ON THE DISCONNECTION OF A DISCRETE CYLINDER
BY A RANDOM WALK

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Preliminary Draft

Abstract

We investigate the large \( N \) behavior of the time the simple random walk on the
discrete cylinder \((\mathbb{Z}/N\mathbb{Z})^d \times \mathbb{Z}\) needs to disconnect the discrete cylinder. We show
that when \( d \geq 3 \), this time is roughly of order \( N^{2d} \) and comparable to the cover
time of the slice \((\mathbb{Z}/N\mathbb{Z})^d \times \{0\}\), but substantially larger than the cover time of the base by the projection of the walk. Further we show that by the time disconnection
occurs, a massive “clogging” typically takes place in the truncated cylinders of height
\( N^{d-\epsilon} \). These mechanisms are in contrast with what happens when \( d = 1 \).

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0 Introduction

Consider simple random walk on an infinite discrete cylinder having a base modelled on a \( d \)-dimensional discrete torus of size \( N \). In this note we investigate the following question of H.J. Hilhorst: what is the asymptotic behavior for large \( N \) of the time needed by the walk to disconnect the cylinder? When \( d = 1 \), it is straightforward to argue that this time is roughly of order \( N^2 \) and comparable to the time for the projection of the process to cover the base. We show here that things behave differently when \( d \geq 3 \), and that in a suitable sense a massive clogging occurs inside the cylinder by the time the disconnection happens.

Before discussing our results any further, we describe the model more precisely. With \( N \geq 1 \), integer, we consider the state space

\[
E = (\mathbb{Z}/N\mathbb{Z})^d \times \mathbb{Z},
\]

that we tacitly endow with its natural graph structure. We say that a finite subset \( S \subseteq E \) disconnects \( E \), if for large \( M \), \((\mathbb{Z}/N\mathbb{Z})^d \times [M, \infty) \) and \((\mathbb{Z}/N\mathbb{Z})^d \times (-\infty, -M] \) are contained in two distinct connected components of \( E \setminus S \).

We denote with \( P_x \), \( x \in E \), the canonical law on \( E^N \) of the simple random walk on \( E \) starting at \( x \), and with \( (X_n)_{n \geq 0} \) the canonical process. We are principally interested in the disconnection time of \( E \):

\[
T_N = \inf\{n \geq 0; X_{[0,n]} \text{ disconnects } E\}.
\]

Under \( P_x \), \( x \in E \), the Markov chain \( X \), is irreducible, recurrent, and it is plain that

\[
T_N < \infty \quad \text{\( P_x \)-a.s., for all } x \in E.
\]

As a comparison consider \( \tilde{C}_N \) the cover time of \((\mathbb{Z}/N\mathbb{Z})^d \) by the projection of \( X \), on the base, i.e. the first time the projection of \( X \) has visited all points of \((\mathbb{Z}/N\mathbb{Z})^d \), as well as \( C_N \) the cover time of \((\mathbb{Z}/N\mathbb{Z})^d \times \{0\} \) by \( X \). It is also plain that:

\[
\tilde{C}_N \leq T_N \leq C_N.
\]

Cover times of finite graphs have been extensively investigated, cf. for instance [1], [2], [4], [5], [6], and the references therein, and one knows that for any \( d \geq 1 \),

\[
\lim_{N \to \infty} \frac{\log \tilde{C}_N}{\log N} \quad \overset{\text{d}}{\longrightarrow} \quad 2
\]

in \( P_0 \)-probability,

(much more is known, see the above references). Our fist main result state that:

**Theorem 1** \((d \geq 3)\)

\[
\lim_{N \to \infty} \frac{\log T_N}{\log N} = \lim_{N \to \infty} \frac{\log C_N}{\log N} = 2d.
\]

In fact, cf. Remark 2.5, (0.6) also holds when \( d = 1 \), as for the case \( d = 2 \), we only present here rather weak bounds, cf. (2.39). As a consequence of (0.5), (0.6), we thus see
that unlike what happens when \( d = 1 \), there is a substantial discrepancy between \( \tilde{C}_N \) and \( T_N \), when \( d \geq 3 \).

