A vanishing theorem for oriented intersection multiplicites

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Abstract

Let $A$ be a regular local ring containing $1/2$, which is either equicharacteristic, or is smooth over a d.v.r. of mixed characteristic. We prove that the product maps on derived Grothendieck-Witt groups of $A$ satisfy the following property: given two elements with supports which do not intersect properly, their product vanishes. This gives an analogue for “oriented intersection multiplicites” of Serre’s vanishing result for intersection multiplicities. It also suggests a Vanishing Conjecture for arbitrary regular local rings containing $1/2$, which is analogous to Serre’s (which was proved independently by Roberts, and Gillet and Soulé).

1 Introduction

Let $A$ be a regular local ring of dimension $d$ and let $M$ and $N$ be finitely generated $A$-modules such that $M \otimes N$ is of finite length. Serre defined the intersection multiplicity $\chi_A(M, N)$ by

$$\chi_A(M, N) = \sum_{i=0}^{d} (-1)^i \ell_A[\text{Tor}^A_i(M, N)].$$

Here $\ell_A$ denotes the length of an $A$-module of finite length. It has been shown that the intersection multiplicity satisfies Serre’s Vanishing Conjecture: if $\dim M + \dim N < d$, then $\chi(M, N) = 0$. Serre proved this in [Se1] for $A$ which is equicharacteristic, or is smooth over a d.v.r. of mixed characteristic; the general case was proved independently by Roberts [Ro], and by Gillet and Soulé [GS1], [GS2].

On the other hand, let $X$ be a smooth variety of dimension $n$ over a field $k$, $V, W$ be irreducible subvarieties of $X$ and $P$ be an irreducible component of $V \cap W$ that has the right codimension, i.e., such that $\text{codim}(P, X) = \text{codim}(V, X) + \text{codim}(W, X)$. If $CH^*(X)$ denotes the Chow ring of $X$ and $\{V\}, \{W\}, \{P\}$ are the classes of the varieties $V, W, P$ in that ring, then it is well known that

$$\{V\} \cdot \{W\} = \chi_{O_X,P}(O_{V,P}, O_{W,P})\{P\} + \beta$$

for a cycle $\beta$ whose support does not contain $P$.

Some years ago, Barge and Morel introduced a generalization of the Chow groups called the oriented Chow groups ([BM]; see also [Fa1] for more details), which admit homomorphisms to the “usual” Chow groups. In a recent paper ([Fa2]), the first author showed that for a smooth variety $X$ over a field of characteristic $\neq 2$, the total oriented Chow group of $X$, denoted by $CH^*(X)$, admits a graded ring structure, such that the natural map $CH^*(X) \to CH^*(X)$ is a ring homomorphism.

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1Here, the local ring $A$ is said to be smooth over a d.v.r. if its completion is a power series ring over the corresponding complete d.v.r.
If $V$ is a closed algebraic subset of $X$ of pure codimension $n$, let $GW^n_V(X)$ denote the $n$-th Grothendieck-Witt group of perfect complexes on $X$ supported in $V$ (see [Wa] for more information). In [Fa2], it is shown that there is a natural homomorphism

$$\alpha_V: GW^n_V(X) \to \widetilde{CH}^n(X);$$

moreover, if $W$ is another closed algebraic subset of pure codimension $m$ such that $V \cap W$ is of pure codimension $m + n$, we have a commutative diagram

$$\begin{array}{ccc}
GW^n_V(X) \times GW^m_W(X) & \xrightarrow{\star} & GW^{m+n}_{V \cap W}(X) \\
\downarrow{\alpha_V \times \alpha_W} & & \downarrow{\alpha_{V \cap W}} \\
\widetilde{CH}^n(X) \times \widetilde{CH}^m(X) & \longrightarrow & \widetilde{CH}^{m+n}(X)
\end{array}$$

Here the bottom row is the multiplication in the oriented Chow ring, and $\star$ denotes the product on the Grothendieck-Witt groups ([Fa2, Theorem 7.6]), induced by the usual tensor structure on perfect complexes. The existence of this commutative diagram is an analogue for oriented Chow groups of the formula (1) for Serre’s intersection multiplicity.

