Multivariate Archimedean Copulas, 
\( d \)-monotone Functions and \( \ell_1 \)-norm Symmetric Distributions

Alexander J. McNeil

*Maxwell Institute for the Mathematical Sciences, Edinburgh, UK*

Johanna Nešlehová

*Department of Mathematics, ETH Zurich, Zurich, Switzerland*

**Abstract**

It is shown that a necessary and sufficient condition for an Archimedean copula generator to generate a \( d \)-dimensional copula is that the generator is a \( d \)-monotone function. Moreover the class of \( d \)-dimensional Archimedean copulas is shown to coincide with the class of survival copulas of \( d \)-dimensional \( \ell_1 \)-norm symmetric distributions that place no point mass at the origin. The \( d \)-monotone Archimedean copula generators may be characterized using a little-known integral transform of Williamson (1956) in an analogous manner to the well-known Bernstein-Widder characterization of completely monotone generators in terms of the Laplace transform. These insights allow the construction of new Archimedean copula families and provide a general solution to the problem of sampling multivariate Archimedean copulas. They also yield useful expressions for the \( d \)-dimensional Kendall function and Kendall’s rank correlation coefficients and facilitate the derivation of results on the existence of densities and the description of singular components for Archimedean copulas. The existence of a sharp lower bound for Archimedean copulas with respect to the positive quadrant dependence ordering is also shown.

**Key words:** Archimedean copula, \( d \)-monotone function, simplex distribution, \( \ell_1 \)-norm symmetric distribution, Laplace transform, Williamson \( d \)-transform, stochastic simulation
1 Introduction

Archimedean copulas are an important class of multivariate dependence models which enjoy considerable popularity in a number of practical applications, including multivariate survival modelling. Their first appearance was in the context of probabilistic metric spaces and a detailed account of this theory may be found, for example, in Schweizer and Sklar (1983). Following Ling (1965), any Archimedean copula permits a simple algebraic form,

\[ C(u_1, \ldots, u_d) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)), \quad (u_1, \ldots, u_d) \in [0, 1]^d \quad (1) \]

where \( \psi \) is a specific function also called a generator of \( C \). As a consequence, many dependence properties of such copulas are relatively easy to establish because they reduce to analytical properties of the generator \( \psi \); see, for example, Genest and McKay (1986a-b), Genest and Rivest (1993), Joe (1997), Nelsen (1999) or Müller and Scarsini (2005). Furthermore, numerous parametric families of Archimedean copulas have been constructed which have attractive stochastic properties and lead to statistically tractable models for continuous data.

While any \( d \)-dimensional Archimedean copula is necessarily of the form (1), the converse of this statement is not true. Somewhat surprisingly, the conditions under which a generator \( \psi \) defines a \( d \)-dimensional copula by means of (1) have not yet been fully clarified except in two cases. Schweizer and Sklar (1983) show that a generator \( \psi \) induces a bivariate copula if and only if it is convex whereas Kimberling (1974) proves that \( \psi \) defines an Archimedean copula in any dimension if and only if it is a completely monotone function, or equivalently, a Laplace transform of a non-negative random variable. Kimberling’s condition is, however, not necessary for a given dimension \( d \geq 3 \) and leads to limited dependence characteristics. Some authors (including Genest and Rivest (1989), Nelsen (2005) or Müller and Scarsini (2005)) have required that \( \psi \) only has derivatives up to order \( d \) which alternate in sign, a condition which, though sufficient and considerably weaker, is still not necessary. The present paper fills this gap by showing that the necessary and sufficient condition is that \( \psi \) should have an analytical property known as \( d \)-monotonicity. This allows the possibility of \( d \)-dimensional Archimedean copulas without densities and reveals the existence of a sharp lower bound on the set of all \( d \)-dimensional Archimedean copulas with respect to the concordance ordering.

The understanding of the necessary and sufficient conditions for \( \psi \) leads to the insight that Archimedean copulas have a very natural geometric interpretation: they are the copulas that are implicit in the survival functions of \( d \)-dimensional \( \ell_1 \)-norm symmetric distributions, also known as simplicially contoured distributions. These were first introduced by Fang and Fang (1988) and comprise scale mixtures of the uniform distribution on the unit \( \ell_1 \)-norm sphere. While it has been observed by Müller (2006) that Archimedean copulas can be constructed using \( \ell_1 \)-norm symmetric distribu-
tions, this paper shows that these distributions are in fact absolutely central to the study of Archimedean copulas.

Formalization of this link requires the consideration of a little-known integral transform which appears in the work of Williamson (1956) and which we call in this paper a Williamson \( d \)-transform. Essentially this transform plays the same role in the study of \( d \)-monotone generators that the Laplace transform plays in the study of completely monotone generators, and knowledge of this link enables us both to propose a general solution to the problem of generating Archimedean copulas with arbitrary given generators, and to propose and analyse a rich variety of new Archimedean copulas.

The paper is organized as follows. Section 2 discusses the necessary and sufficient conditions for a function \( \psi \) to generate a \( d \)-dimensional Archimedean copula. Section 3 establishes the connections between Archimedean copulas, \( \ell_1 \)-norm symmetric distributions and Williamson \( d \)-transforms. The theory is then utilized in Section 4 to prove some properties of Archimedean copulas; this includes establishing conditions for the existence of densities and singular components, extending the notion of the Kendall function (proposed by Genest and Rivest (1993) in the bivariate case) to higher dimensions, and deriving a lower bound for Archimedean copulas with respect to the concordance ordering. In Section 5 we conclude by presenting a series of examples to show how both well-known and entirely new Archimedean copulas may be generated, and suggesting implications for statistical analysis.

2 Archimedean copulas in a given dimension

This section gives necessary and sufficient conditions under which a generator \( \psi \) induces an Archimedean copula via (1). To begin with, we need to establish basic notation. Throughout, \( \mathbf{x} \) denotes a vector \((x_1, \ldots, x_d)\) in \( \mathbb{R}^d \); in particular, \( \mathbf{0} \) is the origin. If not otherwise stated, all expressions such as \( \mathbf{x} + \mathbf{y} \), \( \max(\mathbf{x}, \mathbf{y}) \) or \( \mathbf{x} \leq \mathbf{y} \) are understood as componentwise operations. Furthermore, \([\mathbf{x}, \mathbf{y}]\) refers to the set \([x_1, y_1] \times \cdots \times [x_d, y_d]\) and \( \mathbb{R}^d_+ \) abbreviates the positive quadrant \([0, \infty)^d\). Finally, \( \|\mathbf{x}\|_1 \) denotes the \( \ell_1 \)-norm of \( \mathbf{x} \), i.e. \( \|\mathbf{x}\|_1 = \sum_{i=1}^d |x_i| \) and \( x_+ \) denotes \( \max(x, 0) \).

The symbol \( \mathbf{X} \) will be reserved for a random vector on \( \mathbb{R}^d \) with distribution function \( H \) and survival function \( \overline{H} \), defined, respectively, by

\[
H(\mathbf{x}) = \mathbb{P}(\mathbf{X} \leq \mathbf{x}) \quad \text{and} \quad \overline{H}(\mathbf{x}) = \mathbb{P}(\mathbf{X} > \mathbf{x})
\]

for any \( \mathbf{x} \in \mathbb{R}^d \). Note that \( \overline{H} \) may be (uniquely) retrieved from \( H \) by means of the Sylvester-Poincaré sieve formula and vice versa. In particular, therefore, a probability distribution on \( \mathbb{R}^d \) is uniquely given by \( \overline{H} \) and conversely. Both \( \overline{H} \) and \( H \) can be extended to \( \mathbb{R}^d \) by the usual limiting process and it is in this sense terms like
\( \tilde{H}(x) \) and \( H(x) \) for \( x \in \mathbb{R}^d \) are to be understood. Furthermore, if the support of \( H \) is a subset of \((0, \infty)^d\), \( H \) is uniquely given by its values in \( \mathbb{R}^d_+ \) because obviously \( \tilde{H}(x) = \tilde{H}(\max(x, 0)) \), in which case we refrain from specifying the values of \( \tilde{H} \) outside of \( \mathbb{R}^d_+ \) to ease the presentation. However, it should be kept in mind that if the associated probability distribution places mass on \( \mathbb{R}^d \) on the boundary of \( \mathbb{R}^d_+ \), \( \tilde{H} \) is not uniquely retrievable from its restriction to \( \mathbb{R}^d_+ \).

Finally, we will work with difference operators defined as follows. Let \( f \) be an arbitrary \( d \)-place real function, \( x \in \mathbb{R}^d \) and \( h > 0 \). Then the \( d \)th order difference \( \Delta_h f(x) \) is defined as

\[
\Delta_h f(x) = \Delta_{h_1}^{d} \cdots \Delta_{h_i}^{1} f(x)
\]

where \( \Delta_{h_i}^{1} \) denotes the first order difference operator given by

\[
\Delta_{h_i}^{1} f(x) = f(x_1, \ldots, x_{i-1}, x_i + h_i, x_{i+1}, \ldots, x_d) - f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d).
\]

In the context of distribution functions, \( \Delta_h H(x) \) is the volume assigned by \( H \) to the interval \( [x, x + h] \). This motivates us to denote a function \( f : A \to \mathbb{R} \), \( A \subseteq \mathbb{R}^d \) fulfilling \( \Delta_h f(x) \geq 0 \) for any choice of \( x \) and \( h \) so that all vertices of \( (x, x + h] \) lie in \( A \) as quasi-monotone on \( A \). In other accounts quasi-monotone functions are referred to as \( d \)-increasing, but we prefer the former terminology here to avoid confusion with the notion of \( d \)-monotonicity of real functions which will play a key role later on.

Next, we recall few basic results which will be needed in subsequent discussions.

**Definition 1** A \((d\text{-dimensional})\) copula is a function \( C : [0, 1]^d \to [0, 1] \) satisfying

(i) \( C(u_1, \ldots, u_d) = 0 \) whenever \( u_i = 0 \) for at least one \( i = 1, \ldots, d \).
(ii) \( C(u_1, \ldots, u_d) = u_i \) if \( u_j = 1 \) for all \( j = 1, \ldots, d \) and \( j \neq i \).
(iii) \( C \) is quasi-monotone on \([0, 1]^d\).

Perhaps unfortunately, a survey of Archimedean copulas will amount to the investigation of certain probability distributions on \( \mathbb{R}^d \) given by their survival rather than their distribution functions. To avoid working with the more cumbersome formula for \( H \) in terms of \( \tilde{H} \), it is convenient to formulate the following simple observation.

**Lemma 1** A \( d \)-place function \( \tilde{H} : \mathbb{R}^d \to [0, 1] \) is a survival function of a probability measure on \( \mathbb{R}^d \) if and only if

(i) \( \tilde{H}(-\infty, \ldots, -\infty) = 1 \) and \( \tilde{H}(x) = 0 \) if \( x_i = \infty \) for at least one \( i = 1, \ldots, d \).

(ii) \( \tilde{H} \) is right-continuous, i.e., for all \( x \in \mathbb{R}^d \) it holds that

\[
\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall y \geq x \quad \|y - x\|_1 < \delta \Rightarrow |\tilde{H}(y) - \tilde{H}(x)| < \varepsilon
\]

(iii) The function \( G \) given by \( G(x) = \tilde{H}(-x) \), \( x \in \mathbb{R}^d \), is quasi-monotone on \( \mathbb{R}^d \).

**Proof.** Since the proof is a standard exercise, we only provide a sketch. First, assume \( X \) is a random vector with survival function \( \tilde{H} \). Then (i) and (ii) are immediate and
(iii) is due to the fact that $\bar{H}(-x) = P(-X < x)$. Conversely, suppose $\bar{H}$ satisfies conditions (i) – (iii). Then the right-continuous version $G_+$ of $G$ is a distribution function on $\mathbb{R}^d$. Let $X$ be a random vector with distribution function $G_+$ and observe that $\bar{H}$ is the survival function of $-X$. □

Similarly, it will prove convenient to restate the original result by Sklar (1959) in terms of survival functions.

**Theorem 1** Let $\bar{H}$ be a $d$-dimensional survival function with margins $\bar{F}_i$, $i = 1, \ldots, d$. Then there exists a copula $C$, referred to as the survival copula of $\bar{H}$, such that, for any $x \in \mathbb{R}^d$,

$$\bar{H}(x) = C(\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d)).$$

Furthermore, $C$ is uniquely determined on $D = \{u \in [0,1]^d : u \in \text{ran } \bar{F}_1 \times \cdots \times \text{ran } \bar{F}_d\}$ where $\text{ran } \bar{F}_i$ denotes the range of $\bar{F}_i$. In addition, for any $u \in D$,

$$C(u) = \bar{H}(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))$$

where $F_i^{-1}(u_i) = \inf \{x : \bar{F}_i(x) \leq u_i\}$, $i = 1, \ldots, d$. Conversely, given a copula $C$ and univariate survival functions $\bar{F}_i$, $i = 1, \ldots, d$, $\bar{H}$ defined by (2) is a $d$-dimensional survival function with marginals $\bar{F}_1, \ldots, \bar{F}_d$ and survival copula $C$.

In particular, if $X$ is a random vector with survival function $\bar{H}$ and continuous marginals $F_1, \ldots, F_d$ and $U$ a random vector distributed as the survival copula $C$ of $\bar{H}$, we have that

$$U \overset{d}{=} (\bar{F}_1(X_1), \ldots, \bar{F}_d(X_d)) \quad \text{and} \quad X \overset{d}{=} (F_1^{-1}(U_1), \ldots, F_d^{-1}(U_d)).$$

We are now in position to finally turn our attention to Archimedean copulas. The latter were originally characterized by associativity and the property that $C(u, u) < u$ for any $u \in [0, 1]$; the present paper uses a more common definition based on the generator function $\psi$.

