\alpha_s)
$$

Let $\kappa_1, \cdots, \kappa_s, \kappa'_1, \cdots, \kappa'_s$ be weight factors. Denote by $w$ and $w'$ the weight systems with mass $m$ and decay set $\Omega$ that associate to the field $\alpha_j$ the weight factor $\kappa_j$ and $\kappa'_j$ respectively (see Definition II.5).

(i) If $N_{2m}(J_j; \kappa_j, \kappa'_j) \leq 1$ for $1 \leq j \leq s$, then

$$
\|\tilde{f}\|_w \leq \|f\|_{w'}
$$

In the case that $\Omega = \emptyset$ it suffices to assume that $N_m(J_j; \kappa_j, \kappa'_j) \leq 1$ for $1 \leq j \leq s$.

(ii) Denote by $w''$ the weight system with mass $2m$ and empty decay set that associates to the field $\alpha_j$ the weight factor $\kappa'_j$. If $N_{2m}(J_j; \kappa_j, \kappa'_j) \leq 1$ for all $1 \leq j \leq s$, and $N_{2m}(J_{j_0}; \kappa_{j_0} e^{m d(x, \Omega^c)}, \kappa'_{j_0}) \leq 1$ for some $1 \leq j_0 \leq s$ with $f(\alpha_1, \cdots, \alpha_s)|_{\alpha_{j_0} = 0} = f(0, \cdots, 0)$, then

$$
\|\tilde{f}\|_w \leq \|f\|_{w''}
$$

Here, $\kappa_{j_0} e^{m d(x, \Omega^c)}$ denotes the function $\kappa_{j_0}(x) e^{m d(x, \Omega^c)}$. 

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Proof: Let $a(x_1, \ldots, x_s)$ be a symmetric coefficient system for $f$. Define, for each $n(x_j) = n_j \geq 0$, $1 \leq j \leq s$,

$$
\tilde{a}(x_1, \ldots, x_s) = \sum_{y_1 \in x_1} \cdots \sum_{y_s \in x_s} a(y_1, \ldots, y_s) \prod_{j=1}^{s} \prod_{\ell=1}^{n_j} J_j(y_j, \ell, x_j, \ell)
$$

where $x_j = (x_{j,1}, \ldots, x_{j,n_j})$. Then $\tilde{a}(x_1, \ldots, x_s)$ is a symmetric coefficient system for $\tilde{f}$. Similarly

$$
\tau(\text{supp}(x_1, \ldots, x_s)) \leq \tau(\text{supp}(y_1, \ldots, y_s)) + \sum_{j=1}^{s} \sum_{\ell=1}^{n_j} d(y_j, \ell, x_j, \ell)
$$

we have

$$
e^{m} \tau(\text{supp}(x_1, \ldots, x_s)) \leq e^{m} \tau(\text{supp}(y_1, \ldots, y_s)) \prod_{j=1}^{s} \prod_{\ell=1}^{n_j} e^{m} d(y_j, \ell, x_j, \ell)
$$

Similarly

$$
e^{m} D(\text{supp}(x_1, \ldots, x_s), \Omega^c) \leq e^{m} D(\text{supp}(y_1, \ldots, y_s), \Omega^c) \prod_{j=1}^{s} \prod_{\ell=1}^{n_j} e^{m} d(y_j, \ell, x_j, \ell)
$$

Therefore

$$
w(x_1, \ldots, x_s) \tilde{a}(x_1, \ldots, x_s) = \sum_{\tilde{y}_j \in x_{n_j}^j} \sum_{1 \leq j \leq s} w(\tilde{y}_1, \ldots, \tilde{y}_s) a(\tilde{y}_1, \ldots, \til{y}_s) \prod_{j=1}^{s} \prod_{\ell=1}^{n_j} e^{2m} d(y_j, \ell, x_j, \ell) J_j(y_j, \ell, x_j, \ell) \frac{\kappa_j(x_j, \ell)}{\kappa_j(y_j, \ell)}
$$

For part (i), if there are no source fields, we are to bound

$$
\|\tilde{f}\|_w = \sum_{n_1, \ldots, n_s \geq 0} \max_{x \in X} \max_{\frac{n_j}{n_j} \leq s} \max_{\frac{n_j}{n_j} \leq n_j} \sum_{(x_j)=x} w(x_1, \ldots, x_s) \tilde{a}(x_1, \ldots, x_s)
$$

$$
\leq \sum_{n_1, \ldots, n_s \geq 0} \max_{x \in X} \max_{\frac{n_j}{n_j} \leq s} \max_{\frac{n_j}{n_j} \leq n_j} \sum_{(x_j)=x} \sum_{(\tilde{y}_j)=x} w'(\tilde{y}_1, \ldots, \tilde{y}_s) a(\tilde{y}_1, \ldots, \tilde{y}_s) \prod_{j=1}^{s} \prod_{\ell=1}^{n_j} e^{2m} d(y_j, \ell, x_j, \ell) J_j(y_j, \ell, x_j, \ell) \frac{\kappa_j(x_j, \ell)}{\kappa_j(y_j, \ell)}
$$

(IV.1)