Given this result, it is natural to ask whether a version of Serre’s Vanishing Conjecture is true for the Grothendieck-Witt groups. We may thus formulate:

**Conjecture.** Let $(A, \mathfrak{m})$ be a regular regular local ring of dimension $n$ containing $1/2$. Let $Z$ and $T$ be closed subsets of $\text{Spec}(A)$ such that $\dim Z + \dim T < n$ and $Z \cap T = \mathfrak{m}$. Then the multiplication

$$GW^i_Z(A) \times GW^j_T(A) \to GW^{i+j}_{\mathfrak{m}}(A)$$

is zero for any $i, j \in \mathbb{N}$.

As evidence, we have the following theorem (see Theorems 4.3, 5.1):

**Theorem.** Let $(A, \mathfrak{m})$ be a regular local ring of dimension $n$, such that $1/2 \in A$. Assume further that either $A$ contains a field, or $A$ is smooth over a discrete valuation ring of mixed characteristic.

Let $Z$ and $T$ be closed subsets of $\text{Spec}(A)$ such that $\dim Z + \dim T < n$ and $Z \cap T = \mathfrak{m}$. Then the multiplication

$$GW^i_Z(A) \times GW^j_T(A) \to GW^{i+j}_{\mathfrak{m}}(A)$$

is zero for any $i, j \in \mathbb{N}$.

The principal ingredients of the proof are the “calculus” of derived Grothendieck-Witt and Witt groups, some geometric normalization lemmas, and a result of Gille and Hornbostel giving the vanishing of maps between Witt groups defined by extension of support, in some special situations ([GH, Theorem 0.1]).

We do not recall the definitions of the Grothendieck-Witt groups and Witt groups of a triangulated category with duality. We instead refer the reader to [Ba2] for information on Witt groups, with detailed references, and to [Wa] for the definition, and basic properties of Grothendieck-Witt groups, of triangulated categories with duality. We will also use the products on Grothendieck-Witt and Witt groups defined in [GN], which are naturally induced from the tensor structure on perfect complexes. We remark that the hypothesis that all schemes are over $\mathbb{Z}[1/2]$ is built into the foundations of the theory of Grothendieck-Witt and Witt groups of triangulated categories with a duality, in their present form.

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2 Geometric lemmas

The following theorem is well known (see [Ma], theorem 14.4):

**Theorem 2.1.** Let $(A, m)$ be a $d$-dimensional Noetherian local ring and suppose that $k = A/m$ is an infinite field. Let $q = (u_1, \ldots, u_d)$ be a $m$-primary ideal. Then if $y_i = \sum a_{ij} u_j$ for $1 \leq i \leq d$ are $d$ sufficiently general linear combinations of $u_1, \ldots, u_d$, the ideal $b = (y_1, \ldots, y_d)$ is a reduction of $q$ and $\{y_1, \ldots, y_d\}$ is a system of parameters of $A$.

**Proof.** We give a sketch of the proof in order to make precise what "sufficiently general" means. The proof shows first that there is an homogeneous ideal $Q \subset k[x_1, \ldots, x_s]$ such that

$$
k[x_1, \ldots, x_s]/Q \cong \bigoplus_{n \geq 0} q^n/q^n m = \text{gr}_q(A) \otimes_{A/q} k
$$

and $\dim k[x_1, \ldots, x_s]/Q = d = \dim A$. Now let $X = \text{Spec}(k[z_{ij}])$ with $1 \leq i \leq d$, $1 \leq j \leq s$. Any $k$-rational point $(b_{ij}) \in X(k)$ gives an homomorphism $\varphi_{(b_{ij})} : k[x_1, \ldots, x_d] \to k[z_1, \ldots, z_s]$, defined by $\varphi_{(b_{ij})}(x_i) = \sum b_{ij} z_j$. We say that $(b_{ij})$ is good if $k[x_1, \ldots, x_s]/Q$ is a finite $k[x_1, \ldots, x_d]$-module (under $\varphi_{(b_{ij})}$). The proof shows that there exists a polynomial $D \in k[z_{ij}]$ such that if $U = \text{Spec}(k[z_{ij}]D)$ then any $(b_{ij}) \in U(k)$ is good. Then it is easy to see that $(a_{ij}) \in A$ has the desired property if the residue $(\pi_{ij})$ is in $U(k)$. \hfill \Box

From now on, we assume that every field appearing until the end of Section 3 is infinite.