**Definition 2** A non-increasing and continuous function $\psi : [0, \infty) \to [0, 1]$ which satisfies the conditions $\psi(0) = 1$ and $\lim_{x \to \infty} \psi(x) = 0$ and is strictly decreasing on $[0, \inf \{x : \psi(x) = 0\})$ is called an Archimedean generator. A $d$-dimensional copula $C$ is called Archimedean if it permits the representation

$$C(u) = \psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d), \quad u \in [0, 1]^d$$

for some Archimedean generator $\psi$ and its inverse $\psi^{-1} : (0, 1] \to [0, \infty)$ where by convention $\psi(\infty) = 0$ and $\psi^{-1}(0) = \inf \{u : \psi(u) = 0\}$.

Note that several authors define Archimedean copulas in terms of $\psi^{-1}$ rather than $\psi$. The reason the above definition is favoured here is that it leads, in this context, to simpler expressions as will soon be apparent from the discussions below.
A closer look at (1) readily reveals that \( \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)) \) always satisfies the boundary conditions (i) and (ii) of Definition 1. Therefore, an Archimedean generator \( \psi \) defines a \( d \)-dimensional copula via (1) if and only if \( \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)) \) is quasi-monotone. The latter condition is not necessarily true, as illustrated by the following simple example.

**Example 1** Let \( \psi \) be a function on \([0, \infty)\) given by
\[
\psi(x) = \begin{cases} 
1 - x & \text{if } x \in [0, 1/2], \\
3/2 - 2x & \text{if } x \in [1/2, 3/4], \\
0 & \text{if } x \in [3/4, \infty) \n\end{cases}
\]
which conforms to our definition of an Archimedean generator. However, the 2-place function \( \psi(\psi^{-1}(u) + \psi^{-1}(v)) \) is not quasi-monotone because it assigns negative mass to the rectangle \([11/16, 13/16]^2\):
\[
\Delta_{(1/16, 1/16)} \psi \left( \psi^{-1} \left( \frac{11}{16} \right) + \psi^{-1} \left( \frac{11}{16} \right) \right) \\
= \psi \left( 2\psi^{-1} \left( \frac{13}{16} \right) \right) - 2\psi \left( \psi^{-1} \left( \frac{13}{16} \right) + \psi^{-1} \left( \frac{11}{16} \right) \right) + \psi \left( 2\psi^{-1} \left( \frac{11}{16} \right) \right) \\
= \psi \left( \frac{5}{8} \right) - 2\psi \left( \frac{1}{2} \right) + \psi \left( \frac{3}{8} \right) = \frac{7}{8} - 1 < 0.
\]

The failure of the quasi-monotonicity in Example 1 is due to the non-convexity of the generator, according to the following well-known result.

**Proposition 1** Let \( \psi \) be an Archimedean generator in the sense of Definition 2. Then the function given by
\[
\psi(\psi^{-1}(u) + \psi^{-1}(v))
\]
for \( u, v \in [0, 1] \) is a copula if and only if \( \psi \) is convex.

**Proof.** This result is Theorem 6.3.2 of Schweizer and Sklar (1983).

Proposition 1 does not extend to dimensions \( d \geq 3 \), as illustrated below.

**Example 2** It is a simple matter to check that the function \( \psi(x) = \max(1 - x, 0) \) is a convex Archimedean generator. However,
\[
\psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)) = \max(u_1 + \cdots + u_d - d + 1, 0)
\]
which is the Fréchet-Hoeffding lower bound \( W(u_1, \ldots, u_d) \). As is well known, this function is not a copula for \( d \geq 3 \) because it assigns negative mass to \([1/2, 1]^d\).
Consequently, stronger requirements on \( \psi \) are needed. As will be shown in the sequel, these are based on the notion of \( d \)-monotone functions introduced below.

**Definition 3** A real function \( f \) is called \( d \)-monotone in \( (a, b) \), where \( a, b \in \mathbb{R} \) and \( d \geq 2 \), if it is differentiable there up to the order \( d - 2 \) and the derivatives satisfy

\[
(-1)^k f^{(k)}(x) \geq 0, \quad k = 0, 1, \ldots, d - 2
\]

for any \( x \in (a, b) \) and further if \( (-1)^{d-2} f^{(d-2)} \) is non-increasing and convex in \( (a, b) \). For \( d = 1 \), \( f \) is called \( 1 \)-monotone in \( (a, b) \) if it is non-negative and non-increasing there. If \( f \) has derivatives of all orders in \( (a, b) \) and if \( (-1)^k f^{(k)}(x) \geq 0 \) for any \( x \) in \( (a, b) \), then \( f \) is called completely monotone.

Definition 3 can be extended to functions on not necessarily open intervals.

**Definition 4** A real function \( f \) on an interval \( I \subseteq \mathbb{R} \) is \( d \)-monotone (completely monotone) on \( I \), \( d \in \mathbb{N} \), if it is continuous there and if \( f \) restricted to the interior \( I^o \) of \( I \) is \( d \)-monotone (completely monotone) on \( I^o \).

The reason why the notion of \( d \)-monotonicity is of key importance for the present paper relies on the following result which relates \( d \)-monotonicity to the existence of survival functions.

**Proposition 2** Let \( f \) be a real function on \([0, \infty)\), \( p \in [0, 1] \) and \( \bar{H} \) specified by

\[
\bar{H}(x) = f(\| \max(x, 0) \|_1) + (1 - p)1\{x < 0\}, \quad x \in \mathbb{R}^d.
\]

Then \( \bar{H} \) is a survival function on \( \mathbb{R}^d \) if and only if \( f \) is a \( d \)-monotone function on \([0, \infty)\) satisfying the boundary conditions \( \lim_{x \to \infty} f(x) = 0 \) and \( f(0) = p \).

The proof, together with that of Theorem 2 below, is deferred to Appendix A.

**Theorem 2** Let \( \psi \) be an Archimedean generator. Then \( C : [0, 1]^d \to [0, 1] \) given by

\[
C(u) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)), \quad u \in [0, 1]^d
\]

is a \( d \)-dimensional copula if and only if \( \psi \) is \( d \)-monotone on \([0, \infty)\).

Note that Theorem 2 entails Proposition 1 as a special case. A further straightforward consequence is as follows.

**Corollary 1** Suppose \( \psi \) is an Archimedean generator which has derivatives up to order \( d \) on \((0, \infty)\). Then \( \psi \) generates an Archimedean copula if and only if \( (-1)^k \psi^{(k)}(x) \geq 0 \) for \( k = 1, \ldots, d \).

Obviously, \( d \)-monotonicity of an Archimedean generator \( \psi \) does not imply that \( \psi \) is \( k \)-monotone for \( k > d \). In other words, \( d \)-monotone Archimedean generators do not necessarily generate Archimedean copulas in dimensions higher than \( d \). This observation motivates the following notation.
Definition 5 $\Psi_d$ denotes the class of $d$-monotone Archimedean generators for $d \geq 2$. In addition, $\Psi_\infty$ constitutes of Archimedean generators which can generate an Archimedean copula in any dimension $d \geq 2$.

On the other hand, if a function is $d$-monotone for some $d \geq 2$, it is also $k$-monotone for any $1 \leq k \leq d$. This easy consequence of Definition 2 means that

$$\Psi_2 \supseteq \Psi_3 \supseteq \ldots$$

Before presenting examples which will highlight in particular that $\Psi_d \setminus \Psi_{d+1}$ is nonempty, we note that the verification of the $d$-monotonicity of a generator $\psi$ can be considerably simplified by means of the following result, which is excerpted from Williamson (1956), Theorem 4.

Proposition 3 Let $d \geq 2$ be an integer and $\psi$ be an Archimedean generator in the sense of Definition 2. Then $\psi$ is $d$-monotone on $[0, \infty)$ if and only if $(-1)^{d-2}\psi^{(d-2)}$ exists on $(0, \infty)$ and is non-negative, non-increasing and convex there.

Example 3 Consider the generator $\psi_d^L(x) = (1 - x)^{d-1}$ for some value of $d \geq 2$. This generator recurs throughout the sequel and the reason for the special notation will be apparent in Section 4. We can verify that $\psi_d^L$ is $d$-monotone by computing $(\psi_d^L)^{(d-2)}(x) = (-1)^{d-2}(d-1)!(1 - x)^{d-2}$ and observing that $(-1)^{d-2}(\psi_d^L)^{(d-2)}$ is non-negative, non-increasing and convex on $(0, \infty)$. It is not, however, $(d+1)$-monotone, since $(\psi_d^L)^{(d-1)}$ does not exist at $x = 1$. In other words, $\psi_d^L \in \Psi_d \setminus \Psi_{d+1}$.

Example 4 The Clayton copula family has generator $\psi_\theta(x) = (1 + \theta x)^{-1/\theta}$. It is well known that for $\theta > 0$ this generator is completely monotone and can be used to construct a copula in any dimension. The case $\theta = 0$ should be understood as the limit (from either side) $\psi_\theta(x) = \lim_{\theta \to 0}(1 + \theta x)^{-1/\theta} = \exp(-x)$, which generates the independence copula in any dimension.

The interesting case is $\theta < 0$. When $\theta = -1/(d - 1)$ for some integer $d \geq 2$ then $\psi_\theta(x) = \psi_d^L(\theta x)$ where $\psi_d^L$ is the generator in Example 3. The argument of that example holds and the multiplicative $\theta$ is unimportant; $\psi_\theta$ generates copulas up to dimension $d$ and, in fact, generates exactly the same copulas as $\psi_d^L$. For the general negative case let $\alpha = -1/\theta > 0$, write $\tilde{\psi}_\alpha(x) = \psi_{-1/\alpha}(x)$, and observe first that $\tilde{\psi}_\alpha$ is convex or $2$-monotone when $\alpha \geq 1$. Moreover when $d \geq 3$ and $\alpha \geq d - 1$ then

$$(-1)^{d-2}\tilde{\psi}_\alpha^{(d-2)}(x) = \alpha^{-(d-2)}\prod_{k=0}^{d-3}(\alpha - k) \left(1 - \frac{x}{\alpha}\right)^{\alpha-d+2}$$

exists and is non-negative, non-increasing and convex on $(0, \infty)$, whereas for other $\alpha$ values the derivative will either fail to exist everywhere or fail to have these properties; thus, by Proposition 3, the generator $\tilde{\psi}_\alpha$ is $d$-monotone if and only if $\alpha \geq d - 1$.

Summarizing, the Clayton generator $\psi_\theta$ is $d$-monotone for a particular $d \geq 2$ if and only if $\theta \geq -1/(d - 1)$.
To conclude this section, we can embed the well-known characterization of $\Psi_\infty$ due to Kimberling (1974) in our previous findings.

**Proposition 4** An Archimedean generator $\psi$ belongs to $\Psi_\infty$ if and only if it is completely monotone on $[0, \infty)$.

### 3 Archimedean copulas and $\ell_1$-norm symmetric distributions

For the remaining part of the paper, assume that $d \geq 2$. The beauty of Kimberling’s result on the characterization of $\Psi_\infty$ lies in its combination with the well-known Bernstein-Widder theorem (see, e.g., Widder (1946)). The latter states that an Archimedean generator is completely monotone on $[0, \infty)$ precisely when it is a Laplace transform of a non-negative random variable, say $W$. In this case, as is well-known, the $d$-dimensional Archimedean copula generated by $\psi$ is a survival copula of the survival function

$$\bar{H}(x) = \psi(\|x\|) = E\left( e^{-\|x\|W} \right) = E\left( e^{-W \sum_{i=1}^{d} x_i} \right),$$

which is the survival function of the random vector $X = \frac{1}{W} Y$, where $Y = (Y_1, \ldots, Y_d)$ is a vector of iid exponential variables, independent of $W$. $X$ follows a mixed exponential distributions or frailty model of the kind that is often used to model dependent lifetimes; see Oakes (1989) and Marshall and Olkin (1988).

However it is possible to give another representation for $X$. We can use the well-known facts that the random vector $S_d = Y / \|Y\|$ has a uniform distribution on the $d$-dimensional simplex, and that $S_d$ and $\|Y\|$ are independent to write $X = RS_d$ where $R = \|Y\| / W$ and $R$ is independent of $S_d$. This shows that $X$ follows a so-called $\ell_1$-norm symmetric distribution, a mixture of uniform distributions on simplices, also known as a simplicially contoured distribution.

Archimedean generators that are not completely monotone cannot appear as survival copulas of random vectors following frailty models, but they can all appear as survival copulas of random vectors following $\ell_1$-norm symmetric distributions. These distributions, introduced by Fang and Fang (1988), are formally defined as follows.

**Definition 6** A random vector $X$ on $\mathbb{R}_+^d = [0, \infty)^d$ follows a $\ell_1$-norm symmetric distribution if and only if there exists a non-negative random variable $R$ independent of $S_d$ where $S_d$ is a random vector distributed uniformly on the unit simplex $S_d$,

$$S_d = \{x \in \mathbb{R}_+^d : \|x\|_1 = 1\},$$

so that $X$ permits the stochastic representation

$$X = RS_d.$$
The random variable \( R \) is referred to as the radial part of \( X \) and its distribution as the radial distribution.