Our second main result shows a massive “clogging” in the cylinder by the time disconnection occurs, when \( d \geq 3 \). Let us denote with \( d(x, A) \), for \( x \in E, A \subseteq E \), the minimal length of a nearest neighbor path from \( x \) to \( A \). We have:

**Theorem 2** \((d \geq 3)\)

\[
\text{For all } \epsilon, \eta \in (0, 1), \quad \max_{x \in [\mathbb{Z}/N\mathbb{Z}]^d \times [-N^{d-\epsilon}, N^{d-\epsilon}]} d(x, X_{[0,T_N]}) / N^\eta \xrightarrow{N \to \infty} 0, \tag{0.7}
\]

in \( P_0 \)-probability.

So Theorem 2, (see also Theorem 3.1), shows that by time \( T_N \) the walk pretty much fills up the truncated cylinder \( ([\mathbb{Z}/N\mathbb{Z}]^d \times [-N^{d-\epsilon}, N^{d-\epsilon}] \) when \( N \) is large. Once again this can be contrasted with the \( d = 1 \) situation, where with non-vanishing probability points in \( ([\mathbb{Z}/N\mathbb{Z}]^d \times [-N^{1-\epsilon}, N^{1-\epsilon}] \) at distance of order \( N \) from \( X_{[0,T_N]} \) do occur, cf. Remark 3.2.

We now give some indications on the proofs of Theorem 1 and 2. The proof of Theorem 1 consists of an upper bound, cf. Theorem 1.1, and a lower bound, cf. Theorem 2.1. The upper bound is simpler to prove. It is a direct consequence of the fact that \( T_N \) is smaller than \( C_N \), cf. (0.4), and the estimates we derive on this cover time. It is instructive that this rather primitive strategy captures the correct rough order of magnitude of \( T_N \). The lower bound is more delicate. The rough idea of the proof is that for \( \gamma \in (0, 1) \), and large \( N \), one must find a box of size \( N^\gamma \) in \( E \) containing about \( O(N^{d\epsilon}) \) points of the trajectory \( X_{[0,T_N]} \), since \( X_{[0,T_N]} \) disconnects \( E \), cf. Lemma 2.4. We use here isoperimetric controls of Deuschel-Pisztora [7]. Now if \( \gamma \) is chosen small enough, with high probability \( X \), puts at most about \( (\log N) N^{2\gamma} \) points in any box of size \( N^\gamma \) by times “slightly smaller” than \( N^{2d} \), cf. (2.29). For \( d \geq 3 \), these are much fewer points than the required \( O(N^{d\epsilon}) \) points to produce disconnection. This yields a lower bound on \( T_N \).

As for the proof of the “clogging effect”, cf. Theorem 2 or Theorem 3.1, the main idea is to rely on the lower bound on \( T_N \) of Theorem 2.1 and show that before time \( T_N \) in a uniform fashion for \( x \in ([\mathbb{Z}/N\mathbb{Z}]^d \times [-N^{d-\epsilon}, N^{d-\epsilon}] \), the walk comes “often enough” within distance \( N \) of \( x \), giving each time an opportunity to come closer to \( x \). We use for this matter one of the Ray-Knight theorems, see (3.21).

Let us now explain how this article is organized.

In Section 1, we provide further notations and definitions. The main objective is Theorem 1.1, that provides an upper bound on \( C_N \) and hence also on \( T_N \).

In Section 2, we prove a lower bound on \( T_N \) in Theorem 2.1. We derive controls on excursions of the process in Proposition 2.2 and use a geometric lemma, cf. Lemma 2.4.

In Section 3 we show in Theorem 3.1 that clogging takes place by time \( T_N \), when \( d \geq 3 \).

Let us finally explain the convention we use concerning constants. We denote with \( c \) or \( c' \) positive constants depending on \( d \), with value changing from place to place. The numbered constants \( c_0, c_1, \ldots \) will be fixed and refer to the value at their first appearance in the text. Dependence of constants on additional parameters will appear in the notion; for instance \( c(\delta) \) will denote a positive constant depending on \( d \) and \( \delta \).
1 The upper bound

The main objective of this section is to begin the proof of Theorem 1 of the introduction and more specifically to provide in Theorem 1.1 an upper bound on the disconnection time $T_N$. We begin with some additional notations.