If $k[[z_1, \ldots, z_n]] \to k[[x_1, \ldots, x_n]]$ is a homomorphism between power series algebras over $k$ induced by $z_i \mapsto \sum a_{ij} x_j$ for some $a_{ij} \in k$, we call the induced morphism $\text{Spec}(k[[x_1, \ldots, x_n]]) \to \text{Spec}(k[[z_1, \ldots, z_n]])$ a linear projection. Such linear projections are determined by points in $A^{\text{dim}}(k)$.

The following two corollaries to theorem 2.1 are obvious:

**Corollary 2.2.** Let $X = \text{Spec}(k[[x_1, \ldots, x_n]])$, $Y = \text{Spec}(k[[z_1, \ldots, z_{n-1}]])$ and $Z = V(f) \subset X$. Then any sufficiently general linear projection $p : X \to Y$ has the property that $p|_Z : Z \to Y$ is finite.

**Corollary 2.3.** Let $\pi = (x_1, \ldots, x_n)$ and $\gamma = (y_1, \ldots, y_n)$. Consider an ideal $< I(\pi), J(\gamma) >$ in the ring $k[\pi, \gamma]$. Suppose that

(i) $\dim k[\pi, \gamma]/ < I(\pi), J(\gamma) > \leq n - 1$

(ii) $< I(\pi), J(\gamma), x_1 - y_1, \ldots, x_n - y_n >$ is $< \pi, \gamma >$-primary.

Then for any sufficiently general $a_{ij} \in k$ ($1 \leq i \leq n-1, 1 \leq j \leq n$) the ideal $< I(\pi), J(\gamma), l_1(\gamma), \ldots, l_{n-1}(\gamma) >$ where $l_i(\pi) = \sum_{j=1}^n a_{ij} x_j$ and $l_i(\gamma) = \sum_{j=1}^n a_{ij} y_j$ is $< \pi, \gamma >$-primary.

**Corollary 2.4.** Let $X = \text{Spec}(k[[x_1, \ldots, x_n]])$ and $Y = \text{Spec}(k[[z_1, \ldots, z_{n-1}]])$. Let $Z, T \subset X$ be closed subsets such that $\dim Z + \dim T < \dim X$ and $Z \cap T$ is supported on the closed point. Then for any sufficiently general linear projection $p : X \to Y$ we have that $Z \neq p^{-1}(p(Z))$ and $p^{-1}(p(Z)) \cap W$ are also supported on the closed point.

**Proof.** We may interpret $Z \cap p^{-1}(p(T), T \cap p^{-1}(p(Z)$ in terms of $Z \times_T T$. Thus, let $T$ and $Z$ be respectively defined by ideals $I$ and $J$. Using Corollary 2.3, we see that there is a nonempty open subset $U \subset \text{Spec}(k[z_{ij}])$ such that for any rational point $(b_{ij}) \in U(k)$ the associated projection $p$ has the property that the ideal $< I(\pi), J(\gamma), l_1(\gamma), \ldots, l_{n-1}(\pi) - l_{n-1}(\gamma) >$ is $< \pi, \gamma >$-primary. Therefore $p^{-1}(p(Z)) \cap T$ and $Z \cap p^{-1}(p(T)$ are supported on the closed point. Further, $Z \cap p(Z), T \cap p(T)$ are finite, since the ideal $< I(\pi), l_1(\pi), \ldots, l_{n-1}(\pi) >$ is $< \pi >$-primary and the ideal $< J(\gamma), l_1(\gamma), \ldots, l_{n-1}(\gamma) >$ is $< \gamma >$-primary. Hence $p^{-1}(p(Z) \neq Z$ and $p^{-1}(p(T) \neq T$, since the fibre of $p$ over the closed point of $Y$ is 1-dimensional. \hfill \Box
3 The zero theorem for transfers