The general relationship between Archimedean copulas and \( \ell_1 \)-norm symmetric distributions is quite similar to the relationship between Archimedean copulas with completely monotone generators and frailty models outlined above. The difference is that the role of the Laplace transform is taken by another integral transform, which we call the Williamson \( d \)-transform and define as follows.

**Definition 7** Let \( X \) be a non-negative random variable with distribution function \( F \) and \( d \geq 2 \) an integer. The Williamson \( d \)-transform of \( X \) is a real function on \([0, \infty)\) given by

\[
W_d F(x) = \int_{(x, \infty)} \left(1 - \frac{x}{t}\right)^{d-1} dF(t) = \begin{cases} 
E \left( \frac{1 - t^x}{t} \right)^{d-1} & \text{if } x > 0, \\
1 - F(0) & \text{if } x = 0.
\end{cases}
\]

The class of functions which are Williamson \( d \)-transforms of non-negative random variables will be denoted by \( W_d \).

**Remark 1** The Williamson \( d \)-transform belongs to the general class of Mellin-Stieltjes convolutions and is closely related to the Cesàro means. For more details, refer to Bingham et al. (1989), p. 194 and p. 246.

It is clear that a Williamson \( d \)-transform always exists and is right-continuous at 0 by the dominated convergence theorem. Far less obvious is the following result due to Williamson (1956).

**Proposition 5** Let \( d \geq 2 \) be an arbitrary integer. Then

1. \( W_d \) consists precisely of the real functions \( f \) on \([0, \infty)\) which are \( d \)-monotone on \([0, \infty)\) and satisfy the boundary conditions \( \lim_{x \to \infty} f(x) = 0 \) as well as \( f(0) = p \) for \( p \in [0, 1] \).
2. The distribution of a non-negative random variable is uniquely given by its Williamson \( d \)-transform. If \( f = W_d F \) then, for \( x \in [0, \infty) \), \( F(x) = W_d^{-1} f(x) \) where

\[
W_d^{-1} f(x) = 1 - \sum_{k=0}^{d-2} \frac{(-1)^k x^k f^{(k)}(x)}{k!} - \frac{(-1)^{d-1} x^{d-1} f_{+}^{(d-1)}(x)}{(d - 1)!}.
\]

**Proof.** The proof requires a reformulation of Williamson’s original result in terms of distribution functions. This is carried out in Appendix B. \( \square \)

**Remark 2** The reader may wonder about the limiting behavior of the right-hand side of (5) at 0 and \( \infty \). The latter is clarified by Williamson (1956) who finds that \( \lim_{x \to \infty} x^k f^{(k)}(x) = 0 \) for \( k = 0, \ldots, d - 1 \). On the other hand, \( \lim_{x \to 0} x^k f^{(k)}(x) = 0 \), \( k = 1, \ldots, d - 1 \) results from a careful examination of the proof of Proposition 5.
by induction. By this we mean a successive consideration of the distributions of
the radial parts corresponding to
\[ \bar{H}_k(x) = f(\|\max(x,0)\|_1) + (1 - f(0))1\{x < 0\}, \]
x \in \mathbb{R}^k, k = 1, \ldots, d - 1. In particular, it holds that \( F_R(0+) = 1 - f(0) \). This
confirms the fact that \( F \) has no atom at 0 if and only if \( f(0) = 1 \). It may be noted that
\[ \lim_{x \to 0} x^k f^{(k)}(x) = 0, \quad k = 1, \ldots, d - 1, \]
is clear when \( f^{(k)}(0) \) exists. This however is not always the case, as one may convince oneself by considering the function
\[ f(x) = (1 - \sqrt{x})_+. \]

It may be noted that the proof of Proposition 5 indicates that if \( f \in W_d \) then \( f(0) = p \) if and only if the corresponding distribution function \( F \) satisfies \( F(0) = 1 - p \) (i.e. has an atom at 0 of size \( 1 - p \)). Furthermore, since any \( f \in W_d \)
is strictly decreasing on \([0, \inf\{x : f(x) = 0\}]\), we can conclude the following.

**Corollary 2** \( \Psi_d \) consists of the Williamson \( d \)-transforms of distribution functions \( F \) of non-negative random variables satisfying \( F(0) = 0 \).

The role of the Williamson \( d \)-transform in the study of \( \ell_1 \)-norm symmetric distributions is given in the following Proposition 6, which also recalls two further properties of \( \ell_1 \)-norm symmetric distributions that will be used in this paper. These and many other results on \( \ell_1 \)-norm symmetric distributions have been derived by K.T. Fang, B.Q. Fang and collaborators; see the monograph by Fang et al. (1990).

**Proposition 6** Let \( X \overset{d}{=} RS_d \) be a random vector which follows a \( \ell_1 \)-norm symmetric distribution with radial distribution function \( F_R \). Then

(i) The survival function \( \bar{H} \) of \( X \) is given by
\[
\bar{H}(x) = 2M_d F_R(\|\max(x,0)\|_1) + F_R(0)1\{x < 0\}, \quad x \in \mathbb{R}^d.
\]  

(ii) The density \( X \) exists if and only if \( R \) has a density. In that case, it is specified
as \( h(\|x\|_1) = \Gamma(d)\|x\|^{1-d} f_R(\|x\|_1) \) where \( f_R \) denotes the density of \( R \).

(iii) If \( P(X = 0) = 0 \) then \( R = \|X\|_1 \) and \( S_d = X / \|X\|_1 \).

**Proof.** See Fang et al. (1990), Theorems 5.1, 5.4 and 5.5. \( \square \)

The important implication of this proposition is that the survival functions of \( \ell_1 \)-norm symmetric distributions reduce to \( 2M_d F_R(\|x\|_1) \), a simple function of \( \|x\|_1 \), whenever \( x \in \mathbb{R}^d_+ \). In Proposition 7 below we extend this result to show the converse, namely that whenever a survival function is a function of \( \|x\|_1 \) on \( x \in \mathbb{R}^d_+ \) it must be the survival function of an \( \ell_1 \)-norm symmetric distribution, provided it places no mass on the boundary of \( \mathbb{R}^d_+ \) except possibly at the origin.

**Proposition 7** Let \( X \) be a random vector on \( \mathbb{R}^d_+ \). Then the following are equivalent:

(i) \( X \) has a \( \ell_1 \)-norm symmetric distribution.
(ii) There exists a real function $f$ on $[0, \infty)$ so that the joint survival function $H$ of $X$ satisfies, for any $x \in \mathbb{R}^d$,

$$H(x) = f(\|\max(x, 0)\|_1) + (1 - f(0)) \mathbf{1}\{x < 0\}. \quad (4)$$

Proof. The fact that a $\ell_1$-norm symmetric distribution satisfies (ii) is an immediate consequence of Proposition 6. The reversed implication can be established as follows. First, note that if (ii) holds, $f$ must be $d$-monotone on $[0, \infty)$ and satisfy $f(0) \in [0, 1]$ as well as $\lim_{x \to -\infty} f(x) = 0$ by Proposition 2. By Proposition 5 we have that $f \in W_d$. Let $F_R$ be the distribution function $F_R$ such that $f = W_d F_R$. Then $H$ is precisely of the form (3) and $X$ must have an $\ell_1$-norm symmetric distribution with radial distribution $F_R$.

Remark 3 Proposition 7 is needed to fully establish the connection between $\ell_1$-norm symmetric distributions and Archimedean copulas. Fang and Fang (1988) prove a related result that does not go quite far enough: they introduce a class $T_d$ of distributions on $\mathbb{R}_+^d$ whose survival functions are of the form $f(\|x\|_1)$ for some function $f$ on $[0, \infty)$ and argue that a member of $T_d$ is $\ell_1$-norm symmetric if $f$ is $d$-monotone and $f(0) = 1$. What Proposition 7 shows is that $T_d$ in fact coincides with the class of $\ell_1$-norm symmetric distributions as long we simply exclude certain behaviour on the boundary of $\mathbb{R}_+^d$.

We give a simple example to show how Proposition 7 offers an easy way of checking whether a given survival function corresponds to an $\ell_1$-norm symmetric distribution.

Example 5 A random vector $X$ with iid standard exponential margins has survival function $H(x) = e^{\sum_{i=1}^d x_i}$, $x \in \mathbb{R}_+^d$, which is simply a function of $\|x\|_1$. By Proposition 6 the distribution of the corresponding radial part is that of $\|X\|_1$ which is an Erlang distribution with parameter $d$ or equivalently a gamma distribution with shape parameter $d$ and unit rate parameter, written $\|X\|_1 \sim Ga(d)$.

We are now in a position to bring all the pieces of the argument together in the following, which is the main result of this section.

Theorem 3 (i) Let $X$ have a $d$-dimensional $\ell_1$-norm symmetric distribution with radial distribution $F_R$ satisfying $F_R(0) = 0$. Then $X$ has an Archimedean survival copula with generator $\psi = W_d F_R$.

(ii) Let $U$ be distributed according to the $d$-dimensional Archimedean copula $C$ with generator $\psi$. Then $(\psi^{-1}(U_1), \ldots, \psi^{-1}(U_d))$ has an $\ell_1$-norm symmetric distribution with survival copula $C$ and radial distribution $F_R$ satisfying $F_R = W_d^{-1} \psi$, i.e.

$$F_R(x) = 1 - \sum_{k=0}^{d-2} \frac{(-1)^k x^k \psi^{(k)}(x)}{k!} - \frac{(-1)^{d-1} x^{d-1} \psi^{(d-1)}(x)}{(d-1)!}, \quad x \in [0, \infty). \quad (5)$$
Proof. (i) From Proposition 6, part (i), it follows that the survival function of $X$ satisfies $H(x) = \psi(\|x\|_1)\log(0_+)$ on $\mathbb{R}^d$ where $\psi = \mathcal{W}_dF_R$. By Corollary 2 we know that $\psi \in \Psi_d$. As shown in the proof of Theorem 2, since the marginal survival functions of $H$ are the continuous functions $F_i(x) = \psi(\max(x,0))$ we can conclude that the unique survival copula of $X$ is the Archimedean copula generated by $\psi$.

(ii). The survival function of $(\psi^{-1}(U_1), \ldots, \psi^{-1}(U_d))$ is given by $H(x_1, \ldots, x_d) = C(\psi(x_1), \ldots, \psi(x_d)) = \psi(\max(x,0))$ on $\mathbb{R}^d$. From Proposition 7 we conclude that this is the survival function of an $\ell_1$-norm symmetric distribution and $C$ is its copula. Since $\psi \in \Psi_d$ it must be the Williamson $d$-transform of some distribution function $F_R$ of a non-negative random variable $R$ satisfying $F_R(0) = 0$ and this $R$ is the radial part in the representation $(\psi^{-1}(U_1), \ldots, \psi^{-1}(U_d)) \overset{d}{=} RS_d$. Finally, $F_R = \mathcal{W}_d^{-1}\psi$ follows by (ii) of Proposition 5. \(\square\)

Remark 4 If we consider radial random variables with point mass at the origin and use these to construct $\ell_1$-norm symmetric distributions, then Proposition 6, part (i), shows that these continue to have survival functions determined by the Williamson $d$-transform $f(x) = \mathcal{W}_dF_R(x)$, albeit in a case where $f(0) < 1$ and $f \notin \Psi_d$. The $\ell_1$-norm symmetric distribution does not have a unique copula since it has discontinuous univariate margins. We enter the treacherous world of discrete copulas; see the recent paper by Genest and Nešlehová (2007).

In the light of the second part of Theorem 3 it will be useful to define the notion of an $\ell_1$-norm symmetric distribution associated with a particular Archimedean generator.

Definition 8 Let $\psi \in \Psi_d$ and let $F_R = \mathcal{W}_d^{-1}\psi$ be the inverse Williamson $d$-transform of $\psi$ specified by (5). Then $F_R$ will be known as the radial distribution associated with $\psi$ in dimension $d$. If $R \sim F_R$ is a random variable independent of $S_d$ where $S_d$ is uniformly distributed on $S_d$, the distribution of $RS_d$ will be known as the $\ell_1$-norm symmetric distribution associated with $\psi$ in dimension $d$.

Remark 5 Although there is a one-to-one relationship between $\psi$ and a particular radial distribution $F_R$ and corresponding $\ell_1$-norm symmetric distribution, there is not a one-to-one relationship between the $d$-dimensional Archimedean copulas and the $\ell_1$-norm symmetric distributions. Consider the radial variables $R$ and $R = kR$ for $k > 0$. These have radial distribution functions $F_R(r)$ and $F_R(r) = F_R(r/k)$ respectively and give rise to different $\ell_1$-norm symmetric distributions. Their Williamson $d$-transforms are $\psi(x) = \mathcal{W}_dF_R(x)$ and $\tilde{\psi}(x) = \psi(x/k)$. However these generate exactly the same Archimedean copula in dimension $d$.

Examples 6 and 7 illustrate the derivation of associated radial distributions in two simple cases.