Fix any $n_1, \ldots, n_s \geq 0$ and denote, for each $1 \leq j \leq s$,

$$
J_j(y, x) = e^{2m} d(y, x) J_j(y, \ell, x_j, \ell) \frac{\kappa_j(x_j, \ell)}{\kappa_j(y_j, \ell)}
$$
If there are no source fields also fix a $1 \leq \tilde{j} \leq s$ with $n_{\tilde{j}} \neq 0$ and an $1 \leq \tilde{\ell} \leq n_{\tilde{j}}$. If there are source fields, that is, if $s' > 0$ ($s'$ was specified in Definition II.6) and $\sum_{j=1}^{n_{\tilde{j}}} n_j \geq 1$

$$\max_{x \in X} \sum_{(x_1, \ldots, x_s) \in X^{n_{1}} \times \cdots \times X^{n_{s}}}$$

is replaced by

$$\max_{x_{j, \ell} \in X} \sum_{1 \leq j \leq s'} \sum_{1 \leq \ell \leq n_{j}}$$

We may write both of these max/sums in the form

$$\max_{x_{j, \ell} \in X} \sum_{(j, \ell) \in A} \sum_{(x_1, \ldots, x_s) \in X^{n_{1}} \times \cdots \times X^{n_{s}}}$$

for some nonempty subset $A \subset \{(j, \ell) \mid 1 \leq j \leq n, 1 \leq \ell \leq n_{j}\}$. Fix such an $A$ and an $x_{j, \ell} \in X$ for each $(j, \ell) \in A$.

First, we use that, for each $(j, \ell) \notin A$ and $y_{j, \ell} \in X$,

$$\sum_{x_{j, \ell} \in X} J_j(y_{j, \ell}, x_{j, \ell}) \leq L_{2m}(J_j; \kappa_j, \kappa_j') \leq 1$$

to give

$$\sum_{(x_1, \ldots, x_s) \in X^{n_{1}} \times \cdots \times X^{n_{s}}} \sum_{(x_1, \ldots, x_s) \in X^{n_{1}} \times \cdots \times X^{n_{s}}} w'(\tilde{y}_1, \ldots, \tilde{y}_s)|a(\tilde{y}_1, \ldots, \tilde{y}_s)| \prod_{j=1}^{s} \prod_{\ell=1}^{n_j} J_j(y_{j, \ell}, x_{j, \ell})$$

$$\leq \sum_{(\tilde{y}_1, \ldots, \tilde{y}_s) \in X^{n_{1}} \times \cdots \times X^{n_{s}}} \sum_{1 \leq j \leq s} \sum_{1 \leq \ell \leq n_{j}} w'(\tilde{y}_1, \ldots, \tilde{y}_s)|a(\tilde{y}_1, \ldots, \tilde{y}_s)| \prod_{j=1}^{s} J_j(y_{j, \ell}, x_{j, \ell})$$

$$= \sum_{y_{j, \ell}, (j, \ell) \in A} \left( \sum_{y_{j, \ell}, (j, \ell) \notin A} w'(\tilde{y}_1, \ldots, \tilde{y}_s)|a(\tilde{y}_1, \ldots, \tilde{y}_s)| \prod_{j=1}^{s} J_j(y_{j, \ell}, x_{j, \ell}) \right)$$

$$\leq \left( \max_{y_{j, \ell}, (j, \ell) \in A} \sum_{y_{j, \ell}, (j, \ell) \notin A} w'(\tilde{y}_1, \ldots, \tilde{y}_s)|a(\tilde{y}_1, \ldots, \tilde{y}_s)| \right) \prod_{j=1}^{s} \prod_{1 \leq \ell \leq n_{j}} J_j(y_{j, \ell}, x_{j, \ell})$$

$$\sum_{y_{j, \ell}, (j, \ell) \in A} \prod_{1 \leq \ell \leq n_{j}} J_j(y_{j, \ell}, x_{j, \ell})$$
Then, we use that, for each \((j, \ell) \in A\),

\[
\sum_{y_{j,\ell} \in X} J_{j}(y_{j,\ell}, \bar{x}_{j,\ell}) \leq R_{2m}(J_{j}; \kappa_{j}, \kappa'_{j}) \leq 1
\]

to give

\[
\sum_{(s_{1}, \ldots, s_{n}) \in X^{n_{1}} \times \cdots \times X^{n_{s}}} \sum_{(s_{j})_{j} = \bar{s}_{j}, (j, \ell) \in A} w'((\bar{y}_{1}, \ldots, \bar{y}_{s}), a(y_{1}, \ldots, y_{s}) \prod_{j=1}^{s} \prod_{\ell=1}^{n_{j}} J_{j}(y_{j,\ell}, x_{j,\ell})
\]

\[
\leq \max_{y_{j,\ell}, (j, \ell) \in A} \sum_{y_{j,\ell}, (j, \ell) \notin A} w'((\bar{y}_{1}, \ldots, \bar{y}_{s}), a(y_{1}, \ldots, y_{s})
\]

Maxing over the possible choices of \(A\) and the \(\bar{x}_{j,\ell}\)'s and summing over \(n_{1}, \ldots, n_{j}\) finishes the proof of part (i) of the Lemma.