We recall (see [Gi1], Defn. 2.16) that if $X$ is a Noetherian, Gorenstein $\mathbb{Z}[1/2]$-scheme of finite Krull dimension (e.g. quasi-projective over a field or a complete discrete valuation ring), and $Z \subset X$ is a closed subscheme, then $\widehat{W}_Z^i(X)$ denotes the $i$-th derived Witt group of the full subcategory $D^b_Z(X) \subset D^b(X)$ of the bounded derived category of $X$ consisting of complexes with (coherent) homology supported (set-theoretically) in $Z$, with the natural duality structure (essentially Grothendieck-Serre duality). If $X$ is regular, this coincides with $W^i_Z(X)$, similarly defined using perfect complexes. The trace in duality theory leads to transfer maps in certain situations, with the expected properties (see [Gi2]).

The next theorem is due to Gille and Horobostel ([GH, Theorem 0.1]).

**Theorem 3.1.** Let $R$ be a Gorenstein ring of finite Krull dimension, $t \in R$ a non zero-divisor and $\pi : R \to R/tR$ the quotient map. Suppose that $\pi$ has a flat splitting $q : R/tR \to R$. Then the transfer morphism

$$\text{Tr}_{(R/tR)/R} : \widehat{W}^i(R/tR) \to \widehat{W}^{i+1}(R)$$

is zero for all $i \in \mathbb{Z}$.

In fact, the proof of their result ultimately boils down to Lemma 2.5, proved using a specific computation at the level of forms, given by an isometry. This formula in fact yields a stronger conclusion, for coherent Witt groups with supports (analogous to the stronger assertion Theorem 2.1 in their paper):

**Theorem 3.2.** Let $R$ be a Gorenstein ring of finite Krull dimension, $t \in R$ a non zero-divisor and $\pi : R \to R/tR$ the quotient map. Suppose that $\pi$ has a flat splitting $q : R/tR \to R$. Let $J \subset R$ be an ideal containing $t$, and let $\overline{J} = q(J/tR)R$. Then the transfer morphism

$$\text{Tr}_{(R/tR)/R} : \widehat{W}^i_{J/tR}(R/tR) \to \widehat{W}^{i+1}_{\overline{J}}(R)$$

is zero for all $i \in \mathbb{Z}$.

**Theorem 3.3.** Let $R$ be a Gorenstein ring of finite Krull dimension, $t \in R$ a non zero-divisor and $J \subset R$ an ideal containing $t$. Then for any $i \in \mathbb{N}$ the transfer morphism

$$\text{Tr}_{(R/tR)/R} : \widehat{W}^i_{J/t}(R/tR) \to \widehat{W}^{i+1}_{J}(R)$$

is an isomorphism.

**Proof.** See [Gi3, Theorem 3.2].

**Corollary 3.4.** Let $R$, $t$ be as in Theorem 3.2. Let $X = \text{Spec}(R)$, $Y = \text{Spec}(R/tR)$ and $Z$ a closed subscheme of $Y$. Let $p : X \to Y$ be the flat splitting of the inclusion $i : Y \to X$. Then the extension of support

$$e : \widehat{W}^i_Z(X) \to \widehat{W}^i_{p^{-1}(Z)}(X)$$

is zero.

**Proof.** We have the following commutative diagram:

$$
\begin{array}{ccc}
\widehat{W}^i_Z(X) & \xrightarrow{e} & \widehat{W}^i_{p^{-1}(Z)}(X) \\
\text{Tr}_{Y/X} \downarrow & & \uparrow \text{Tr}_{Y/X} \\
\widehat{W}^{i-1}_Z(Y) & & \\
\end{array}
$$

The diagonal transfer $\text{Tr}_{Y/X} : \widehat{W}^{i-1}_Z(Y) \to \widehat{W}^i_{p^{-1}(Z)}(X)$ is zero by theorem 3.2 and the vertical transfer $\text{Tr}_{Y/X} : \widehat{W}^{i-1}_Z(Y) \to \widehat{W}^i_Z(X)$ is an isomorphism by the above theorem. \qed
Finally, we have the following proposition, analogous to the key step in Quillen’s proof of the Gersten conjecture (see also [GH], [Sr] or [Qu]):

**Proposition 3.5.** Let \( X = \text{Spec}(k[[x_1, \ldots, x_n]]) \), \( Z \subset X \) a proper closed subset and \( Y = \text{Spec}(k[[z_1, \ldots, z_{n-1}]]) \). Then for any \( i \in \mathbb{N} \) and any sufficiently general linear projection \( p : X \to Y \) the extension of support

\[
\widetilde{W}_Z^i(X) \to \widetilde{W}_{p^{-1}(Z)}(X)
\]

is zero.