We denote with $\pi_N$ the canonical projection from $\mathbb{Z}^{d+1}$ on $E$, cf. (0.1). For $x \in \mathbb{Z}^{d+1}$, resp. $x \in E$, we let $x^{d+1}$ stand for the last component, resp. the projection on $\mathbb{Z}$, of $x$. We denote with $| \cdot |$ and $| \cdot |_{\infty}$ the Euclidean and $\ell_\infty$-distances on $\mathbb{Z}^{d+1}$, or the corresponding induced distances on $E$. We write $B(x, r)$ or $B_\infty(x, r)$ for the corresponding open balls with radius $r > 0$ and center $x \in \mathbb{Z}^{d+1}$, or $x \in E$. For $A$ and $B$ subsets of $E$ or of $\mathbb{Z}^{d+1}$ we denote with $A + B$ the set of points of the form $x + y$, with $x$ in $A$ and $y$ in $B$. For a subset $U$ of $\mathbb{Z}^{d+1}$ or $E$, we denote with $|U|$ the cardinality of $U$ and with $\partial U$ the boundary of $U$:

\[ \partial U = \{ x \in U^c; \exists y \in U, \, |x - y| = 1 \}. \]

We let $(\theta_n)_{n \geq 0}$, and $(\mathcal{F}_n)_{n \geq 0}$, respectively stand for the canonical shift and filtration for the process $(X_n)_{n \geq 0}$ on $E^\mathbb{N}$. For $U \subseteq E$, $H_U$, $T_U$ are the entrance time and exit time in or from $U$:

\[ H_U = \inf \{ n \geq 0, X_n \in U \}, \quad T_U = \inf \{ n \geq 0, X_n \notin U \}. \]

For simplicity we write $H_{x}$ in place of $H_{\{x\}}$. We denote with $Q_x, x \in \mathbb{Z}^{d+1}$, the canonical law on $(\mathbb{Z}^{d+1})^\mathbb{N}$ of the simple random walk on $\mathbb{Z}^{d+1}$. We will use, when this causes no confusion, the same notations as above for the canonical process, the canonical shift, the entrance or exit times for the simple random walk on $\mathbb{Z}^{d+1}$.

We now turn to the main objective of this section: the derivation of an upper bound on the disconnection time $T_N$. As explained in the Introduction, we simply use the fact that $T_N$ is smaller than the cover time by $X$ of $(\mathbb{Z}/N\mathbb{Z})^d \times \{0\}$, and estimate from above this cover time.

**Theorem 1.1.** $(d \geq 2)$

\[ \forall \delta > 0, \quad \lim_{N \to \infty} P_0 \left[ \frac{\log T_N}{\log N} \leq \frac{\log C_N}{\log N} \leq 2d + \delta \right] = 1, \]

(as a matter of fact (1.3) also holds for $d = 1$, cf. Remark 1.4 below).

**Proof.** We introduce two subsets of $E$, namely the truncated cylinders

\[ B = (\mathbb{Z}/N\mathbb{Z})^d \times [-N,N], \quad \tilde{B} = (\mathbb{Z}/N\mathbb{Z})^d \times [-2N + 1,2N - 1]. \]

We then consider the sequence of successive returns to $B$ and departures from $\tilde{B}$ of the walk:

\[ R_1 = H_B, \quad D_1 = T_{-B} \circ \theta_{R_1} + R_1, \quad \text{and for} \quad k \geq 1, \]

\[ R_{k+1} = R_1 \circ \theta_{D_k} + D_k, \quad D_{k+1} = D_1 \circ \theta_{D_k} + D_k, \]

so that

\[ 0 \leq R_1 \leq D_1 \leq \cdots \leq R_k \leq D_k \leq \cdots \leq \infty, \]

and these inequalities except maybe for the first one are strict, $P_x$-a.s., for any $x \in E$.