(ii) Since \(\tilde{f}(0, \ldots, 0) = f(0, \ldots, 0)\), we may assume, without loss of generality, that \(f(0, \ldots, 0) = 0\). Let the coefficient systems \(a(\bar{x}_{1}, \ldots, \bar{x}_{s})\) and \(\bar{a}(\bar{x}_{1}, \ldots, \bar{x}_{s})\) be as in part (i). Since \(f(\alpha_{1}, \ldots, \alpha_{s})|_{\alpha_{j} = 0} = 0\), it suffices to consider \(n(\bar{x}_{j_0}) = n_{j_0} > 0\). Let \((\bar{x}_{1}, \ldots, \bar{x}_{s}) \in X^{n_{1}} \times \cdots \times X^{n_{s}}\) and choose any \(x \in \text{supp}(\bar{x}_{1}, \ldots, \bar{x}_{s})\) with \(d(x, \Omega^{c}) = D(\text{supp}(\bar{x}_{1}, \ldots, \bar{x}_{s}), \Omega^{c})\). Then

\[
D(\text{supp}(\bar{x}_{1}, \ldots, \bar{x}_{s}), \Omega^{c}) = d(x, \Omega^{c}) \leq d(x, x_{j_0,1}) + d(x_{j_0,1}, \Omega^{c})
\]

\[
\leq \tau(\text{supp}(\bar{x}_{1}, \ldots, \bar{x}_{s})) + d(x_{j_0,1}, \Omega^{c})
\]

\[
\leq \tau(\text{supp}(\bar{y}_{1}, \ldots, \bar{y}_{s})) \sum_{j_1=1}^{s} \sum_{\ell=1}^{n_{j}} d(y_{j,\ell}x_{j,\ell}) + d(x_{j_0,1}, \Omega^{c})
\]

Consequently

\[
e^{m} \tau(\text{supp}(\bar{x}_{1}, \ldots, \bar{x}_{s})) e^{m} D(\text{supp}(\bar{x}_{1}, \ldots, \bar{x}_{s}), \Omega^{c})
\]

\[
\leq e^{2m} \tau(\text{supp}(\bar{y}_{1}, \ldots, \bar{y}_{s})) \prod_{j=1}^{s} \prod_{\ell=1}^{n_{j}} e^{2m} d(y_{j,\ell}x_{j,\ell}) e^{md(x_{j_0,1}, \Omega^{c})}
\]

and

\[
w(\bar{x}_{1}, \ldots, \bar{x}_{s})|_{\bar{a}(\bar{x}_{1}, \ldots, \bar{x}_{s})}
\]

\[
\leq \sum_{(s_{j})_{j} \in X^{n_{j}}} w''((\bar{y}_{1}, \ldots, \bar{y}_{s}), a(y_{1}, \ldots, y_{s}) \prod_{j=1}^{s} \prod_{\ell=1}^{n_{j}} e^{2m} d(y_{j,\ell}x_{j,\ell}) J_{j}(y_{j,\ell}, x_{j,\ell}) \frac{c_{j}(x_{j,\ell})}{c_{j}(y_{j,\ell})} e^{md(x_{j_0,1}, \Omega^{c})}
\]

The proof may now continue as in part (i).
Corollary IV.3 Let $h(\gamma_1, \cdots, \gamma_r)$ be an analytic function on a neighbourhood of the origin in $C^ r |X|$, and let $\Gamma_i^j$ be operators on $C^ X$ with kernels $\Gamma_i^j(x, y)$, $1 \leq i \leq s$, $1 \leq j \leq r$. Set

$$\tilde{h}(\alpha_1, \cdots, \alpha_s) = h\left(\sum_{i=1}^s \Gamma_i^1 \alpha_i, \cdots, \sum_{i=1}^s \Gamma_i^s \alpha_i\right)$$

Furthermore let $\kappa_i$, $\lambda_j : X \to (0, \infty]$ for $1 \leq i \leq s$, $1 \leq j \leq r$ be weight factors. Let $w$ be the weight system with mass $m$ and decay set $\Omega$ that associates the weight factor $\kappa_i$ to the field $\alpha_i$, and let $w'$ be the weight system with mass $m$ and decay set $\Omega$ that associates the weight factor $\lambda_j$ to the field $\gamma_j$.

(i) If, for $1 \leq j \leq r$,

$$\sup_{x \in X} \sum_{y \in X} e^{2m d(x, y)}|\Gamma_j^i(x, y)| \frac{\kappa_i(y)}{\lambda_j(x)} \leq 1 \quad \text{and} \quad \sup_{y \in X} \sum_{x \in X} e^{2m d(x, y)}|\Gamma_j^i(x, y)| \frac{\kappa_i(y)}{\lambda_j(x)} \leq 1$$

then

$$\|\tilde{h}\| \leq \|h\|_{w'}$$

In the case that $\Omega = \emptyset$ it suffices to assume

$$\sup_{x \in X} \sum_{y \in X} e^{m d(x, y)}|\Gamma_j^i(x, y)| \frac{\kappa_i(y)}{\lambda_j(x)} \leq 1 \quad \text{and} \quad \sup_{y \in X} \sum_{x \in X} e^{m d(x, y)}|\Gamma_j^i(x, y)| \frac{\kappa_i(y)}{\lambda_j(x)} \leq 1$$

for $j = 1, \cdots r$.