**Proof.** As \( Z \) is a proper closed subset of \( X \) there exists a non-zero non-unit \( t \in k[[x_1, \ldots, x_n]] \) such that \( Z \subset V(t) \). Let \( j : V(t) \to X \) be the inclusion. Any sufficiently general linear projection \( p : X \to Y \) is flat and has the property that \( p_{(V(t))} : V(t) \to Y \) is finite. Consider the following fibre product:

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
p' \downarrow & & \downarrow p \\
V(t) & \xrightarrow{p_{(V(t))}} & Y.
\end{array}
\]

The inclusion \( j : V(t) \to X \) induces a closed immersion \( i' : V(t) \to X' \) such that \( fi' = j \). Observe that \( V(t) \) is also a principal divisor in \( X' \) ([Sr, Theorem 5.23]). As closed subsets, we have \( p^{-1}(p(Z)) = f(p')^{-1}(Z) \) and then it is enough to show that \( \widetilde{W}_Z^i(X) \to \widetilde{W}_{f(p')^{-1}(Z)}^i(X) \) is zero to get the result. We have the following commutative diagram:

\[
\begin{array}{ccc}
\widetilde{W}_Z^i(X) & = & \widetilde{W}_Z^i(X) \\
\downarrow j_* & & \downarrow f_* \\
\widetilde{W}_Z^{-1}(V(t)) & \xrightarrow{(i')_*} & \widetilde{W}_{f(p')^{-1}(Z)}^i(X', L) \\
\downarrow f_* & & \downarrow f_* \\
\widetilde{W}_{f(p')^{-1}(Z)}^i(X', L) & \xrightarrow{e} & \widetilde{W}_{f(p')^{-1}(Z)}^i(X', L)
\end{array}
\]

where \( e \) is the extension of support, and \( L \) is the relative dualizing sheaf for \( f \) (see the appendix of [DF] for more information on functoriality properties of the transfer). By Theorem 3.3, we know that \( j_* \) is an isomorphism. The above corollary 3.4 shows that \( e : \tilde{W}_{i'(Z)}^i(X') \to \tilde{W}_{f(p')^{-1}(Z)}^i(X') \) is zero.

### 4 The theorem in the equicharacteristic case

We first recall the following computation of Witt groups with support from [BW], §6.

**Lemma 4.1.**  
(i) For any regular local ring \((A, m)\), we have isomorphisms

\[
W^n_m(A) \cong \begin{cases} 0 & \text{if } n \not\equiv \dim A \pmod{4} \\ W(A/m) & \text{if } n \equiv \dim A \pmod{4}. \end{cases}
\]

(ii) If \((A, m) \to (B, n)\) is a flat homomorphism of regular local rings with \( mB = n \), so that \( \dim A = \dim B \), then for any \( n \equiv \dim A \pmod{4} \), there is a commutative diagram

\[
\begin{array}{ccc}
W^n_m(A) & \cong & W(A/m) \\
\downarrow & & \downarrow \\
W^n_m(B) & \cong & W(B/n)
\end{array}
\]
Theorem 4.2. Let \((A, m)\) be a regular local ring of dimension \(n\) containing a field of characteristic \(\neq 2\). Let \(Z\) and \(T\) be closed subsets of \(\text{Spec}(A)\) such that \(\dim Z + \dim T < n\) and \(Z \cap T = m\). Then the multiplication

\[
W^i_Z(A) \times W^j_T(A) \to W^{i+j}_m(A)
\]

is zero for any \(i, j \in \mathbb{N}\).

Proof. Let \(\hat{A}\) be the completion of \(A\) (for the \(m\)-adic valuation).