The proof of Theorem 1.1 will use the next

3
Lemma 1.2.

For any \( N \geq 1, y \in B, \) and \( x \in (\mathbb{Z}/N\mathbb{Z})^d \times \{0\} \),
\begin{equation}
P_y[H_x < T_B] \geq cN^{-(d-1)}.
\end{equation}

Proof. We define the subset of \( \mathbb{Z}^{d+1} \):
\begin{equation}
U = (-2N, 2N)^{d+1} \cap \mathbb{Z}^{d+1}.
\end{equation}
The probability in (1.6) is bigger than
\begin{equation}
Q_v[H_u < T_U],
\end{equation}
where \( v \in \{0, \ldots, N-1\}^d \times \{-N, \ldots, N\} \) and \( u \in \{0, \ldots, N-1\}^d \times \{0\} \) satisfy \( \pi_N(v) = y, \) \( \pi_N(u) = x \).

For \( D \) a subset of \( \mathbb{Z}^{d+1} \), we denote with \( g_D(\cdot, \cdot) \) the Green function of the simple random walk killed when exiting \( D \):
\begin{equation}
g_D(w, w') = E^Q_w \left[ \sum_{n=0}^{T_{D-1}} 1\{X_n = w'\} \right], \text{ for } w, w' \in \mathbb{Z}^{d+1},
\end{equation}
and for simplicity write \( g(\cdot, \cdot) \) for \( g_{\mathbb{Z}^{d+1}}(\cdot, \cdot) \). It follows from the strong Markov property at the stopping time \( H_u \wedge T_U \), that:
\begin{equation}
Q_v(H_u < T_U) = \frac{g_U(v, u)}{g_U(u, u)}.
\end{equation}
Now by standard estimates
\begin{equation}
c g(w, w') \leq g_U(w, w') \leq g(w, w'), \text{ for all } w, w' \in \{-N, \ldots, N\}^{d+1} \subseteq U,
\end{equation}
(the second inequality is immediate, for the first inequality we refer to (1.82) and (1.83) in Antal [3], when \( |w - w'| \leq cN \), with \( c \) small, the general case follows for instance with the invariance principle), and (1.6) then follows from classical bounds on \( g(\cdot, \cdot) \), cf. Lawler [12], p. 31.

We now return to the proof of Theorem 1.1. Consider \( x \in (\mathbb{Z}/N\mathbb{Z})^d \times \{0\} \). From the strong Markov property, we see that for \( k \geq 0 \), with the convention \( D_0 = 0 \),
\begin{equation}
P_0[H_x > D_{k+1}] = E_0[H_x > R_{k+1}, P_{X_{R_{k+1}}}[H_x > T_B]]
\end{equation}
\begin{equation}
\leq P_0[H_x > R_{k+1}](1 - cN^{-(d-1)}) \leq P_0[H_x > D_k](1 - cN^{-(d-1)})
\end{equation}
\begin{equation}
\text{induction}
\leq (1 - cN^{-(d-1)})^{k+1}.
\end{equation}
We thus have the following estimate on the cover time of \( (\mathbb{Z}/N\mathbb{Z})^d \times \{0\} \):
\begin{equation}
P_0 \left[ \max_{x \in (\mathbb{Z}/N\mathbb{Z})^d \times \{0\}} H_x > D_k \right] \leq N^d(1 - cN^{-(d-1)})^k, \text{ for } k \geq 1.
\end{equation}
On the complement of the event inside the above probability, we have:
\begin{equation}
T_N \leq C_N = \max_{x \in (\mathbb{Z}/N\mathbb{Z})^d \times \{0\}} H_x \leq D_k,
\end{equation}

and hence for $\epsilon > 0$ and large $N$,
\[
P_0[T_N \leq C_N \leq D_{N^d}] \geq 1 - N^d e^{-cN^\epsilon} \xrightarrow{N \to \infty} 1.
\]