(ii) Let $w''$ be the weight system with mass $2m$ and empty decay set that associates the weight factor $\lambda_j$ to the field $\gamma_j$. If, for all $1 \leq j \leq r$,

$$\sup_{x \in X} \sum_{y \in X} e^{2m d(x, y)}|\Gamma_j^i(x, y)| \frac{\kappa_i(y)}{\lambda_j(x)} \leq 1 \quad \text{and} \quad \sup_{y \in X} \sum_{x \in X} e^{2m d(x, y)}|\Gamma_j^i(x, y)| \frac{\kappa_i(y)}{\lambda_j(x)} \leq 1$$

and for some $1 \leq j_0 \leq r$ with $h(\gamma_1, \cdots, \gamma_r)\big|_{\gamma_{j_0} = 0} = h(0, \cdots, 0)$,

$$\sup_{x \in X} \sum_{y \in X} e^{2m d(x, y)}e^{m d(y, \Omega^c)}|\Gamma_{j_0}^i(x, y)| \frac{\kappa_i(y)}{\lambda_{j_0}(x)} \leq 1$$

$$\sup_{y \in X} \sum_{x \in X} e^{2m d(x, y)}e^{m d(y, \Omega^c)}|\Gamma_{j_0}^i(x, y)| \frac{\kappa_i(y)}{\lambda_{j_0}(x)} \leq 1$$

then

$$\|\tilde{h}\| \leq \|h\|_{w''}$$
Proof: We prove part (i). The proof of part (ii) is similar.
Set \( \tilde{X} = X \times \{0, 1, \ldots, s\} \). Define the distance between \((x, i), (y, i') \in \tilde{X}\) to be \(d(x, y)\).
For \(j = 1, \ldots, r\), let \(J_j\) be the operator on \(C^{\tilde{X}}\) defined by
\[
(J_j \beta)(x, i') = \begin{cases} 
\sum_{y \in X} \sum_{i = 1, \ldots, a} \Gamma^i_j(x, y) \beta(y, i) & \text{if } i' = 0 \\
0 & \text{if } i' = 1, \ldots, s
\end{cases}
\]
The kernel of \(J_j\) is
\[
J_j((x, i'), (y, i)) = \begin{cases} 
\Gamma^i_j(x, y) & \text{if } i' = 0 \text{ and } i > 0 \\
0 & \text{if } i' > 0 \text{ or } i = 0
\end{cases}
\]
We also introduce the weight system \(\tilde{\omega}\) that associates to the field \(\beta_j(x, i), 1 \leq j \leq r, 0 \leq i \leq s\), the weight factor
\[
\tilde{\omega}_j(x, i) = \begin{cases} 
\lambda_j(x) & \text{if } i = 0 \\
\kappa_i(x) & \text{if } i = 1, \ldots, s
\end{cases}
\]
and has mass \(m\) and decay set \(\Omega \times \{0, 1, \ldots, s\}\) and set \(\tilde{\omega}' = \tilde{\omega}\). Then
\[
L_{2m}(J_j; \tilde{\omega}_j, \tilde{\omega}_j) = \sup_{x \in X} \sum_{y \in X} e^{2m d(x, y)} |\Gamma^i_j(x, y)| \frac{\tilde{\omega}_j(y)}{\lambda_j(x)} \leq 1
\]
\[
R_{2m}(J_j; \tilde{\omega}_j, \tilde{\omega}_j) = \sup_{y \in X} \sum_{x \in X} e^{2m d(x, y)} |\Gamma^i_j(x, y)| \frac{\tilde{\omega}_j(x)}{\lambda_j(x)} \leq 1
\]
For fields \(\beta_1, \ldots, \beta_r\) on \(\tilde{X}\), set
\[
\tilde{h}(\beta_1, \ldots, \beta_r) = h(\beta_1(\cdot, 0), \ldots, \beta_r(\cdot, 0))
\]
\[
\tilde{h}(\beta_1, \ldots, \beta_r) = \tilde{h}(J_1 \beta_1, \ldots, J_r \beta_r)
\]
By Lemma IV.2
\[
\|\tilde{h}\|_{\tilde{\omega}} \leq \|\tilde{h}\|_{\tilde{\omega}'} \tag{IV.2}
\]
If we write
\[
\beta_j(x, i) = \begin{cases} 
\gamma_j(x) & \text{for } i = 0 \\
\alpha_i(x) & \text{for } i = 1, \ldots, s
\end{cases}
\]
then
\[
\tilde{h}(\beta_1, \ldots, \beta_r) = h(\gamma_1, \ldots, \gamma_r), \quad \|\tilde{h}\|_{\tilde{\omega}'} = \|h\|_{\omega'}
\]
\[
\tilde{h}(\beta_1, \ldots, \beta_r) = \tilde{h}(\alpha_1, \ldots, \alpha_s), \quad \|\tilde{h}\|_{\tilde{\omega}} = \|\tilde{h}\|_{\omega}
\]
Therefore the claim of the Lemma follows from (IV.2).
Remark IV.4