Example 6 Consider the Archimedean generator $\psi_d^d(x) = (1 - x)_+^{d-1}$ introduced in Example 3. The survival function of the associated $d$-dimensional $\ell_1$-norm symmetric distribution is given by $H(x) = (1 - \|x\|_1)_+^{d-1}$, $x \in \mathbb{R}^d_+$. We now apply (5) in order to
calculate the distribution of the radial part. It is clear that \( F_R(x) = 1 \) and \( F_R(x) = 0 \), if, respectively, \( x \geq 1 \) and \( x < 0 \). For \( x \in [0, 1) \) Formula (5) yields

\[
F_R(x) = 1 - \sum_{k=0}^{d-1} \frac{(-1)^k x^k}{k!} (d-1) \ldots (d-k)(1-x)^{d-1-k} = 1 - \sum_{k=0}^{d-1} \left( \frac{d-1}{k} \right) x^k (1-x)^{d-1-k} = 1 - (x + (1-x))^{d-1} = 0.
\]

Consequently, \( R = 1 \) a.s. This confirms the result obtained by de Finetti that \( \bar{H} \) is precisely the joint survival function of \( S_d \); see Fang et al. (1990), Theorem 5.2. ■

**Example 7** The Archimedean generator \( \psi(x) = e^{-x} \) is a member of \( \Psi_\infty \) and, as is well-known, induces the independence copula \( \Pi \), i.e. \( \Pi(u) = \prod_{i=1}^d u_i, u \in [0,1]^d \). Furthermore, the corresponding \( \ell_1 \)-norm symmetric distribution has survival function \( \bar{H}(x) = e^{-\sum_{i=1}^d x_i} \) which is the survival function of the vector \( X \) of iid standard exponential variables. As has been discussed in Example 5, the radial distribution of \( X \) is gamma with shape parameter \( d \). This can also be obtained using (5). Since \( (-1)^k \psi^{(k)}(x) = e^{-x} \), it follows that

\[
F_R(x) = 1 - e^{-x} \sum_{k=0}^{d-1} \frac{x^k}{k!}
\]

for \( x \in [0, \infty) \) and \( F_R(x) = 0 \) otherwise. The density of \( F_R \) is

\[
f_R(x) = e^{-x} \frac{x^{d-1}}{(d-1)!}, \quad x \in [0, \infty)
\]

so that \( R \sim Ga(d) \). ■

4 Archimedean copulas in new perspective

In general the close relationship between Archimedean copulas and \( \ell_1 \)-norm symmetric distributions delineated in Theorem 3 can be exploited in various ways. The important ingredient is the fact that the distribution \( F_R \) of the radial part of an \( \ell_1 \)-norm symmetric distribution can be explicitly retrieved from \( \mathcal{W}_d F_R \) and hence from \( \bar{H} \) by means of (5), the inversion formula for Williamson \( d \)-transforms. From this point of view, working with Williamson \( d \)-transforms is more convenient than working with Laplace transforms for which the inversion formula may not be possible to evaluate explicitly.

To begin with, the simple stochastic structure of \( \ell_1 \)-norm symmetric distributions can help us to explore analytical as well as dependence properties of Archimedean
copulas. We shall see several results of this kind in Sections 4.1 – 4.3. Secondly, the connection between Archimedean copulas and \( \ell_1 \)-norm symmetric distributions can be beneficial for statistical modelling. This will be discussed in Section 5.

4.1 Singular components of \( d \)-dimensional Archimedean copulas

The well-known Lebesgue decomposition theorem yields that any copula \( C \) can be written as

\[
C = C_A + C_S
\]

where \( C_A \) and \( C_S \) are, respectively, the absolutely continuous and singular component of \( C \). In other words, \( C_A \) is a distribution function of a (finite) measure on \( \mathbb{R}^d \) which is absolutely continuous w.r.t. the Lebesgue measure on \( \mathbb{R}^d \), i.e. it has Lebesgue density. On the other hand, the measure induced by \( C_S \) is singular, i.e. it is concentrated on a set of Lebesgue measure zero.

Generally, the study of either component of a copula is not easy. However, in view of maximum likelihood estimation and theoretical statistics it may be convenient to know whether a copula \( C \) has a density. In this section, we shall see that more concrete results can be obtained when \( C \) is Archimedean. Furthermore, Genest and MacKay (1986b) and Nelsen (1999) discuss singular components of Archimedean copulas in the bivariate case; we shall extend these results to higher dimensions.

For the start however, we would like to warn too optimistic readers against a fallacy. Let \( X \) denote a random vector with survival function \( \bar{H} \) and continuous marginal survival functions \( \bar{F}_1, \ldots, \bar{F}_d \). Then, according to Sklar’s theorem, there exists a unique survival copula \( C \) of \( \bar{H} \). Furthermore, as mentioned in Section 2, if \( U \) is a random vector whose distribution function is \( C \), then

\[
U \overset{d}{=} (\bar{F}_1(X_1), \ldots, \bar{F}_d(X_d)) \quad \text{and} \quad X \overset{d}{=} (\bar{F}_1^{-1}(U_1), \ldots, \bar{F}_d^{-1}(U_d)).
\]

In other words, if \( P^C \) and \( P^H \) denote, respectively, the probability measures induced by \( C \) and \( \bar{H} \), then \( P^C \) is an image measure of \( P^H \) with respect to a certain transformation and vice versa. Therefore, it may be tempting to believe that if \( P^H \) is absolutely continuous than so is \( P^C \) and vice versa. This may be true when \( \bar{F}_i \) and \( \bar{F}_i^{-1} \) obey the usual smoothness conditions required by the classical change of variable formula for Lebesgue integrals. In general however, it is only guaranteed that \( \bar{F}_i \) is differentiable almost everywhere and to investigate the relationship between the continuity properties of \( P^C \) and \( P^H \) requires more involved change of variable formulas which surpass the scope of this paper. To readers who are more interested in this subject we recommend to consult Hoffmann-Jørgensen (1994) for further information and references. We do provide a simple counterexample however.

**Example 8** Take \( F_C \) to be the Cantor function (c.f. Hewitt and Stromberg (1975), Example 18.8). As is well-known, \( F_C : [0, 1] \to [0, 1] \) is strictly increasing, continu-
ous, fulfills $F_C(0) = 0$ as well as $F_C(1) = 1$ but also that $F_C'(t) = 0$ a.e. in $[0, 1]$. In particular, therefore, $F_C$ provides an example of a singular univariate distribution function. Now, consider the following survival function $\bar{H}$ on $\mathbb{R}^d$:

$$\bar{H}(x) = \prod_{i=1}^{d} \bar{F}_C(x_i), \quad x \in \mathbb{R}^d.$$ 

It is immediate that $\bar{H}$ does not induce an absolutely continuous probability measure. On the other hand, the marginals $\bar{F}_C$ of $\bar{H}$ are continuous and there exists a unique survival copula corresponding to $\bar{H}$ by Sklar’s theorem. In fact, the latter is the independence copula $\Pi$ specified as $\Pi(u) = \prod_{i=1}^{d} u_i$. Clearly, $\Pi$ has a density. □

We chose to explain the fallacy in terms of survival functions because it is more convenient for the discussions below; it may be noted however that the more traditional setup of copulas corresponding to distribution functions is merely another side of the same coin. This is due to the fact that a copula is absolutely continuous precisely when its survival counterpart is.

In the case of Archimedean copulas however, absolute continuity can be lead back to absolute continuity of the associated $\ell_1$-norm symmetric distribution as the following Proposition shows.

**Proposition 8** Let $C$ be a $d$-dimensional Archimedean copula with generator $\psi \in \Psi_d$. Let further $H$ stand for the distribution function of the $\ell_1$-norm symmetric distribution associated with $C$. Then

(i) $C$ is absolutely continuous if and only if $H$ is.

(ii) If $\psi \in \Psi_{d+1}$ then $C$ is absolutely continuous.

**Proof.** Refer to Appendix C. □

It may be noted that Proposition 8 in particular implies that all lower dimensional margins of an Archimedean copula are absolutely continuous.

Now, Proposition 6 states that an $\ell_1$-norm symmetric distribution is absolutely continuous if and only if its radial part is. Furthermore, the distribution function of the radial part is retrievable from $\psi$ using the inversion formula (5) for Williamson $d$-transforms. These observations yield the following result.

**Proposition 9** Let $C$ be a $d$-dimensional Archimedean copula with generator $\psi$. Then $C$ is absolutely continuous if and only if $\psi^{(d-1)}$ exists and is absolutely continuous in $(0, \infty)$. Furthermore, if $C$ is absolutely continuous then its density $c$ is given by

$$c(u) = \frac{\psi^{(d)}(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))}{\psi'(\psi^{-1}(u_1)) \cdots \psi'(\psi^{-1}(u_d))}$$

for almost all $u \in (0, 1)^d$. 16
**Proof.** Verification of this claim may again be found in Appendix C. □

Note that, in particular, Archimedean generators considered in Corollary 1 obey conditions of Proposition 9.

Let us now summarize where we stand. If \( C \) is a \( d \)-dimensional Archimedean copula, its generator \( \psi \) belongs to either \( \Psi_{d+1} \) or \( \Psi_{d} \setminus \Psi_{d+1} \). In the former case, \( C \) has a density by means of Proposition 8. In the latter however, the density of \( C \) may or may not exist. Examples below illustrate situations that arise for \( \psi \in \Psi_{d} \setminus \Psi_{d+1} \).

**Example 9** \( \psi^{(d-1)} \) **is absolutely continuous on** \((0, \infty)\). **In this case,** Proposition 9 **ensures that** \( C \) **has a density.** **An example in the bivariate case is as follows.** Take \( \psi \) on \([0, \infty)\) **given by**

\[
\psi(x) = \begin{cases} 
\frac{8}{7}x^2 - \frac{16}{7}x + 1 & \text{if } x \in [0, 0.5), \\
\frac{16}{7}x^2 - \frac{24}{7}x + \frac{9}{7} & \text{if } x \in [0.5, 0.75), \\
0 & \text{if } x \in [0.75, \infty).
\end{cases}
\]

Because \( \psi'(x) = \frac{16}{7}(x - 1) \) for \( x \in [0, 0.5) \) and \( \psi'(x) = \frac{16}{7}(2x - \frac{3}{2}) \) for \( x \in [0.5, 0.75) \), \( -\psi' \) is non-increasing and non-negative on \((0, \infty)\). In particular, therefore, \( \psi \in \Psi_2 \). \( \psi' \) is further absolutely continuous in \([0, \infty)\) and the bivariate Archimedean copula induced by \( \psi \) is absolutely continuous. On the other hand however, \( -\psi' \) is not convex as can be easily seen by noting that \( \frac{-7}{16} \psi' \) is exactly the function discussed in Example 1. Consequently, \( \psi \) is not 3-monotone on \((0, \infty)\) and hence not in \( \Psi_3 \). ■

**Example 10** \( \psi^{(d-1)} \) **is continuous but not absolutely continuous on** \((0, \infty)\). **We again provide an example in the bivariate case.** Consider the function

\[
\psi(x) = \begin{cases} 
1 - \frac{1}{c} \int_t^x F_\mathcal{C}(1 - t)dt & \text{if } t \in [0, 1], \\
0 & \text{if } t > 1,
\end{cases}
\]

where \( c = \int_0^1 F_\mathcal{C}(1-t)dt \) and \( F_\mathcal{C} \) is the Cantor function introduced in Example 8. It is first clear that \( \psi(0) = 1, \psi(0) = 0 \) and that \( \psi \) is continuous on \([0, \infty)\). Furthermore,

\[
\psi'(x) = \begin{cases} 
-\frac{1}{c}F_\mathcal{C}(1 - x) & \text{if } t \in [0, 1], \\
0 & \text{if } t > 1,
\end{cases}
\]

for any \( x \in [0, \infty) \). In particular, \( -\psi' \) is continuous, non-negative and non-increasing and thus \( \psi \in \Psi_2 \). However, \( \psi' \) is singular on \([0, 1] \). It may be noted that the radial part of the associated \( \ell_1 \)-norm symmetric distribution is also purely singular because, for almost all \( x \in [0, 1] \), \( F_R^C(x) = \frac{1}{c}F_\mathcal{C}(1 - x) = 0 \). ■

**Example 11** \( \psi^{(d-1)}_+ \) **has jumps.** **Situations of this kind arise easily upon considering radial parts with atoms.** To provide an example, set \( R \) to be a random variable independent of \( S_d \) which follows the geometric distribution, i.e. \( P(R = i) = p(1-p)^{i-1} \).
for \( i \in \mathbb{N} \). The random vector \( \mathbf{X} \overset{d}{=} \mathbf{R} \mathbf{S}_d \) is then \( \ell_1 \)-norm symmetrically distributed and its survival copula \( C \) is Archimedean with generator

\[
\psi(x) = \mathcal{M}_d F_R(x) = \sum_{i=1}^{\infty} \left( 1 - \frac{x}{i} \right)^{d-1} + p(1-p)^i, \quad x \in [0, \infty).
\]

In particular, because \( F_R \) has jumps and \( \psi \) and \( \psi^{(k)}, k = 1, \ldots, d-2 \) are continuous, formula (5) implies that \( \psi^{(d-1)}_+ \) has jumps. In particular, \( \psi^{(d-1)}_+ \) is not absolutely continuous. This confirms that \( C \) does not have density which is already clear from the fact that \( R \) has atoms.

So far, we have used the relationship between \( \ell_1 \)-norm symmetric distributions and Archimedean copulas in order to examine the existence of densities. As the rest of this section shows, the latter can be successfully employed for the investigation of the singular part of Archimedean copulas as well. We begin with a simple Example.