Using lemma 4.1, we see that \(W^n_m(A) \simeq W^n_m(\hat{A})\) for all \(n\), and the following diagram commutes,

\[
\begin{array}{ccc}
W^i_Z(A) \times W^j_T(A) & \longrightarrow & W^{i+j}_m(A) \\
\downarrow & & \downarrow \\
W^i_Z(\hat{A}) \times W^j_T(\hat{A}) & \longrightarrow & W^{i+j}_m(\hat{A})
\end{array}
\]

where the vertical arrows are induced by the completion. Hence it is enough to prove the result for a complete regular local ring. Therefore we can suppose that \(A = k[[x_1, \ldots, x_n]]\) for a field \(k\).

Next, we observe that if \(k'\) is an infinite algebraic extension of \(k\) which is an increasing union of finite algebraic subextensions of odd degree, then the natural map \(W(k) \to W(k')\) is injective, since the map on Witt groups for a finite extension of fields of odd degree is injective (standard transfer argument). By applying lemma 4.1(ii) to \(k[[x_1, \ldots, x_n]] \to k'[[x_1, \ldots, x_n]]\), we thus further reduce to the case when the field \(k\) is infinite.

Let \(B = k[[z_1, \ldots, z_{n-1}]]\). Then using Corollary 2.4 and Proposition 3.5, we see that there exists a linear projection \(p : \text{Spec}(A) \to \text{Spec}(B)\) such that:

1. The extension of support \(p : W^i_Z(A) \to W^i_{p^{-1}(p(Z)))}(A)\) is zero.
2. \(p^{-1}(p(Z)) \cap T = m\).

The conclusion follows from the following commutative diagram:

\[
\begin{array}{ccc}
W^i_Z(A) \times W^j_T(A) & \longrightarrow & W^{i+j}_m(A) \\
\downarrow & & \downarrow \\
W^i_{p^{-1}(p(Z)))}(A) \times W^j_T(A) & \longrightarrow & W^{i+j}_m(A).
\end{array}
\]

As an immediate corollary, we get the following theorem.

Theorem 4.3. Let \((A, m)\) be a regular local ring of dimension \(n\) containing a field of characteristic \(\neq 2\). Let \(Z\) and \(T\) be closed subsets of \(\text{Spec}(A)\) such that \(\dim Z + \dim T < n\) and \(Z \cap T = m\). Then the multiplication

\[
GW^i_Z(A) \times GW^j_T(A) \to GW^{i+j}_m(A)
\]

is zero for any \(i, j \in \mathbb{N}\).

Proof. Let \(\pi : GW^i_m(A) \to W^i_m(A)\) be the natural projection. By Theorem 4.2, the following composition is zero:

\[
GW^i_Z(A) \times GW^j_T(A) \longrightarrow GW^i_m(A) \longrightarrow W^i_m(A).
\]
Let $f : GW_{m}^{i+j}(A) \to K_0(D_k^b(P(A)))$ be given by forgetting the symmetric form. Clearly $K_0(D_k^b(P(A))) \cong \mathbb{Z}$ and it is enough to show that the composition

$$GW_{k}^{i+j}(A) \times GW_{k}^{j}(A) \longrightarrow GW_{m}^{i+j}(A) \overset{f}{\longrightarrow} K_0(D_k^b(A))$$

is zero to prove the corollary. Since the following diagram commutes

$$\begin{array}{c}
GW_{k}^{i+j}(A) \times GW_{k}^{j}(A) \\
\downarrow_{f \times f} \quad \quad \quad \quad \quad \downarrow_{f} \\
K_0(D_k^b(A)) \times K_0(D_k^b(A)) \longrightarrow K_0(D_k^b(A))
\end{array}$$

we can use [Se1, Chapter 5, Theorem 1] to conclude the result. \hfill \Box

5 The case of a regular local ring smooth over a d.v.r. of mixed characteristic

The proof in the case of a regular local ring smooth over a d.v.r. of mixed characteristic is similar in many respects to that in the equicharacteristic case. Hence, we will be sketchy, except at points where there are some new features in the proof in this case.