(a) The hypotheses of Corollary IV.3.i are fulfilled if, for all $j = 1, \cdots, r$

$$\sum_{i=1,\cdots,s} N_{2m}(\Gamma_j^i; \kappa_i, \lambda_j) \leq 1, \text{ resp., in the case } \Omega = \emptyset, \sum_{i=1,\cdots,s} N_m(\Gamma_j^i; \kappa_i, \lambda_j) \leq 1,$$

or, in the case $\Omega = \emptyset$,

$$\sum_{1 \leq i \leq s} \sum_{\Lambda_j \cap \Gamma_j^i \neq 0} N_{2m}(\Gamma_j^i; \kappa_i, \lambda_j) \leq 1 \quad \text{and} \quad \sum_{1 \leq i \leq s} \sum_{\Lambda_j \cap \Gamma_j^i \neq 0} N_m(\Gamma_j^i; \kappa_i, \lambda_j) \leq 1$$

(b) The hypotheses of Corollary IV.3.ii are fulfilled if, for all $1 \leq j \leq r$,

$$\sum_{i=1,\cdots,s} N_{2m}(\Gamma_j^i; \kappa_i, \lambda_j) \leq 1$$

and for some $1 \leq j_0 \leq r$,

$$\sum_{i=1,\cdots,s} N_{2m}(\Gamma_j^{i_0}; \kappa_i, \lambda_j) \leq 1$$

(c) In the situation of Lemma IV.2.i, assume that $f(\alpha_1, \cdots, \alpha_s)$ is a polynomial that is homogeneous of degree $d_j$ in $\alpha_j$ for each $1 \leq j \leq s$. Then

$$\|\tilde{f}\|_w \leq \|f\|_{w'} \prod_{j=1}^s N_{2m}(J_j; \kappa_j, \kappa'_j)^{d_j}$$

Similar results hold for the other statements of Lemma IV.2 and for Corollary IV.2.

In [BFKT2] we use some elementary properties of the operator norms of Definition IV.1.
Remark IV.5

(i) Let $J$ be a linear map from $\Phi^X$ to $\Phi^X$, and let $m_1, m_2$ be masses with $m_1 - m_2 \geq m$. Then

$$N_m(J; \kappa, \kappa') \leq N_{m_1}(J; 1, 1) \left( \sup_{x, y \in X, J(x, y) \neq 0} e^{-m_2 d(x, y)} \frac{\kappa(y)}{\kappa(x)} \right) \left( \sup_{x \in X} \frac{\kappa(x)}{\kappa'(x)} \right)$$

$$N_m(J; \kappa, \kappa') \leq N_{m_1}(J; 1, 1) \left( \sup_{x, y \in X, J(x, y) \neq 0} e^{-m_2 d(x, y)} \frac{\kappa'(y)}{\kappa'(x)} \right) \left( \sup_{y \in X} \frac{\kappa(y)}{\kappa'(y)} \right)$$

(ii) Let $J_1, J_2$ be a linear maps from $\Phi^X$ to $\Phi^X$. Furthermore let $\kappa_1, \kappa_2, \kappa_3$ be weight factors and $Y$ a subset of $X$. We consider the functions $e^{m_1 d(x, y)}$ and $e^{-m_3 d(x, y)}$ as weight factors. Then

$$N_{m_2}(J_2 Y J_1; \kappa_1 e^{m_1 d(x, y)}, \kappa_3 e^{-m_3 d(x, y)}) \leq N_{m_1 + m_2}(Y J_1; \kappa_1, \kappa_2) \cdot N_{m_2 + m_3}(J_2 Y; \kappa_2, \kappa_3)$$

By abuse of notation, we use $Y$ to also denote the operator that multiplies by the characteristic function of the set $Y$.

(iii) Let $J$ be an operator on $\Phi^X$, and let $m', m_1, m_2$ be masses with $m_1 - m_2 \geq m$. Furthermore let $\kappa, \kappa'$ be weight factors and $Y$ and $Z$ subsets of $X$. Then

$$N_m(Y J; \kappa, \kappa' e^{m d(x, Y)}) \leq e^{-m d(Y, Z)} N_m(Y J; \kappa, \kappa')$$

$$N_m(Y J; \kappa, \kappa' e^{m_2 d(x, Y)}) \leq e^{-m_2 d(Y, Z)} N_m(Y J; \kappa, \kappa')$$

$$N_m(Y J Z; \kappa, \kappa') \leq e^{-m_2 d(Y, Z)} N_m(Y J Z; \kappa, \kappa')$$

**Proof:** (i) follows from the inequalities

$$|J(x, y)| e^{m d(x, y)} \frac{\kappa(y)}{\kappa'(y)} \leq |J(x, y)| e^{m_1 d(x, y)} \kappa'(x) \kappa(y)$$

$$|J(x, y)| e^{m d(x, y)} \kappa(y) \kappa'(x) \leq |J(x, y)| e^{m_1 d(x, y)} \kappa'(y) \kappa(y)$$