We now control the tail of $D_k$ for large $k$. To this end we note that:
\[
\begin{equation}
\text{under } P_0, X^{d+1}_t \text{ has same distribution as } Y, \text{ the random walk on } \mathbb{Z}, \text{ starting in 0, with jump distribution } \frac{1}{2(d+1)} (\delta_{-1} + \delta_1) + \frac{d}{d+1} \delta_0.
\end{equation}
\]
Note that the above random walk $Y$, is obtained by delaying a simple random walk on $\mathbb{Z}$ with a geometric clock with parameter $\frac{1}{d+1}$ at each site of $\mathbb{Z}$. Coming back to $P_0$, the strong Markov property yields that
\[
\begin{equation}
\text{under } P_0, D_1, D_2 - D_1, \ldots, D_{k+1} - D_k, \ldots \text{ are independent variables and for } k \geq 1, D_{k+1} - D_k \text{ have the same distribution as the sum of two independent variables respectively distributed like the entrance time of } Y \text{ in } N \text{ and } T^*_B \circ \theta_{R_2} \text{ under } P_0.
\end{equation}
\]
As an immediate consequence we see that for $k \geq 1$,
\[
\begin{equation}
\text{under } P_0, D_{k+1} - D_1 \text{ has the same law as the sum of two independent variables } U_k \text{ and } V_k \text{ respectively distributed as the entrance time of } Y \text{ in } kN \text{ and the sum of } k \text{ independent variables } T^*_B \circ \theta_{R_2} \text{ under } P_0.
\end{equation}
\]
We then note that:
\[
\begin{equation}
\sup_{z \in B} E_x \left[ \exp \left\{ \frac{c}{N^2} T^*_B \right\} \right] \leq c', \text{ for } N \geq 1.
\end{equation}
\]

**Proof.** This is a consequence of Kha\v{m}inskii’s lemma, cf. [10], and the estimate
\[
\begin{equation}
\sup_{z \in B} E_x[T^*_B] \leq cN^2,
\end{equation}
\]
see for instance Lemma 1.1 of [14], p. 292. \hfill \Box

With standard Cramer-type estimates, it now follows from (1.18), (1.19), that for some positive constant $c$ and any $\epsilon > 0$:
\[
\begin{equation}
P_0[V_{N^d} > c N^d (d+\epsilon)] \to 0.
\end{equation}
\]
It also follows from the remark below (1.16) and Example 6.6 in Chapter 7 of Durrett [8], p. 369, that with hopefully obvious notations:
\[
\begin{equation}
P^Y \left[ H_{N^d} > N^{2d+3\epsilon} \right] \to 0.
\end{equation}
\]
We thus find that for $\epsilon > 0$ and large $N$:
\[
\begin{equation}
P_0[T_N \leq C_N \leq N^d + \epsilon] \geq P_0[D_{N^d} \leq N^{2d+4\epsilon}] - N^d e^{-cN^\epsilon} \xrightarrow{N \to \infty} 1,
\end{equation}
\]
using (1.21), (1.22) in the last step, together with (1.19) and the fact that $D_1 = T^*_B$, $P_0$-a.s. Since $\epsilon$ is an arbitrary positive number, the claim (1.3) now follows. \hfill \Box
Remark 1.4.

1) When $d = 1$, Theorem 1.1 remains true. One only needs to replace $N^{-(d-1)}$ with $(\log N)^{-1}$ in (1.6) of Lemma 1.2, see for instance Proposition 1.6.7 of [12]. Inserting this new lower bound in (1.13), (1.14), the proof of Theorem 1.1 goes otherwise unchanged.

2) We refer to Dembo-Peres-Rosen-Zeitouni [5], and also to Lawler [11], where the asymptotic analysis of the cover time of a ball of radius $N$ by the two-dimensional simple random walk is analyzed. This problem has some common flavor with the investigation of the large $N$ behavior of $C_N$, which in this note comes as a subsidiary issue to the asymptotic analysis of $T_N$. \hfill \Box

2 The lower bound

The main object of this section is to derive a lower bound on $T_N$, cf. Theorem 2.1, and thus complete the proof of Theorem 1 of the Introduction. As a consequence of the results of this section, we will see that when $d \geq 3$, the cover time of $(\mathbb{Z}/N\mathbb{Z})^d \times \{0\}$ that was used to bound $T_N$ from above in the last section, is in principal order comparable to $T_N$.

Theorem 2.1. ($d \geq 3$)

\begin{equation}
\forall \delta > 0, \lim_{N \to \infty} P_0 \left[ \frac{\log T_N}{\log N} \geq 2d - \delta \right] = 1,
\end{equation}

(as a matter of fact (2.1) holds also for $d = 1$, cf. Remark 2.5 below).