First, as in the proof of Theorem 4.3, it suffices to prove the analogous vanishing result for products of Witt groups, since the corresponding result for Grothendieck groups is a consequence of Serre’s original result on vanishing of intersection multiplicities for local rings in mixed characteristic. We clearly then reduce to the case of a complete local ring, i.e., we may assume that $A = \Lambda[[x_1, \ldots, x_n]]$ where $\Lambda$ is a complete d.v.r. of mixed characteristic, and $1/2 \in \Lambda$.

Next, we claim that it suffices to treat the case when $\Lambda$ has an infinite residue field. This is similar to the argument in the equicharacteristic case. If $\Lambda$ has a finite residue field $k$, and $k'$ is a finite extension of $k$ of odd degree, then (from Cohen structure theory, for example) we can find an over-ring $\Lambda'$ which is also a complete discrete valuation ring, finite and unramified over $\Lambda$, with residue field $k'$. Since the map of Witt groups $W(k) \to W(k')$ is injective (since we have a transfer here as well), it suffices to obtain the result for $A \otimes_\Lambda \Lambda'$.

We may pass to a direct limit over a tower of such odd extensions, and obtain a new local ring, smooth over a d.v.r. with infinite residue field, and it suffices to prove the result for the completion of this new local ring.

Thus, we have reduced the proof of the Main Theorem in the mixed characteristic case to the following result.

**Theorem 5.1.** Let $\Lambda$ be a complete d.v.r. of mixed characteristic containing $1/2$, with infinite residue field, and let $A = \Lambda[[x_1, \ldots, x_n]]$, with maximal ideal $m$. Let $Z$ and $T$ be closed subsets of $\text{Spec}(A)$ such that $\dim Z + \dim T < n + 1 = \dim A$ and $Z \cap T = m$. Then the multiplication

$$W_{Z}^{i}(A) \times W_{T}^{j}(A) \to W_{m}^{i+j}(A)$$

is zero for any $i, j \in \mathbb{N}$.

**Proof.** We first consider the case when $Z, T$ are both contained in the closed fiber $\overline{X}$ of $X \to \text{Spec}(A)$. Let $f : \overline{X} \to X$ be the inclusion. Then we have isomorphisms (since $\overline{X}$ is a principal divisor)

$$f_* : W_{Z}^{i-1}(\overline{X}) \cong W_{Z}^{i}(X), \quad f_* : W_{T}^{j-1}(\overline{X}) \to W_{T}^{j}(X).$$

If $\alpha \in W_{Z}^{i-1}(\overline{X}), \beta \in W_{T}^{j-1}(\overline{X})$, then by the projection formula ([DF, Theorem A.15]), we have

$$f_*(\alpha) \cdot f_*(\beta) = f_*(\alpha \cdot f^*f_*\beta) \in W_{m}^{i+j}(X).$$
So it suffices to prove that the composition
\[ W^j_{T^{-1}}(X) \xrightarrow{p^*} W^j_T(X) \xrightarrow{f_*} W^{j+1}_T(X). \]
is 0. For this, it suffices prove the vanishing of the further composition with the isomorphism
\[ f_* : W^j_T(X) \rightarrow W^{j+1}_T(X). \]
Now if \( 1 \in W(X) \) is the unit form, then
\[ f_* f_* = f_*(f^* f_* 1) = f_* f_* = f_* f_*(f^* f_* 1). \]
So it suffices to show that \( f^* f_* (1) \in W(\overline{X}) \) vanishes. But, regarding \( X \) as a scheme over \( \text{Spec}(\Lambda) \), clearly \( 1 \in W(X) \) is the pullback of \( 1 \in W(\text{Spec}(k)) \), where \( k \) is the residue field of \( \Lambda \), and the element \( f^* f_* (1) \) is similarly the pullback of the corresponding element in \( W(\text{Spec}(k)) \).

But \( W(\text{Spec}(k)) = 0 \).

Thus, in the case when \( Z, T \) are both contained in the closed fiber \( \overline{X} \), the theorem holds.

So we may assume that (say) \( Z \) is not contained in the closed fiber; in particular, \( n > 0 \).

Let \( Z = Z' \cup Z'' \) where all irreducible components of \( Z' \) dominate \( \text{Spec}(\Lambda) \), and \( Z'' = Z \cap \overline{X} \).