(ii)

$$L_{m_2}(J_2 Y J_1; \kappa_1 e^{m_1 d(x, y)}, \kappa_3 e^{-m_3 d(x, y)})$$

$$\leq \sup_{x \in X} \sum_{y' \in Y} e^{m_2 d(x, y')} e^{m_2 d(y', y)} |J_2(x, y')| |J_1(y', y)| \frac{\kappa_1(y)}{\kappa_3(x)} e^{m_1 d(y, Y)} e^{m_3 d(x, Y)}$$

$$\leq \sup_{x \in X} \sum_{y' \in Y} e^{(m_2 + m_3) d(x, y')} e^{(m_1 + m_2) d(y', y)} |J_2(x, y')| |J_1(y', y)| \frac{\kappa_1(y)}{\kappa_2(y')} \frac{\kappa_2(y')}{\kappa_3(x)}$$

$$\leq \sup_{x \in X} \sum_{y' \in Y} e^{(m_2 + m_3) d(x, y')} |J_2(x, y')| \frac{\kappa_2(y')}{\kappa_3(x)} \sum_{y \in X} e^{(m_1 + m_2) d(y', y)} |J_1(y', y)| \frac{\kappa_1(y)}{\kappa_2(y')}$$

$$\leq L_{m_1 + m_2}(Y J_1; \kappa_1, \kappa_2) \cdot L_{m_2 + m_3}(J_2 Y; \kappa_2, \kappa_3)$$

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Similarly

\[ R_{m_2}(J_2 Y; \kappa_1 e^{m_1 d(x,Y)}, \kappa_3 e^{-m_3 d(x,Y)}) \leq R_{m_1+m_2}(Y; \kappa_1, \kappa_2) \cdot R_{m_2+m_3}(J_2 Y; \kappa_2, \kappa_3) \]

(iii) The first inequality follows from the observation that, for all \( x \in Y, y \in X \)

\[ |J(x, y)| \frac{\kappa(y)}{\kappa'(x) e^{m d(x, y)}} \leq e^{-m' d(Y, Z)} |J(x, y)| \frac{\kappa(y)}{\kappa'(x)} \]

The second inequality follows from the fact that, for all \( x \in X, y \in Y \)

\[ e^{-m_2 d(x, y)} |J(x, y)| \frac{\kappa(y)}{\kappa'(x) e^{m_2 d(x, y)}} \leq e^{-m_2 d(Y, Z)} |J(x, y)| \frac{\kappa(y)}{\kappa'(x)} \]

The third inequality follows from the fact that, for all \( x \in Y, y \in Z \)

\[ e^{m d(x, y)} |J(x, y)| \frac{\kappa(y)}{\kappa'(x)} \leq e^{-m_2 d(Y, Z)} e^{m_1 d(x, y)} |J(x, y)| \frac{\kappa(y)}{\kappa'(x)} \]

In [BFKT2], we use a more sophisticated

**Lemma IV.6** Let

- \( m_0, m_1, m_2, m_3, m_4, m_5 \geq 0 \)
- \( O, O', L, L_1, L_2, L_3 \subset X \)
- \( J_1, J_2 \) be linear operators on \( \Phi^X \)
- \( R > 0 \) and \( \kappa : X \to (0, \infty) \) be a weight factor that obeys

\[ \kappa(x) \leq Re^{m_5 d(x, L)} \]

for all \( x \in L_3 \).

In the event that \( m_2 > 0 \), assume that \( L_j \subset O^c \) for at least one \( j \in \{1, 2, 3\} \). We set, for any operator \( A \) on \( \Phi^X \),

\[ \|A\| = N_{m_0}(A; 1, 1) \]

(i) Assume that \( m_1 + m_2 + m_4 + m_5 \leq m_0 \). If

(a) \( L_j \subset L \) for at least one \( j \in \{1, 2, 3\} \) and

\[ D = \max \{d(L_1, L_2), d(L_1, L_3), d(L_1, L_3)\} \]

or if

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(b) \( m_3 \geq m_4 \) and \( L_j \subset L \) for at least one \( j \in \{1, 2, 3\} \) and

\[
D = \max \{ d(L_1, L_2), d(L_1, L_3), d(L_1, L_3), d(L_1, O'), d(L_2, O'), d(L_3, O') \}
\]

or if

(c) \( m_3 \geq m_4 + m_5 \) and \( O' \subset L \) and

\[
D = \max \{ d(L_1, L_2), d(L_1, L_3), d(L_1, L_3), d(L_1, O'), d(L_2, O'), d(L_3, O') \}
\]

then

\[
N_{m_1}(L_1 J_1 L_2 J_2 L_3; \kappa e^{m_2 d(x, O')}, e^{m_3 d(x, O')}) \leq \Re^{-m_4 D} \| L_1 J_1 L_2 \| \| L_2 J_2 L_3 \| ^4
\]

(ii) Assume that \( m_1 + m_2 + m_3 + m_4 + m_5 \leq m_0 \) and that \( L_j \subset L \) for at least one \( j \in \{1, 2, 3\} \). If \( m_3 \geq 0 \), further assume that \( L_1 \subset O' \) for at least one \( j \in \{1, 2, 3\} \). Set

\[
D = \max \{ d(L_1, L_2), d(L_1, L_3), d(L_1, L_3) \}
\]