Proof. We denote with $P^N$ the law of the walk with initial distribution $\nu_N$ the uniform measure on $B$, see (1.4). Thanks to translation invariance,

\begin{equation}
T_N \text{ has same distribution under } P_0 \text{ and } P^N.
\end{equation}

The claim (2.1) will hence follow if we replace $P_0$ with $P^N$. We introduce the positive numbers

\begin{equation}
\delta, \gamma \in (0, 1) \text{ and } \delta' = \frac{\delta}{3},
\end{equation}

and for $N \geq 1, x \in E$, the subsets of $E$, see (1.4) and the beginning of Section 1 for the notations,

\begin{equation}
B(x) = x + B, \quad \tilde{B}(x) = x + \tilde{B},
\end{equation}

\begin{equation}
C(x) = B_{\infty}(x, [N^\gamma]), \quad \tilde{C}(x) = B_{\infty}(x, 2N^\gamma).
\end{equation}

Let us briefly explain the strategy of the proof. We are first going to show that when $\gamma$ is small, cf. (2.25), with probability tending to 1 as $N$ goes to infinity, the time spent by the trajectory $X_{[0,N^{2d-\delta}]}$ in any $C(x)$ is at most $O(N^{2\gamma} \log N)$, cf. (2.29).

Then we will see, cf. Lemma 2.4, that when $d \geq 3$, for large $N$, any set disconnecting $E$ has at least $O(N^{2\gamma})$ points in some $C(x)$. This and (2.29) will show that with probability tending to 1 as $N$ goes to infinity $T_N$ is bigger or equal to $N^{2d-\delta}$.
Our first goal is to prove (2.29). We denote with $R_k^x \leq D_k^x$, $k \geq 1$, the successive times of return to $B(x)$ and departure from $\tilde{B}(x)$, defined analogously as in (1.5) with $B$ and $\tilde{B}$ replaced by $B(x)$ and $\tilde{B}(x)$. In what follows, when this causes no confusion, we will simply drop the superscript $x$ and write $R_k, D_k$ in place of $R_k^x, D_k^x$, for simplicity.

We want to investigate the number of returns to $C(x)$ and departures from $\tilde{C}(x)$ performed by $X$, during each time interval $[R_k, D_k - 1]$, keeping in mind that for large $N$,

$$X, \text{ lies in } B(x)^c \subset \tilde{C}(x)^c \text{ during each time interval } [0, R_1 - 1], \text{ and } [D_k, R_{k+1} - 1], k \geq 1.$$ (2.6)

We thus define the sequence of stopping times:

$$R_1' = D_1 \wedge (H_C(x) \circ \theta_{R_1} + R_1), \quad D_1' = D_1 \wedge (T_{\tilde{C}(x)} \circ \theta_{R_1'} + R_1),$$

and for $m \geq 1$,

$$R_{m+1}' = D_1 \wedge (H_C(x) \circ \theta_{D_m} + D_m'),\quad D_{m+1}' = D_1 \wedge (T_{\tilde{C}(x)} \circ \theta_{R_{m+1}'} + R_{m+1}').$$

The number of returns to $C(x)$ and departures from $\tilde{C}(x)$ during $[R_1, D_1 - 1]$ is then:

$$N_1^x = \sum_{m \geq 1} 1\{D_m' < D_1\}$$ (2.8)

and the corresponding number during $[R_k, D_k - 1]$, $k \geq 2$, is

$$N_k^x = N_1^x \circ \theta_{R_k^x},$$

where the above equality holds matter-of-factly for $k = 1$ as well. We will use the following, (see (2.3) for the notation):

**Proposition 2.2.** $(d \geq 2, \delta \in (0, 1), 0 < \gamma \leq \frac{\theta}{(d-1)}$)

There is a constant $c_0 \geq 1$, such that

$$\lim_{N \to \infty} P_N^x \left[ \sup_{x \in E} \sum_{k \geq 1} N_k^x \{ R_k^x \leq N^{2d-\delta} \} \geq c_0(\log N) \right] = 0.$$ (2.10)