Let \( \overline{Z} = Z' \cap \overline{X} \) be the closed fiber of \( Z' \rightarrow \text{Spec}(\Lambda) \), so that \( \dim \overline{Z} = \dim Z' - 1 \). Let \( T \) be the closed fiber of \( T \rightarrow \text{Spec}(\Lambda) \).

Let \( Y = \text{Spec}(\Lambda[[z_1, \ldots, z_{n-1}]]) \). Let \( Y \) denote the closed fiber of \( Y \) over \( \text{Spec}(\Lambda) \).

We consider morphisms \( p : X \rightarrow Y \) of \( \Lambda \)-schemes induced by continuous homomorphisms with \( z_i \mapsto \sum_j a_{ij} x_j \), with \( a_{ij} \in \Lambda \). For \( a \in \Lambda \), let \( \overline{a} \) denote its image in the residue field \( k \).

By Corollary 2.3 applied to the ideals of \( \overline{Z} \) and \( T \) in \( \overline{X} = \text{Spec}(k[[x_1, \ldots, x_n]]) \), we see that for general \( a_{ij} \in \Lambda \), if \( p : X \rightarrow Y \) is the corresponding morphism, then \( p^{-1}(\overline{Z}) \cap T = \{ m \} \), and \( p_{|\overline{Z}} : \overline{Z} \rightarrow Y \) is finite (since the fiber of \( Z' \rightarrow Y \) over the closed point is the corresponding fiber of \( \overline{Z} \rightarrow Y \), it is quasi-finite).

Let \( \overline{p^{-1}(\overline{Z})} \) be the closed fiber of \( p^{-1}(\overline{Z}) \), and let \( \overline{Z} = Z'' \cup \overline{p^{-1}(\overline{Z})} \). Then \( \overline{Z} \) is the closed fiber of \( p^{-1}(\overline{Z}) \cup Z'' \), and it has dimension at most that of \( Z \).

We now claim that the image of the extension of support map
\[ W^i_Z(X) \rightarrow W^i_{p^{-1}(\overline{Z}) \cup Z''}(X) \]
has image contained in that of the similar map
\[ W^i_{\overline{Z}}(X) \rightarrow W^i_{p^{-1}(\overline{Z}) \cup Z''}(X), \]
From the exact sequence
\[ W^i_Z(X) \rightarrow W^i_{p^{-1}(\overline{Z}) \cup Z''}(X) \rightarrow W^i_{p^{-1}(\overline{Z}) \setminus \overline{Z}}(X \setminus \overline{Z}), \]
and the excision isomorphism
\[ W^i_{p^{-1}(\overline{Z}) \cup Z'' \setminus \overline{Z}}(X \setminus \overline{Z}) \rightarrow W^i_{p^{-1}(\overline{Z}) \setminus \overline{X}}(X \setminus \overline{X}), \]
it suffices to show that the map
\[ W^i_{\overline{Z}}(X) \rightarrow W^i_{p^{-1}(\overline{Z}) \setminus \overline{X}}(X \setminus \overline{X}) \]
is 0. This in turn follows from the fact that
\[ W^i_{\overline{Z} \setminus \overline{X}}(X \setminus \overline{X}) \rightarrow W^i_{p^{-1}(\overline{Z}) \setminus \overline{X}}(X \setminus \overline{X}) \]
is 0, by Proposition 3.5 applied to the affine scheme \( X \setminus \overline{X} \).
From the commutative diagram

\[
\begin{array}{ccc}
W_i^j(A) \times W_j^j(A) & \rightarrow & W_m^{i+j}(A) \\
\downarrow \downarrow & & \downarrow \downarrow \\
W^{i-1}_{p^{-1}(p(Z)), Z'}(A) \times W_l^j(A) & \rightarrow & W_m^{i+j}(A)
\end{array}
\]

we thus see that it suffices to prove the result with \(Z\) replaced by \(\tilde{Z}\), i.e., in the special case when \(Z\) is contained in the closed fiber. By a similar argument, we further reduce to the case when \(T\) is also contained in the closed fiber; now we are in the first case, already dealt with.

\[
\square
\]

References