Then

\[
N_{m_1}(L_1 J_1 L_2 J_2 L_3; \kappa e^{m_2 d(x, O')}, e^{m_3 d(x, O')}) \leq \Re^{-m_4 D} \| L_1 J_1 L_2 \| \| L_2 J_2 L_3 \| ^4
\]

Proof: (i) Write the kernel of the operator \( L_1 J_1 L_2 J_2 L_3 \) as

\[
(L_1 J_1 L_2 J_2 L_3)(x, y) = \sum_{z \in X} (L_1 J_1 L_2)(x, z) (L_2 J_2 L_3)(z, y)
\]

For each \( x, y, z \in X \), bound

\[
e^{-m_3 d(x, O')} |(L_1 J_1 L_2)(x, z)| |(L_2 J_2 L_3)(z, y)| \kappa(y) e^{m_2 d(y, O')}
\]

\[
\leq e^{-m_3 d(x, O')} |(L_1 J_1 L_2)(x, z)| |(L_2 J_2 L_3)(z, y)| \kappa(y) e^{m_2 d(x, z) + d(z, y)}
\]

\[
\leq e^{-m_3 d(x, O')} e^{m_2 d(x, z)} |(L_1 J_1 L_2)(x, z)| e^{m_2 d(z, y)} |(L_2 J_2 L_3)(z, y)| \kappa(y)
\]

\[
\leq e^{-m_3 d(x, O')} e^{(m_2 + m_4) d(x, z)} |(L_1 J_1 L_2)(x, z)| e^{m_2 + m_4 d(z, y)} |(L_2 J_2 L_3)(z, y)| e^{-m_4 D}
\]

\[
\leq e^{-m_3 d(x, O')} e^{(m_2 + m_4) d(x, z)} |(L_1 J_1 L_2)(x, z)| e^{m_2 + m_4 d(z, y)} |(L_2 J_2 L_3)(z, y)| e^{-m_4 D}
\]

\[
\leq e^{(m_2 + m_4) d(x, z)} |(L_1 J_1 L_2)(x, z)| e^{m_2 + m_4 d(z, y)} |(L_2 J_2 L_3)(z, y)| e^{-m_4 D}
\]

\[
\leq e^{(m_2 + m_4 + m_5) d(x, z)} |(L_1 J_1 L_2)(x, z)| e^{m_2 + m_4 + m_5 d(z, y)} |(L_2 J_2 L_3)(z, y)| e^{-m_4 D}
\]

\[
\leq e^{m_2 + m_4 + m_5 d(x, z)} |(L_1 J_1 L_2)(x, z)| e^{m_2 + m_4 + m_5 d(z, y)} |(L_2 J_2 L_3)(z, y)| e^{-m_4 D}
\]

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If we fix any \( \mathbf{x} \in X \) and then sum over \( \mathbf{y}, \mathbf{z} \in X \), we arrive at the desired bound on 
\[ L_{m_1} (L_{J_1} L_{J_2} L_3; \kappa e^{m_2 d(\mathbf{x}, \mathbf{O}^c)}, e^{m_3 d(\mathbf{x}, \mathbf{O}^c)}) \]. Fixing any \( \mathbf{y} \in X \) and summing over \( \mathbf{x}, \mathbf{z} \in X \), we get the desired bound on 
\[ R_{m_1} (L_{J_1} L_{J_2} L_3; \kappa e^{m_2 d(\mathbf{x}, \mathbf{O}^c)}, e^{m_3 d(\mathbf{x}, \mathbf{O}^c)}) \].

(ii) As in part (i), bound
\[
e^{m_3 d(\mathbf{x}, \mathbf{O}^c)} \left| (L_{J_1} L_{J_2} L_3) (\mathbf{x}, \mathbf{z}) \right| \left| (L_{J_2} L_3) (\mathbf{y}, \mathbf{z}) \right| \kappa (\mathbf{y}) e^{m_2 d(\mathbf{y}, \mathbf{O}^c)}
\]
\[
\leq e^{m_3 [d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})]} \left| (L_{J_1} L_{J_2}) (\mathbf{x}, \mathbf{z}) \right| \left| (L_{J_2} L_3) (\mathbf{y}, \mathbf{z}) \right| \left| (L_{J_2} L_3) (\mathbf{z}, \mathbf{y}) \right| e^{-m_4 D}
\]
\[
\leq e^{(m_2 + m_3 + m_4) d(\mathbf{x}, \mathbf{z})} \left| (L_{J_1} L_{J_2}) (\mathbf{x}, \mathbf{z}) \right| e^{(m_2 + m_3 + m_4) d(\mathbf{z}, \mathbf{y})} \left| (L_{J_2} L_3) (\mathbf{z}, \mathbf{y}) \right| e^{-m_4 D}
\]
\[
\leq e^{(m_2 + m_3 + m_4 + m_5) d(\mathbf{x}, \mathbf{z})} \left| (L_{J_1} L_{J_2}) (\mathbf{x}, \mathbf{z}) \right| e^{(m_2 + m_3 + m_4 + m_5) d(\mathbf{z}, \mathbf{y})} \left| (L_{J_2} L_3) (\mathbf{z}, \mathbf{y}) \right| e^{-m_4 D}
\]
\[
\leq e^{(m_2 + m_3 + m_4 + m_5) d(\mathbf{x}, \mathbf{z})} \left| (L_{J_1} L_{J_2}) (\mathbf{x}, \mathbf{z}) \right| e^{(m_2 + m_3 + m_4 + m_5) d(\mathbf{z}, \mathbf{y})} \left| (L_{J_2} L_3) (\mathbf{z}, \mathbf{y}) \right| e^{-m_4 D}
\]