**Example 12** Similar situation as in Example 11 arises when we consider \( d = 2 \) and a radial part satisfying \( P(R = 1) = 2/3 \) and \( P(R = 2) = 1/3 \). The Archimedean generator of the survival copula of \( R \mathbf{S}_2 \) is then

\[
\psi(x) = \mathcal{M}_d F_R(x) = \frac{2}{3} (1 - x)_+ + \frac{1}{3} \left( 1 - \frac{x}{2} \right)_+.
\]

By the same argument as in Example 11, the bivariate Archimedean copula \( C \) induced by \( \psi \) does not have a density. Figure 1 illustrates that the \( \ell_1 \)-norm symmetric distribution with radial part \( R \) is concentrated on two simplexes,

\[
\mathcal{S}_2(1) = \left\{ \mathbf{x} \in \mathbb{R}^2_+ : \|\mathbf{x}\|_1 = 1 \right\} \quad \text{and} \quad \mathcal{S}_2(2) = \left\{ \mathbf{x} \in \mathbb{R}^2_+ : \|\mathbf{x}\|_1 = 2 \right\}.
\]

\( C \) is also purely singular as can be seen with the middle panel of Figure 1. It is not
difficult to verify that the support of \( C \) is \( A_1 \cup A_2 \) where

\[
A_i = \{(u_1, u_2) \in [0, 1]^2 : \psi^{-1}(u_1) + \psi^{-1}(u_2) = i\}, \quad i = 1, 2.
\]

Note that if \( U \sim C \), then \( \Pr(U \in A_1) = 2/3 \) and \( \Pr(U \in A_2) = 1/3 \). ■

The findings of Example 12 can be generalized. To do so, we first need to introduce additional notation. A level set \( L(s) \) of a copula \( C \) is given by

\[
L(s) = \left\{ u \in [0, 1]^d : C(u) = s \right\}
\]

for \( s \in [0, 1] \). For a \( d \)-dimensional Archimedean copula the level sets take the form

\[
L(s) = \begin{cases} 
\left\{ u \in [0, 1]^d : \sum_{i=1}^d \psi^{-1}(u_i) = \psi^{-1}(s) \right\} & \text{if } s \in (0, 1], \vspace{1em} \\
\left\{ u \in [0, 1]^d : \sum_{i=1}^d \psi^{-1}(u_i) \geq \psi^{-1}(0) \right\} & \text{if } s = 0.
\end{cases}
\]

Proposition 10 below, which is a high-dimensional extension of the results due to Genest and MacKay (1986b) and Alsina, Frank und Schweizer (c.f. Nelsen (1999), Section 4.3), indicates the mass placed by an Archimedean copula on its level sets.

**Proposition 10** Let \( C \) be a \( d \)-dimensional Archimedean copula with generator \( \psi \) and let \( P^C \) stand for the probability measure on \([0, 1]^d\) induced by \( C \). Then

\[
P^C(L(s)) = \frac{(-1)^{d-1}(\psi^{-1}(s))^{d-1}}{(d-1)!} \left( \psi^{(d-1)}(\psi^{-1}(s)) - \psi^{(d-1)}(\psi^{-1}(0)) \right)
\]

for \( s \in (0, 1] \). Furthermore, if \( \psi^{-1}(0) = \infty \) then \( P^C(L(0)) = 0 \). Otherwise,

\[
P_C(L(0)) = P^C \left\{ u \in [0, 1]^d : \sum_{i=1}^d \psi^{-1}(u_i) = \psi^{-1}(0) \right\} =
\]

\[
\frac{(-1)^{d-1}(\psi^{-1}(0))^{d-1}}{(d-1)!} \psi^{(d-1)}(\psi^{-1}(0)).
\]

**Proof.** Refer to Appendix C. □

4.2 The Kendall function in \( d \) dimensions

For a bivariate Archimedean copula \( C \) and a random vector \((U_1, U_2) \equiv C\), the Kendall function \( K_C \) (also known as the bivariate probability integral transform) defined as the distribution function of \( C(U_1, U_2) \) is a corner stone of non-parametric inference for Archimedean copulas as introduced by Genest and Rivest (1993). Although this paper is not devoted to estimation, it is nonetheless worth noting that Theorem 3 allows an easy generalization of several of the findings about \( K_C \) to...
higher dimensions. Results from this section also nicely complement those of Barbe et al. (1996), Example 3.

Let \( C \) be a \( d \)-dimensional Archimedean copula with generator \( \psi \) and \( U \sim C \) be a random vector. The function \( K_C \) is, in analogy to the bivariate case, given as the distribution function of the random variable \( C(U) \), i.e.

\[
K_C(x) = P(C(U_1, \ldots, U_d) \leq x).
\]

Clearly, \( C(U) \) is concentrated on \([0, 1]\). For bivariate Archimedean copulas, \( K_C(x) \) can be given explicitly in terms of the generator \( \psi \) and its left-hand derivative (see Genest and Rivest (1993), Proposition 1.1). To establish an analogous result in the \( d \)-dimensional case, observe first that

\[
P(C(U_1, \ldots, U_d) \leq x) = P\left( \sum_{i=1}^{d} \psi^{-1}(U_i) \geq \psi^{-1}(x) \right).
\]

Furthermore, let \( X \) be a random vector which follows the \( \ell_1 \)-norm symmetric distribution associated with \( \psi \). Sklar’s theorem then implies

\[
X \overset{d}{=} (\psi^{-1}(U_1), \ldots, \psi^{-1}(U_d)).
\]

In particular, Proposition 6 yields that the radial part \( R \) of \( X \) satisfies

\[
R \overset{d}{=} \|X\|_1 = \sum_{i=1}^{d} \psi^{-1}(U_i)
\]

and is independent of \( X/\|X\|_1 \). The consequences are now immediate.

**Proposition 11** Let \( C \) be a \( d \)-dimensional Archimedean copula with generator \( \psi \) and let \( U \) be a random vector with \( U \sim C \). Then

\[
\sum_{i=1}^{d} \psi^{-1}(U_i) \quad \text{and} \quad \left( \frac{\psi^{-1}(U_1)}{\sum_{i=1}^{d} \psi^{-1}(U_i)}, \ldots, \frac{\psi^{-1}(U_d)}{\sum_{i=1}^{d} \psi^{-1}(U_i)} \right)
\]

are independent. Moreover, for any \( j = 1, \ldots, d \), the random variable

\[
V_j = \left( 1 - \frac{\psi^{-1}(U_j)}{\sum_{i=1}^{d} \psi^{-1}(U_i)} \right)^{d-1}
\]

is standard uniform.

**Proof.** What remains to be shown is the uniformity of \( V_j, j = 1, \ldots, d \). This follows easily by observing the fact that the univariate marginals of the uniform distribution on the simplex \( S_d \) follow the Beta\((1, d - 1)\) distribution, i.e. their survival functions are given by \( \psi_d^L(x) = (1 - x)^{d-1} \).

\[\square\]
**Proposition 12** Let $C$ be a $d$-dimensional Archimedean copula generated by $\psi$. Then

$$K_C(x) = \begin{cases} 
\frac{(-1)^d(\psi^{-1}(0))^{d-1}}{(d-1)!} & \text{if } x = 0, \\
\sum_{k=0}^{d-2} \frac{(-1)^k(\psi^{-1}(x))^k}{k!} + \frac{(-1)^d(\psi^{-1}(x))^{d-1}}{(d-1)!} & \text{if } x \in (0, 1].
\end{cases}$$

**Proof.** Because $K_C(x) = P(R \geq \psi^{-1}(x))$ where $R$ denotes the radial part of the $\ell_1$-norm symmetric distribution associated with $\psi$, the assertion follows directly from Theorem 3 in combination with (5) and from Proposition 10. □

### 4.3 Archimedean copulas are bounded below with respect to the PQD ordering

Recall that the positive quadrant (or concordance) ordering is a partial ordering on the set of all $d$-dimensional copulas given as follows:

$$C_1 \preceq_{\text{PQD}} C_2 \iff C_1(u) \leq C_2(u) \text{ for any } u \in [0, 1]^d.$$  

It is a well-established fact that any bivariate copula $C$ satisfies $W \preceq_{\text{PQD}} C$ where $W$ is the Fréchet-Hoeffding lower bound specified in Example 2. In dimensions greater than two, it still holds that $W \preceq_{\text{PQD}} C$ with the unfortunate exception that $W$ is no longer a copula. Furthermore, it can be shown that for $d \geq 3$, no sharp lower bound of the set of all copulas with respect to the PQD ordering exists (see Nelsen (1999), Theorem 2.10.13).

The situation for Archimedean copulas is different however. As is well-known, an Archimedean copula $C$ whose generator is completely monotone is positive quadrant dependent, i.e. $\Pi \preceq_{\text{PQD}} C$ where $\Pi$ is the independence copula given by $\Pi(u) = \prod_{i=1}^d u_i$ (see Nelsen (1999), Corollary 4.6.3.). Proposition 13 below shows that similar statement can be made in the general case also.

**Proposition 13** Let $C$ be a $d$-dimensional Archimedean copula. Then

$$C_d^L \preceq_{\text{PQD}} C$$

where $C_d^L$ is a $d$-dimensional Archimedean copula with generator $\psi^L_d$ of Example 3.

**Proof.** Refer to Appendix C. □

Figure 2 below illustrates the lower bound $C_d^L$ in dimension $d = 3$.

A useful implication of Proposition 13 is that if $C$ is the bivariate marginal distribution function of a $d$-dimensional Archimedean copula with generator $\psi \in \Psi_d$,

$$C(u_1, u_2) = \psi(\psi^{-1}(u_1) + \psi^{-1}(u_2)), \quad (u_1, u_2) \in [0, 1],$$

21
and $\rho$ an arbitrary measure of concordance in the sense of Scarsini (1984) then

$$\rho(\mathcal{C}_d^L) \leq \rho(C)$$

where $\rho(\mathcal{C}_d^L)$ refers to $\rho$ applied to the bivariate marginal distribution of $\mathcal{C}_d^L$. This applies in particular to Kendall’s $\tau$ which is a concordance measure given by

$$\tau(C) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1 = 4 E(C(U_1, U_2)) - 1$$

where $(U_1, U_2)$ refers to a random vector distributed according to $C$.

For bivariate Archimedean copulas, Genest and Rivest (1993) discovered that $\tau(C)$ can be expressed by means of the corresponding Archimedean generator,

$$\tau(C) = 4 \int_0^1 \psi^{-1}(t) \psi'((\psi^{-1}(t)) dt + 1 = 1 - 4 \int_{\text{inf}\{x: \psi(x) = 0\}}^{\text{inf}\{x: \psi(x) = 0\}} t [\psi'(t)]^2 dt \quad (6)$$

where the second expression results from the first by change of variable. The following simple observation offers a geometric interpretation in terms of the associated $\ell_1$-norm symmetric distribution.

**Proposition 14** Let $C$ be a bivariate Archimedean copula with generator $\psi$ and $R$ be the radial part of the $\ell_1$-norm symmetric distribution corresponding to $C$ in the sense of Definition 8. Then

$$\tau(C) = 4 E(\psi(R)) - 1.$$

**Proof.** Let $(U_1, U_2)$ be a random vector distributed as $C$. The assertion is then an immediate consequence of the fact that $\psi((\psi^{-1}(U_1) + \psi^{-1}(U_2)) \overset{d}{=} \psi(R)$ which has been established in Section 4.2. $\square$

The lower bound on Kendall’s tau $\tau(C)$ is now straightforward.
Corollary 3  Let $C$ be the bivariate margin of a $d$-dimensional Archimedean copula with generator $\psi$. Then $\tau(C) \geq -\frac{1}{2d-3}$.

Proof. The proof follows from (6) using elementary calculus.  \qed

5 Examples and Practical Implications

In this section we give some examples showing further practical implications of the connection between Archimedean copulas and $\ell_1$-norm symmetric distributions and give some brief discussion of diagnostic tests for copulas and estimation of copulas.

5.1 Identification of $R$ and stochastic simulation

For a given Archimedean generator $\psi \in \Psi_d$ generating a copula $C$ in a particular dimension $d$, it is of interest to determine the distribution of the radial part $R$ of the corresponding $d$-dimensional $\ell_1$-norm symmetric distribution in the sense of Definition 8. Knowledge of this distribution can be helpful for stochastic simulation, calculation of dependence measures like Kendall’s rank correlation and testing of data for a dependence structure consistent with a given Archimedean copula.

Example 13  Consider the Clayton copula family of Example 4 with generators $\psi_\theta(x) = (1+\theta x)^{-1/\theta}$. Assume that a particular $d \geq 2$ and a particular $\theta \geq -1/(d-1)$ are given. Suppose first that $\theta < 0$, set $\alpha = -1/\theta > 0$ and consider the generator $	ilde{\psi}_\alpha(x) = \psi_{-1/\alpha}(x)$. It is straightforward to use Theorem 3 in combination with (5) to calculate that for $x \in [0, \alpha)$ the radial distribution must satisfy

$$F_R(x) = 1 - \sum_{k=0}^{d-1} \frac{(-1)^k x^k \tilde{\psi}_\alpha^{(k)}(x)}{k!}$$

$$= 1 - \sum_{k=0}^{d-1} \frac{x^k}{k! \alpha^k} \prod_{j=0}^{k-1}(\alpha - j) \left(1 - \frac{x}{\alpha}\right)^{\alpha-k}$$

$$= 1 - \sum_{k=0}^{d-1} \binom{\alpha}{k} \left(\frac{x}{\alpha}\right)^k \left(1 - \frac{x}{\alpha}\right)^{\alpha-k}$$

(7)

where $\binom{\alpha}{k}$ denotes the extended binomial coefficient. Moreover, $F_R(x) = 0$ and $F_R(x) = 1$ if, respectively, $x < 0$ and $x \geq \alpha$. Observe firstly that, if $\alpha = d - 1$, then this distribution simplifies to point mass at $x = d - 1$ in an analogous manner to Example 6. The copula generated is also $C^d_{L_d}$ as in that example. It is also an easy limiting calculation to establish that, as $\alpha \to \infty$, the distribution function of $R$ converges pointwise to a gamma distribution with parameter $d$ as in Example 7.
If $\theta > 0$ we can proceed in a similar manner by differentiating $\psi_\theta(x)$ to show that for $x \in [0, \infty)$ we have

$$F_R(x) = 1 - \sum_{k=0}^{d-1} \prod_{j=0}^{k-1} \frac{(1 + j\theta)}{k!} x^k (1 + \theta x)^{-\left(1/\theta + k\right)}.$$  \hspace{1cm} (8)

Again, it is easy to verify that this converges pointwise to the gamma distribution function in Example 7 as $\theta \to 0$.