To convert tree decay into decay from a subset \( \Omega^c \), we use

**Lemma IV.7** Assume that \( X \) is a metric space. Let \( \Omega \subset X \) be a decay set, \( m > 0 \) a mass, and \( \kappa_j : X \to (0, \infty) \) for \( j = 1, \cdots, s \) weight factors. Let \( w \) be the weight system with mass \( m \) and decay set \( \Omega \) that associates the weight factor \( \kappa_j \) to the field \( \alpha_j \), \( j = 1, \cdots, s \). (See Definition II.5.) Also, let \( w' \) be the weight system with mass \( 2m \) and empty decay set that again associates the weight factor \( \kappa_j \) to the field \( \alpha_j \), \( j = 1, \cdots, s \). Furthermore let \( h(\alpha_1, \cdots, \alpha_s) \) be an analytic function.

(i) \[
\left\| h(\alpha_1, \cdots, \alpha_s) - h(\alpha_1, \cdots, \alpha_s) \right\|_{\alpha_j(x) = 0 \text{ for } x \in \Omega^c} \leq \left\| h \right\|_{w'}
\]

(ii) Let \( r \leq s \) and let \( w'' \) be the weight system with mass \( 2m \) that associates the weight factors \( \kappa_j \) to the field \( \alpha_1, \cdots, \alpha_r \), and the weight factors \( e^{m d(\mathbf{x}, \mathbf{O}^c)} \kappa_j(\mathbf{x}) \) to the fields \( \alpha_{r+1}, \cdots, \alpha_s \), respectively. Assume in addition that \( h(\alpha_1, \cdots, \alpha_r, 0, \cdots, 0) = 0 \). Then
\[
\left\| h \right\|_{w} \leq \left\| h \right\|_{w''}
\]

**Proof:**

(i) The coefficients \( a(\tilde{x}_1, \cdots, \tilde{x}_s) \) of \( h(\alpha_1, \cdots, \alpha_s) - h(\alpha_1, \cdots, \alpha_s) \) \( \alpha_j(x) = 0 \text{ for } x \in \Omega^c \) are non-zero only if \( \text{supp}(\tilde{x}_1, \cdots, \tilde{x}_s) \cap \Omega^c \neq \emptyset \). In this case choose any \( \mathbf{y} \in \text{supp}(\tilde{x}_1, \cdots, \tilde{x}_s) \cap \Omega^c \).
Also choose $\mathbf{x} \in \text{supp}(\bar{x}_1, \ldots, \bar{x}_s)$ such that $d(\mathbf{x}, \Omega^c) = D(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s), \Omega^c)$. Then

$$D(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s), \Omega^c) = d(\mathbf{x}, \Omega^c) \leq d(\mathbf{x}, \mathbf{y}) \leq \tau(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s))$$

so that

$$mD(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s), \Omega^c) + m\tau(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s)) \leq 2m\tau(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s))$$

(ii) In this case the coefficients $a(\bar{x}_1, \ldots, \bar{x}_s)$ of $h$ vanish unless $\text{supp}(\bar{x}_r+1, \ldots, \bar{x}_s) \neq \emptyset$. In this case choose any $\mathbf{y} \in \text{supp}(\bar{x}_r+1, \ldots, \bar{x}_s)$. Also choose $\mathbf{x} \in \text{supp}(\bar{x}_1, \ldots, \bar{x}_s)$ such that

$$d(\mathbf{x}, \Omega^c) = D(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s), \Omega^c)$$

Then

$$D(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s), \Omega^c) = d(\mathbf{x}, \Omega^c) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \Omega^c) \leq \tau(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s)) + d(\mathbf{y}, \Omega^c)$$

so that

$$w(\bar{x}_1, \ldots, \bar{x}_s) = e^{m\tau(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s))}e^{mD(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s), \Omega^c)}\prod_{j=1}^{s}\prod_{\ell=1}^{n(\bar{x}_j)}\kappa_j(\mathbf{x}_{j,\ell})$$

$$\leq e^{2m\tau(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s))}e^{mD(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s), \Omega^c)}\prod_{j=1}^{s}\prod_{\ell=1}^{n(\bar{x}_j)}\kappa_j(\mathbf{x}_{j,\ell})$$

$$\leq e^{2m\tau(\text{supp}(\bar{x}_1, \ldots, \bar{x}_s))}\prod_{j=1}^{r}\prod_{\ell=1}^{n(\bar{x}_j)}\kappa_j(\mathbf{x}_{j,\ell})\prod_{j=r+1}^{s}\prod_{\ell=1}^{n(\bar{x}_j)}e^{mD(\mathbf{x}_{j,\ell}, \Omega^c)}\kappa_j(\mathbf{x}_{j,\ell})$$

$$= w''(\bar{x}_1, \ldots, \bar{x}_s)$$

\[\blacksquare\]