For illustration consider the case when $d = 3$. If $\theta = -0.5$ the distribution of $R$ is point mass at $x = 2$. For $\theta > -0.5$ $R$ has a density and the copula is absolutely continuous. The radial densities are given in Figure 3 for the cases when $\theta$ takes the values -0.3334, -0.3, -0.2, 0 and 0.2.

![Fig. 3. The radial densities for the Clayton copula when $\theta$ takes the values -0.3334, -0.3, -0.2, 0 and 0.2.](image)

A method of stochastic simulation for Archimedean copulas based upon our knowledge of the distribution of $R$ is straightforward in principle. To generate a random vector from the $d$-dimensional copula $C$ we would perform the following steps.

(1) Generate a random vector $S = (S_1, \ldots, S_d)$ uniformly distributed on the $d$-dimensional simplex $S_d$. One way of doing this (based on Example 5 and Proposition 6, part (iii)) is to generate a vector of iid standard exponential variates $Y = (Y_1, \ldots, Y_d)$ and to return $S_i = Y_i/\|Y\|_1$ for $i = 1, \ldots, d.$

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(2) Generate a univariate random variable $R$ having the radial distribution of the $\ell_1$-norm symmetric distribution associated with $C$ in the sense of Definition 8.

(3) Return $U = (U_1, \ldots, U_d)$ where $U_i = \psi(RS_i)$ for $i = 1, \ldots, d$.

The second step is the only practically challenging part of the algorithm. In Example 13 we could proceed by using the inversion method (although the necessary evaluation of the distribution functions (7) and (8) is slow for larger values of $d$).

We now give an example where the method works particularly efficiently and allows us to generate samples from an attractive bivariate family.

**Example 14** Consider the generator $\psi(t) = (1 - t^{1/\theta})_+$ where $\theta \geq 1$; this is the second family considered by Nelsen (1999) (see Table 4.1). Since $\psi$ is convex it follows that $\psi \in \Psi_2$ but $\psi \notin \Psi_3$ because $\psi^{(1)}(t)$ is not defined at $t = 1$. Since $\psi^{(1)}_+(t) = -\theta^{-1}t^{1/\theta-1}$ for $t \in [0, 1)$ and $\psi^{(1)}_+(t) = 0$ otherwise, we can use Theorem 3 and (5) to show that

$$F_R(x) = \begin{cases} \left(1 - \frac{1}{\theta}\right)x^{1/\theta} & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x \geq 1, \end{cases}$$

which has a jump at $x = 1$ and yields a bivariate copula with a singular component. The distribution of $R$ is easily sampled by inversion. In Figure 4 we show four examples of 2000 points generated from this copula for different values of $\theta$. Obviously $\theta = 1$ corresponds to the Fréchet lower bound copula generated by $\psi_2^L$ and as $\theta \to \infty$ we approach perfect positive dependence. It is easy to use Proposition 14 to calculate that the Kendall’s rank correlation of this copula is given by $\tau(C) = 1 - 2/\theta$ which implies that the points in the second picture ($\theta = 2$) are taken from a copula with Kendall’s rank correlation equal to zero. ■

5.2 Construction of new Archimedean copulas

By choice of a radial distribution $F_R$ for $R$ we can construct a copula in a given dimension $d$ with generator given by the Williamson $d$-transform $\mathfrak{W}_d F_R$ of $F_R$. This technique of copula construction has already been used in Examples 11 and 12. Obviously, if we choose a distribution $F_R$ that we can sample then we can easily generate data from the new copula; we simply have to be able to evaluate $\psi(x) = \mathfrak{W}_d F_R(x)$.

If the copula should have a density then we should choose a radial distribution with a density. One further example is given below.

**Example 15** Suppose we begin with the radial density

$$f_R(x) = \begin{cases} \frac{ab}{b-a}x^{-2} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise}, \end{cases}$$

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Fig. 4. Four examples of 2000 points generated from the copula of Example 14 for different values of $\theta$.

where $0 < a < b$. This is the density of the reciprocal of a uniform distribution on the interval $[b^{-1}, a^{-1}]$. The Williamson $d$-transform may be calculated to be

$$w_dF_R(x) = \int_a^b \frac{ab}{b-a} t^{-2} \left(1 - \frac{x}{t}\right)^{d-1}_+ \, dt$$

$$= \frac{ab}{xd(b-a)} \left(1 - \frac{x}{b}\right)_+^d - \left(1 - \frac{x}{a}\right)_+^d .$$

In Figure 5 we show four examples of 2000 points generated from the corresponding bivariate copula for $a = 1$ and different values of $b$. 

5.3 Diagnostic tests for Archimedean copulas and estimation

The insights of this paper and, in particular, Proposition 11 can be used to devise diagnostic tests for particular hypothesized copulas. Suppose we have $d$-dimensional data vectors $\mathbf{U}_1, \ldots, \mathbf{U}_n$ that are believed to form a random sample from an Archimedean copula with generator $\psi$. (Alternatively they might be observations from the empirical copula of some multivariate data, derived by applying the marginal
Clearly there are numerous possible ways of deriving numerical or graphical tests of \( SU \).

We can form the quantities \( a = 1; b = 1.2 \) \( a = 1; b = 1 \) \( a = 1; b = 5 \) \( a = 1; b = 1000 \)

\[ \begin{align*}
  \text{Fig. 5. Four examples of 2000 points generated from the copula of Example 15 for } a = 1 \\
  \text{and different values of } b. 
\end{align*} \]

empirical distribution functions to the data as in Genest et al. (1995).) For each vector observation \( U_k = (U_{k1}, \ldots, U_{kd}) \) we can form the quantities \( R_k = \sum_{i=1}^d \psi^{-1}(U_{ki}) \) and \( S_k = (\psi^{-1}(U_{k1}), \ldots, \psi^{-1}(U_{kd}))/R_k \). We then need to test for three things:

1. The compatibility of the data \( (R_1, \ldots, R_n) \) with the radial distribution \( F_R \) of the \( \ell_1 \)-norm symmetric distribution associated with the \( d \)-dimensional Archimedean copula generated by \( \psi \);
2. The compatibility of the data \( (S_1, \ldots, S_n) \) with the hypothesis of uniformity on the \( d \)-dimensional simplex;
3. The compatibility of the pairs \( \{(R_k, S_k), k = 1, \ldots, n\} \) with the hypothesis of independence between \( R \) and \( S \).

Clearly there are numerous possible ways of deriving numerical or graphical tests of these hypotheses.

The correspondence between the Archimedean copula family and the \( \ell_1 \)-norm symmetric distributions, and the simple structure of the latter, suggest that it should be possible to reduce the estimation of Archimedean copulas to a one-dimensional problem. Ideas developed by Genest and Rivest (1993) for the bivariate case may be extended to the general multivariate case. In that paper a non-parametric margin-
A free estimation procedure is proposed for the distribution function

\[ K_C(x) = P(C(U_1, U_2) \leq x), \]

where \((U_1, U_2)\) are random variables distributed according to the unknown copula \(C\). This procedure may be easily extended to derive a non-parametric estimate of \(K_C(x) = P(C(U_1, \ldots, U_d) \leq x)\) in the general multivariate case. At the same time the distribution \(K_C\) may be computed under parametric assumptions for the copula generator \(\psi\) using Proposition 12. This makes it possible to estimate the copula by choosing the parameter values that give the best correspondence between the parametric model for \(K_C\) and the non-parametric estimate.

**Appendix A: Proofs from Section 2**

Theorem 2 relies on several results on \(d\)-monotone functions. First is the following Lemma which summarizes Theorems 4, 5, and 6 of Chapter IV of Widder (1946).

**Lemma 2** Let \(d \geq 1\) be an integer and \(f\) a non-negative real function on \((a, b)\), \(a, b \in \mathbb{R}\). If \(f\) satisfies

\[ (\Delta_h)^k f(x) \geq 0 \]

for any \(k = 1, \ldots, d\), any \(x \in (a, b)\) and any \(h > 0\) so that \((x + kh) \in (a, b)\), then

(i) \(f\) is non-decreasing on \((a, b)\)
(ii) if \(d \geq 2\), \(f\) is convex and continuous on \((a, b)\)
(iii) if \(d \geq 2\), the left-hand and right-hand derivatives of \(f\) exist everywhere in \((a, b)\).

Moreover, for any \(a < x < y < b\),

\[ f'_-(x) \leq f'_+(x) \leq f'_+(y). \]

(iv) for any \(k = 1, \ldots, (d-1)\) and any \(a < x \leq y < b\), \((\Delta_h)^k f(x) \leq (\Delta_h)^k f(y)\) whenever \(h\) is chosen small enough so that \(y + kh < b\).

Second result which has proved of key importance is the following characterization of \(d\)-monotone functions.

**Proposition 15** Let \(f\) be a real function on \((a, b)\), \(a, b \in \mathbb{R}\) and let \(\tilde{f}\) denote a function on \((-b, -a)\) given by \(\tilde{f}(x) = f(-x)\). Further let \(d \geq 1\) be an integer. Then the following statements are equivalent:

(i) \(f\) is \(d\)-monotone in \((a, b)\)
(ii) \(f\) is non-negative and satisfies, for any \(k = 1, \ldots, d\), any \(x \in (-b, -a)\) and any \(h_i > 0, i = 1, \ldots, k\) so that \((x + h_1 + \cdots + h_k) \in (-b, -a)\),

\[ \Delta_{h_k} \cdots \Delta_{h_1} \tilde{f}(x) \geq 0 \]
where \( \Delta_{h_1} \ldots \Delta_{h_d} \) denotes a sequential application of the first order difference operator \( \Delta_h \) given by \( \Delta_h f(x) = f(x + h) - f(x) \) whenever \( x, (x + h) \in (-b, -a) \).

(iii) \( f \) is non-negative and satisfies, for any \( k = 1, \ldots, d \), any \( x \in (-b, -a) \) and any \( h > 0 \) so that \( (x + kh) \in (-b, -a) \),

\[
(\Delta_h)^k f(x) \geq 0
\]

where \((\Delta_h)^k\) denotes the \( k \)-monotone sequential use of the operator \( \Delta_h \).

**Proof of Proposition 15.**

**Proof of (ii) ⇔ (iii).** First note that (iii) is a special case of (ii). For the reverse implication, assume without the loss of generality that \( d \geq 2 \) and fix an arbitrary \( x \) in \((-b, -a)\) and \( h_i > 0 \), \( i = 1, \ldots, d \) such that \( x + h_1 + \cdots + h_d \in (-b, -a) \). Now, let \( \tilde{f}_\ell \), \( \ell = 1, \ldots, d \), denote a function on \((-b, -a - (h_d + \cdots + h_\ell))\) given by

\[
\tilde{f}_\ell(y) = \Delta_{h_d} \ldots \Delta_{h_\ell} f(y), \quad y \in (-b, -a - (h_d + \cdots + h_\ell)).
\]

Observe in particular that any \( \tilde{f}_\ell \), \( \ell = 1, \ldots, d \), is non-negative. One can even show that the functions \( \tilde{f}_\ell \), \( \ell = 2, \ldots, d \), satisfy (9) of Lemma 2 for any \( k = 1, \ldots, \ell - 1 \). This claim can be verified by induction: because \( \tilde{f} \) satisfies the assumptions of Lemma 2, the assertion (iv) thereof yields that \( (\Delta_h)^k \tilde{f}(y) \leq (\Delta_h)^k \tilde{f}(y + h_d) \) for any \( k = 1, \ldots, d - 1 \) whenever \( h \) is small enough so that \( y + kh \in (-b, -a - h_d) \). But

\[
(\Delta_h)^k \tilde{f}(y + h_d) - (\Delta_h)^k \tilde{f}(y) = (\Delta_h)^k \tilde{f}_d(y)
\]

for any \( k = 1, \ldots, d - 1 \) and hence the beginning of the induction is established.

Now suppose that \( \tilde{f}_{\ell+1} \) satisfies (9) for any \( k = 1, \ldots, \ell \). The fact that \( \tilde{f}_\ell(y) = \tilde{f}_{\ell+1}(y + h_\ell) - \tilde{f}_{\ell+1}(y) \) for any \( y \in (-b, -a - (h_d + \cdots + h_\ell)) \) yields

\[
(\Delta_h)^k \tilde{f}_\ell(y) = (\Delta_h)^k \tilde{f}_{\ell+1}(y + h_\ell) - (\Delta_h)^k \tilde{f}_{\ell+1}(y)
\]

for any \( y \in (-b, -a - (h_d + \cdots + h_\ell)) \) and any \( k = 1, \ldots, \ell - 1 \). The right-hand side is however non-negative by (iv) of Lemma 2. This immediately yields that \( \tilde{f}_\ell \) satisfies (9) for any \( k = 1, \ldots, \ell - 1 \).

Now, application of Lemma 2 to \( \tilde{f}_2 \) implies that \( \tilde{f}_2 \) is non-decreasing on \((-b, -a - (h_d + \cdots + h_2))\). Because \( x, x + h_1 \in (-b, -a - h_d - \cdots - h_2) \) by assumption and

\[
\tilde{f}_2(x + h_1) - \tilde{f}_2(x) = \Delta_{h_d} \ldots \Delta_{h_2} \tilde{f}(x + h_1) - \Delta_{h_d} \ldots \Delta_{h_2} \tilde{f}(x) = \Delta_{h_d} \ldots \Delta_{h_1} \tilde{f}(x),
\]

it follows that \( \Delta_{h_d} \ldots \Delta_{h_1} \tilde{f}(x) \geq 0 \). Since \( \Delta_{h_k} \ldots \Delta_{h_1} \tilde{f}(x) \geq 0 \) can be shown along the same lines for any \( k = 1, \ldots, d - 1 \), the desired implication follows. \( \Box \)

**Proof of (i) ⇔ (iii).** First recall that if \( d = 1 \) and \( d = 2 \), respectively, \( d \)-monotone monotonicity of \( f \) reduces to \( f \) non-increasing and non-increasing and convex, respectively. It is therefore sufficient to restrict the discussion to the case \( d > 2 \).
That (i) implies (ii) can be established by the midpoint theorem. Note first that if $f$ is $d$-monotone on $(a, b)$, then, for $k = 1, \ldots, d - 2$, $\tilde{f}^{(k)}$ exists on $(-b, -a)$ and is non-negative there. Consequently, for any $k = 1, \ldots, d - 2$ and any $x \in (-b - a)$ and any $h > 0$ so that $x + kh \in (-b, -a)$ there exists $x^* \in (x, x + kh)$ so that $(\Delta_h)^k \tilde{f}(x) = \tilde{f}^{(k)}(x^*) \geq 0$.

The reverse implication can be established using the same argument as Theorem 7, Chapter IV of Widder (1946). To do so, pick an arbitrary $x \in (-b, -a)$. The assertion (iv) of Lemma 2 yields that, for any $k = 1, \ldots, d - 1$ and any $h > 0$ such that $x + kh \in (-b, -a)$, $(\Delta_h)^k \left( \tilde{f}(x) - \tilde{f}(x - \ell)/\ell \right) \geq 0$ whenever $\ell > 0$ is sufficiently small so that $x - \ell \in (-b, -a)$. By letting $\ell \to 0$, it then follows that the left-hand derivative $\tilde{f}'_-$ fulfills (9) for any $k = 1, \ldots, d - 1$. The assertion (ii) of Lemma 2 in particular gives that $\tilde{f}'_-$ is continuous on $(-b, -a)$. Yet another application of Lemma 2, this time of the statement (iii) on $\tilde{f}$, then guarantees the existence of $\tilde{f}'$ on $(-b, -a)$.

The same chain of arguments can be applied successively on $\tilde{f}^{(k)}$. In the last step, this yields that $\tilde{f}^{(d - 2)}$ exists on $(-b, -a)$ and fulfills (9) for $k = 1, 2$. In particular, $\tilde{f}^{(d - 2)}$ is continuous, non-decreasing and convex. Put together, the derivatives $\tilde{f}^{(k)}$, $k = 1, \ldots, d - 2$, exist on $(-b, -a)$ and are continuous, non-negative, non-decreasing and convex there. Because $(-1)^k \tilde{f}^{(k)}(x) = \tilde{f}^{(k)}(-x)$, the proof is complete. □

**Remark 6** After finishing this manuscript, Alfred Müller (personal communication) brought the notion of $d$-convexity as defined and studied by (Hopf, 1926), (Popoviciu, 1933) and Popoviciu (1944) to our attention. It is not our aim to discuss the various concepts of higher order convexity here; nonetheless, we would like to mention the paper (Boas and Widder, 1940) containing in particular the following results. First, a continuous function $f$ on $(a, b)$ which is weakly $d$-convex (meaning that $(\Delta_h)^d f(x) \geq 0$ for any $h > 0$ so that $x + dh \in (a, b)$) fulfills $\Delta_{h_1} \ldots \Delta_{h_d} f(x) \geq 0$ whenever $x + \sum_{i=1}^d h_i \in (a, b)$. Second, if a function $f$ is continuous and weakly $d$-convex on $(a, b)$, $f^{(d - 2)}$ exists and is convex on $(a, b)$. These results would offer an alternative proof to parts of Proposition 15. However, we favour our proof of the latter proposition which is self-contained and more accessible. Furthermore, Proposition 15 relates $d$-monotonicity, which a stronger concept than weak $d$-convexity, to non-negativity of higher order differences. This link is absolutely central to the study of quasi-monotonicity of survival functions as delineated in the proofs of Proposition 2 and Theorem 2 below.

The fact that Proposition 15 characterizes $d$-monotonicity in terms of $\tilde{f}$ rather than $f$ may seem somewhat less elegant. The reason for this is that $d$-monotonicity is related to the existence of survival rather than distribution functions. How exactly is described by Proposition 2 which we prove next.

**Proof of Proposition 2**. Assume first that there exists a random vector $X$ on $\mathbb{R}^d$ so that $\bar{H}(x) = P(X > x)$ for any $x \in \mathbb{R}^d_+$. It is clear that $\lim_{x \to -\infty} f(x) = 0$ and
Now, define \( f(x) \) by \( f(-x) \) for any \( x \in (-\infty, 0] \) as in Proposition 15 and observe that for any \( x \in (-\infty, 0) \), any \( k = 1, \ldots, d \) and any \( h > 0 \) so that \( x + kh < 0 \),

\[
(\Delta_h)^{k} f(x) = \sum_{\ell=0}^{k}(-1)^{k-\ell} \binom{k}{\ell} f(x + \ell h)
\]

\[
= \sum_{\ell=0}^{k}(-1)^{k-\ell} \sum_{J \subseteq \{1, \ldots, k\}, \#J = \ell} f\left(-\frac{x}{k} - \ell \left(\frac{x}{k} + h\right)\right)
\]

\[
= \sum_{\ell=0}^{k}(-1)^{k-\ell} \sum_{J \subseteq \{1, \ldots, k\}, \#J = \ell} P\left(\bigcap_{i \in J}\left\{-X_i < \frac{x}{k} + h\right\} \cap \bigcap_{i \notin J}\left\{-X_i < \frac{x}{k}\right\}\right)
\]

\[
= P\left(\bigcap_{i=1}^{k}\left\{X_i < \left(-\frac{x}{k} - h, -\frac{x}{k}\right)\right\}\right).
\]

The last expression is however clearly non-negative in which case Proposition 15 implies that \( f \) is \( d \)-monotone on \((0, \infty)\). Because \( f \) is right-continuous at 0, it is \( d \)-monotone even on \([0, \infty)\).

To establish the converse, note that \( f \) is in particular continuous on \([0, \infty)\) by Lemma 2. Therefore, \( \bar{H} \) is right-continuous. Because condition \((i)\) of Lemma 1 is clearly fulfilled as well, the only assertion which needs to be verified is the quasi-monotonicity of \( \bar{H}(-x) \). Since \( \bar{H} \) is right-continuous and places no mass outside of \([0, \infty)^d\) and because \( 1\{x < 0\} \) is quasi-monotone, this in turn amounts to the confirmation of \( \Delta_h f(-\|x\|_1) \geq 0 \) for any \( x \in (-\infty, 0)^d \) and \( h > 0 \) so that \( x + h < 0 \) However, this is immediately guaranteed by \((ii)\) of Proposition 15 upon setting \( x = \|x\|_1 \). Hence, the proof is complete. \( \Box \)

With Proposition 2, the proof of Theorem 2 is straightforward.

**Proof of Theorem 2.** As already discussed in Section 3, \( C \) always satisfies the boundary conditions \((i)\) and \((ii)\) of Definition 1. Therefore, what remains to be shown is that \( C \) is quasi-monotone if and only if \( \psi \) is \( d \)-monotone on \((0, \infty)\). However, because any Archimedean generator is continuous and fulfills \( \psi(0) = 1 \), \( \bar{F} \) given by \( \bar{F}(x) = \psi(x) \) for \( x \geq 0 \) and by \( \bar{F}(x) = 1 \) otherwise, is a continuous univariate survival function.

Suppose first \( \psi \) is \( d \)-monotone on \((0, \infty)\). Then, by Proposition 2, the function \( \bar{H} \) defined as

\[
\bar{H}(x) = \psi(\max(x, 0))
\]

for any \( x \in \mathbb{R}^d \) is a survival function in \( \mathbb{R}^d \) with continuous (identical) margins \( \bar{F} \). Theorem 1 then ensures that the function \( C \) associated with \( \bar{H} \) via

\[
C(u) = \bar{H}(\bar{F}^{-1}(u_1), \ldots, \bar{F}^{-1}(u_d))
\]

\(31\)
for any \( u \in [0, 1]^d \) is a copula. Furthermore, \( \bar{F}^{-1}(u_i) = \inf\{x : \psi(x) \leq u\} = \psi^{-1}(u_i) \) for \( u_i \in [0, 1] \) and \( \bar{F}^{-1}(1) = \inf\{x : F(x) \geq 0\} = -\infty \). However, \( \max(\bar{F}^{-1}(1), 0) = 0 = \psi^{-1}(1) \) and hence

\[
C(u) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)), \quad u \in [0, 1]^d.
\]

Conversely, assume that \( C \) is a copula. By virtue of Theorem 1 it therefore follows that \( \bar{H}(x) = C(\bar{F}(x_1), \ldots, \bar{F}(x_d)), x \in \mathbb{R}^d, \) is a joint survival function with (identical) marginal survival functions \( F \). Furthermore, it is easy to convince oneself that \( \bar{H}(x) = \psi(x_1 + \cdots + x_d) \) for any \( x \in \mathbb{R}^d_+ \). Therefore, \( \bar{H} \) satisfies the hypothesis of Proposition 2 and it follows that \( \psi \) is \( d \)-monotone on \([0, \infty)\).

**Appendix B: Proofs from Section 3**

**Proof of Proposition 5.** Throughout, \((\mathcal{S})\) indicates Stieltjes integration; any other integral is to be understood in the usual Lebesgue-Stieltjes sense. The original result by Williamson (1956) states that a function \( f \) is \( d \)-monotone on \((0, \infty)\) if and only if there exists a non-negative, non-decreasing and bounded below real function \( \gamma \) on \([0, \infty)\) such that

\[
f(t) = (\mathcal{S}) \int_0^\infty (1 - ut)^{d-1} d\gamma(u)
\]

for any \( t \in (0, \infty) \). Furthermore, \( \gamma \) is uniquely determined at its points of continuity and fulfills \( \gamma(0) = 0 \) as well as \( \gamma(0+) = f(\infty) \).

**Proof of (i).** Suppose \( f \) is a Williamson \( d \)-transform of a non-negative random variable \( X \) with distribution function \( F \). Then \( f(0+) = f(0) = 1 - F(0) \) and \( \lim_{x \to \infty} f(x) = 0 \) by dominated convergence. That \( f \) is \( d \)-monotone on \((0, \infty)\) is then clear from Williamson’s result.

To establish the reverse implication, observe first that in the situation of Proposition 5, Williamson’s result ensures the existence of a unique right-continuous, non-negative, non-decreasing and bounded below function \( G \) on \( \mathbb{R} \) so that

\[
f(t) = \int_0^\infty (1 - ut)^{d-1} dG(u)
\]

for any \( t \in (0, \infty) \) and \( G(0-) = G(0+) = 0 \) as well as \( \lim_{u \to \infty} G(u) = p \). The transformation formula for Lebesgue-Stieltjes integrals then yields

\[
f(t) = \int_0^\infty \left(1 - \frac{t}{u}\right)^{d-1} d\nu(u)
\]

where \( \nu \) is an image measure of the measure induced by \( G \) with respect to the mapping \( \phi \) given by \( \phi(u) = 1/u \) for \( u \in (0, \infty) \) and by 0 otherwise. A possible
distribution function of $\nu$ is
\[ \tilde{G}(u) = \begin{cases} 0 & \text{if } u \in (-\infty, 0), \\ p - G(1/u -) & \text{if } u \in [0, \infty). \end{cases} \]

The disturbing fact that $\nu$ is not necessarily a probability measure can be easily remedied upon setting
\[ F(u) = \begin{cases} 0 & \text{if } u \in (-\infty, 0), \\ 1 - G(1/u -) & \text{if } u \in [0, \infty). \end{cases} \]

It then still holds that $f(t) = \int_0^\infty \left(1 - \frac{t}{u}\right)^{d-1} dF(u)$ whenever $t > 0$ but in addition also that $f(t) = \mathcal{M}_d F(t)$ for any $t \in (0, \infty)$. Since both $f$ and $\mathcal{M}_d F$ are right-continuous at 0, the reversed implication is established.

**Proof of (ii).** The fact that the relationship between $f$ and $F$ is one-to-one is clear from the original result by Williamson in combination of (i). Theorem 3 of Williamson (1956), which is the inversion formula for Williamson $d$-transforms, further ensures that $F$ is of the form (5) at all points of continuity. Furthermore, the one-sided derivatives of $(-1)^{d-2} f^{(d-2)}$ exist everywhere and are non-decreasing by Lemma 2. A version of the mid-point theorem for continuous functions with one-sided derivatives (see e.g. Miller and Vyborny (1986)) easily yields that $f^{(d-1)}_+$ is right-continuous on $(0, \infty)$ with left-hand limit $f^{(d-1)}_-$. \qed

**Appendix C: Proofs from Section 4**

**Proof of Proposition 8.** Throughout, let $X \sim H$ and $U \sim C$ be random vectors and $\lambda_d$ be the Lebesgue measure on $\mathbb{R}^d$.

**Proof of (i).** It is sufficient to consider $d = 2$. In the case $d \geq 3$ the situation is simpler because then $\psi'$ exists everywhere in $(0, \infty)$ and one can argue by virtually the same arguments. Now, suppose $H$ has a density. Then obviously $P(X \in (0, \psi^{-1}(0))^d) = 1$. Furthermore, consider the transformation $T : (0, \psi^{-1}(0))^d \to \mathbb{R}^d$ given by
\[ T(x_1, \ldots, x_d) = (\psi(x_1), \ldots, \psi(x_d)), \quad \text{if } x \in (0, \psi^{-1}(0))^d. \]

Then, $T$ is injective and $U \overset{d}{=} T(X)$. Because $\psi'$ exists a.e. in $(0, \psi^{-1}(0))$, $T$ is a.e. regular (meaning that the set of points where $T$ is not differentiable has Lebesgue measure zero). Furthermore, if $\psi'(t)$ exists, then necessarily $\psi' > 0$.

To see this, assume for the moment the converse. Because $\psi$ is convex and decreasing on $(0, \psi^{-1}(0))$, $\psi'_+$ is non-positive and non-decreasing. If now $0 < t < \psi^{-1}(0)$ would be such that $\psi'(t) = 0$, there would exist an $\varepsilon > 0$ so that $\psi'(x) = 0$ for almost all
$x \in [t, \psi^{-1}(0) - \varepsilon]$. However, $\psi$ is absolutely continuous by assumption so $\psi'(x) = 0$ a.e. in $[t, \psi^{-1}(0) - \varepsilon]$ would imply that $\psi$ is constant there which is obviously not the case.

Put together, $D = \{u \in (0, 1)^d : \prod_{i=1}^d |\psi'((u_i))| = 0\}$ is an empty set. In particular, $\lambda_d(D) = 0$ meaning that $T$ is a.e. smooth. By means of Sard’s theorem and the Second transformation theorem (c.f. Hoffmann-Jørgensen (1994), Sections 8.9 and 8.10) it then follows that $C$ has a density.

The converse can be established similarly by considering a transformation $\tilde{T} : (0, 1)^d \to \mathbb{R}^d$ given by

$$\tilde{T}(u_1, \ldots, u_d) = (\psi^{-1}(u_1), \ldots, \psi^{-1}(u_d)), \quad x \in (0, 1)^d$$

Clearly, $P(U \in (0, 1)^d) = 1$ and $X \overset{d}{=} \tilde{T}(U)$ (for the latter claim, see the proof of Theorem 2). Again, one can easily convince oneself that $\tilde{T}$ is a.e. regular. Furthermore, $\psi^{-1}$ is (strictly) increasing, continuous and convex on $(0, 1)$ and therefore in particular absolutely continuous in any $[a, b] \subset (0, 1)$. By the same argument as above, it can then be established that $\psi^{-1}(t) > 0$ whenever it exists. Consequently, $\tilde{T}$ is a.e. smooth and $X$ has a density again by Sard’s theorem and the Second transformation theorem.

**Proof of (ii).** Suppose now $\psi \in \Psi_{d+1}$ and let $\tilde{C}$ be the $(d + 1)$-dimensional Archimedean copula with generator $\psi$. Furthermore, fix a random vector $\tilde{X}$ following the $\ell_1$-norm symmetric distribution associated with $\tilde{C}$ in the sense of Theorem 3. Section 5.2.3 in Fang et al. (1990) then yields that all lower dimensional margins of $\tilde{X}$ have densities. In particular, this applies to $Y = (\tilde{X}_1, \ldots, \tilde{X}_d)$. However, $Y \overset{d}{=} X$ and $C$ has a density by means of (i). \(\Box\)

**Proof of Proposition 9.** Because of (i) of Proposition 8 and (iii) of Lemma 2, we only need to verify that the radial part of the $\ell_1$-norm symmetric distribution associated with $C$ has a density if and only if $\psi^{\ell_1(d-1)}$ is absolutely continuous on $(0, \infty)$. Before we do so, observe that $F_R$ may be, for $x \in [0, \infty)$, rewritten as

$$F_R(x) = F_Q(x) - \frac{(-1)^{d-1}x^{d-1}f^{\ell_1(d-1)}(x)}{(d-1)!}$$

where $F_Q$ refers to the distribution function of the radial part of the $\ell_1$-norm distribution associated with the $(d - 1)$-dimensional margin of $C$, i.e. with $\psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_{d-1}))$. Furthermore, Proposition 8 guarantees that $F_Q$ is absolutely continuous.

Assume first that $F_R$ is absolutely continuous and consider $[a, b] \subset (0, \infty)$. Because $f^{\ell_1(d-1)}(x) = (-1)^{d-1}(d-1)!\frac{F_Q(x) - F_R(x)}{x}$ on $[a, b]$, $f^{\ell_1(d-1)}$ is absolutely continuous there.

To establish the reverse implication, observe that because $f^{\ell_1(d-1)}$ is continuous on $(0, \infty)$, $F_Q$ is continuous on $\mathbb{R}^d$, and $F_R$ is continuous in 0, $F_R$ is continuous in $\mathbb{R}^d$. Now, fix $[a, b] \subset (0, \infty)$. By a similar argument as above it than clearly holds that $F_R$
is absolutely continuous in \([a, b]\). Because \(F_R\) is continuous and of bounded variation, it is absolutely continuous even in \([0, b]\). Since \(F_R\) is clearly absolutely continuous on \((-\infty, 0]\), the absolute continuity of \(F_R\) on the entire \(\mathbb{R}^d\) is immediate.

The last claims follows from the fact that if \(H\) is an absolutely continuous distribution function then its density \(h\) satisfies

\[
h(x) = \frac{\partial^d}{\partial x_1 \ldots \partial x_d} H(x)
\]

for almost all \(x \in \mathbb{R}^d\). However clear this statement may seem, it is far from obvious, see Saks (1937), p. 115 or Easton et al. (1967). □

**Proof of Proposition 10.** First recall that if \(U \sim C\) then the random vector \(X \overset{d}{=} (\psi^{-1}(U_1), \ldots, \psi^{-1}(U_d))\) follows a \(\ell_1\)-norm symmetric distribution associated with \(C\). Denote the radial part of \(X\) by \(R\) and its distribution function by \(F_R\). Then, for any \(s \in (0, 1]\),

\[
P^C(L(s)) = P\left(\sum_{i=1}^d \psi^{-1}(U_i) = \psi^{-1}(s)\right) = P\left(\sum_{i=1}^d X_i = \psi^{-1}(s)\right) = P(R = \psi^{-1}(s))
\]

where the last equality follows from \((iii)\) of Proposition 6. Inversion formula (5) and the fact that \(\psi^{(k)}\) is continuous for any \(k = 1, \ldots, d - 2\) in turn imply that

\[
P(R = \psi^{-1}(s)) = F_R(\psi^{-1}(s)) - F_R(\psi^{-1}(s)) - \frac{(-1)^{d-1}(\psi^{-1}(s))^{d-1}}{(d-1)!} \left(\psi^{(d-1)}(\psi^{-1}(s)) - \psi^{(d-1)}(\psi^{-1}(s))\right).
\]

Regarding \(L(0)\), one similarly argues that \(P^C(L(0)) = P(R \geq \psi^{-1}(0))\). From this it is first immediate that \(P^C(L(0)) = 0\) whenever \(\psi^{-1}(0) = \infty\). Suppose now \(\psi^{-1}(0) < \infty\). Because all derivatives \(\psi^{(k)}(x), k = 1, \ldots, d - 2\), as well as \(\psi^{(d-1)}(x)\) vanish for \(x \in [\psi^{-1}(0), \infty)\), it ensues from (5) that \(F_R(x) = 1\) for any \(x \in [\psi^{-1}(0), \infty)\). Therefore,

\[
P^C(L(0)) = P\left(\sum_{i=1}^d \psi^{-1}(U_i) = \psi^{-1}(0)\right).
\]

Along the same lines as in the case of \(L(s), s \in (0, 1]\), one reasons that

\[
P^C(L(0)) = P(R = \psi^{-1}(0)) = \frac{(-1)^{d-1}(\psi^{-1}(0))^{d-1}}{(d-1)!} \psi^{(d-1)}(\psi^{-1}(0))
\]

which concludes the proof. □

**Proof of Proposition 13.** First, recall that \(C^L_d\) is Archimedean as detailed in Example 3. Furthermore, \(\psi^L_2(x) = (1 - x)_+\) and \(C^L_2\) coincides with the Fréchet-Hoeffding lower bound. Therefore, assume that \(d \geq 3\) and let \(\psi\) be the generator
Assume first \( d \geq 4 \). In that case, \( f \) is twice differentiable on \((0, \psi^{-1}(0))\) with

\[
f^{(2)}(x) = \frac{1}{d-1} \psi^{(2)}(x)\psi(x)^{-\frac{d-2}{d-1}} + \frac{d-2}{(d-1)^2} (\psi'(x))^2 \psi(x)^{-\frac{d-2}{d-1}} - \frac{1}{d-1} \psi'(x) - \frac{1}{d-1(d-2)} \psi^{(2)}(x)
\]

To establish that \( f^{(2)}(x) \leq 0 \) for \( x \in (0, \psi^{-1}(0)) \), we consider the matrix

\[
A = \begin{pmatrix}
\psi(x) & -\frac{1}{d-1} \psi'(x) \\
-\frac{1}{d-1} \psi'(x) & -\frac{1}{d-1} \psi'(x) - \frac{1}{d-1(d-2)} \psi^{(2)}(x)
\end{pmatrix}
\]

and show that it is positive semi-definite for all \( x \in (0, \psi^{-1}(0)) \). Since \( \psi \) is \( d \)-monotone, dominated convergence theorem ensures that, for \( x \in (0, \psi^{-1}(0)) \),

\[
\psi^{(k)}(x) = (d-1)\ldots(d-k) \int_0^\infty (-1)^k \frac{1}{t^k} \left(1 - \frac{x}{t}\right)^{d-1-k} dF_R(t), \quad k = 1, \ldots, d-2
\]

where \( F_R \) is the distribution function of a non-negative random variable whose Williamson \( d \)-transform is \( \psi \). Now, consider \( a_i \in \mathbb{R}, i = 1, 2 \). Then

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} a_i a_j A_{ij} = (a_1)^2 \int_0^\infty \left(1 - \frac{x}{t}\right)^{d-1} dF_R(t) + 2a_1a_2 \int_0^\infty \frac{1}{t} \left(1 - \frac{x}{t}\right)^{d-2} dF_R(t) + (a_2)^2 \int_0^\infty \frac{1}{t^2} \left(1 - \frac{x}{t}\right)^{d-3} dF_R(t)
\]

\[
= \int_x^\infty \left(1 - \frac{x}{t}\right)^{d-3} \left(a_1 \left(1 - \frac{x}{t}\right) + a_2 \right)^2 dF_R(t) \geq 0.
\]

Consequently, \(|A| \geq 0\) which in turn implies \( f^{(2)}(x) \leq 0 \) for \( x \in (0, \psi^{-1}(0)) \).

For \( d = 3 \) we can use the fact that \( \psi^{(2)}_+ \) exists everywhere in \((0, \psi^{-1}(0))\). Consequently, \( f^{(2)}_+(x) \) exists for any \( x \in (0, \psi^{-1}(0)) \) and

\[
f^{(2)}_+(x) = \frac{(d-2)(\psi'(x))^2 - (d-1)\psi^{(2)}_+(x)\psi(x)}{(d-1)^2 \psi(x)^{\frac{d-2}{d-1}} + 1}.
\]

Concavity of \( f \) then follows from \( f^{(2)}_+ \leq 0 \) on \((0, \psi^{-1}(0))\); see Theorem 1 of Miller and Vyborny (1986). Now consider \( h > 0 \) and \( x \in (0, \psi^{-1}(0)) \). Then
\[
\frac{\psi'(x + h) - \psi'(x)}{h} = -2 \left[ \int_{x+h}^{\infty} \frac{1}{t} \left( 1 - \frac{x+h}{t} \right) dF_R(t) - \int_x^{\infty} \frac{1}{t} \left( 1 - \frac{x}{t} \right) dF_R(t) \right]
= 2 \left[ \int_{(x+h, \infty)} t^2 dF_R(t) + \int_{[x, x+h]} \frac{1}{th} \left( 1 - \frac{x}{t} \right) dF_R(t) \right].
\]

Because \( \frac{1}{th} \left( 1 - \frac{x}{t} \right) \leq \frac{1}{x(x+h)} \) for \( t \in (x, x+h] \), dominated convergence theorem yields

\[
\psi'(x + h) - \psi'(x) = 2 \int_{(0, \infty)} t^2 dF_R(t).
\]

The inequality \( f'_{(x)}(x) \leq 0 \) for \( x \in (0, \psi^{-1}(0)) \) can now be established by exactly the same arguments as in the case \( d \geq 4 \). \( \square \)

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