**Proof.** Note that for $x \in E$, $k \geq 1$, $D_k^x - D_1^x$ has the same distribution under $P_N^x$ as $D_{k+1}^x - D_1^x$ in (1.18). Hence for large $N$, for any $x \in E$, using the strong Markov property for the random walk $Y$ of (1.16) at the entrance times in $k$, we find

$$P_N^x \left[ R_k^x \leq N^{2d-\delta} \right] \leq P_Y^x \left[ H_N \leq N^{(d-1)(1-\gamma)} \right] \leq N^{2d-\delta}$$

$$\leq P_Y^x \left[ H_1 \leq N^{2d-\delta} \right] P_Y^x \left[ N \leq N^{(d-1)(1-\gamma)} \right] \leq (1 - cN^{-(d-\delta/2)})^N N^{1+(d-1)(1-\gamma)}$$

$$\leq \exp \left\{ -cN^{1+(d-1)(1-\gamma)-d+\delta/2} \right\} \leq \exp \left\{ -cN^{d/2} \right\},$$

thanks to our assumption on $\gamma$, as well as (3.4) of Chapter 3 of Durrett [8], and the remark below (1.16).

Then observe that $P_N^x$-a.s., up to time $N^{2d-\delta}$, the $\mathbb{Z}$-component of $X$, remains bounded in absolute value by $N + N^{2d-\delta}$. Hence for large $N$, the sum inside the probability in (2.10) vanishes for any $x \in E$ with $|x^{d+1}| \geq N^{2d}$. From this remark and (2.11), we see that the claim (2.10) follows from:

$$\lim_{N \to \infty} P_N^x \left[ \sup_{x:|x^{d+1}| \leq N^{2d}} \sum_{k \leq N^{(d-1)(1-\gamma)}} N_k^x \geq c_0(\log N) \right] = 0.$$ (2.12)

The proof of (2.12) will rely on the next
Lemma 2.3. \((d \geq 2, 0 < \gamma < 1)\)

There are positive constants \(c_1, c_2 > 0\), such that for large \(N\), \(0 \leq \lambda \leq c_1\), and \(x \in E\):

i) \(P^N\)-a.s., for all \(k \geq 2\),

\[
E_{X_{R_k}}[e^{\lambda N_k^x}] \leq 1 + c_2 \lambda N^{-(d-1)(1-\gamma)} \tag{2.13}
\]

ii) \(E^N[e^{\lambda N_k^x}] \leq 1 + c_2 \lambda N^{-(d-1)(1-\gamma)}\).

Proof. First observe that for \(k \geq 1\), \(\lambda \geq 0\), \(x \in E\), dropping the superscripts from the stopping times as mentioned below (2.5), we find:

\[
E_{X_{R_k}}[e^{\lambda N_k^x}] = 1 + (e^\lambda - 1) \sum_{m \geq 0} e^{\lambda m} P_{X_{R_k}}[N^x_I > m], \tag{2.14}
\]

and with \(N\) large, for \(x \in E\), \(m, k \geq 1\):

\[
P_{X_{R_k}}[N^x_I > m] \leq P_{X_{R_k}}[D_{m+1}^x < D_1] \leq P_{X_{R_k}}[R_{m+1}^x < D_1] = \tag{2.15}
\]

\[
E_{X_{R_k}}[D_{m}^x < D_1, P_{X_R}^x[H_C(x) < D_1]],
\]

using the strong Markov property at \(D_m^x\) in the last step.

Then note that the simple random walk on \(\mathbb{Z}^{d+1} \simeq \mathbb{Z}^d \times \mathbb{Z}\), for large \(N\), when starting at \(y \in \partial B_\infty(0,2[N^\gamma])\) has a probability bigger than \(c > 0\) of first reaching \(B_\infty(0,[N^\gamma])\) without entering \(B_\infty(0,[N^\gamma]) + N \mathbb{Z}^d \times \{0\}\) and then exiting \(\mathbb{Z}^d \times [-2N,2N]\) without entering \(B_\infty(0,[N^\gamma]) + N \mathbb{Z}^d \times \{0\}\), as follows from instance for the invariance principle and standard estimates on the Green function. Hence for large \(N\), and any \(x \in E\),

\[
\forall y \in \partial \tilde{C}(x), P_y[H_C(x) < D_1] \leq (1 - c). \tag{2.16}
\]

Inserting this inequality in the last line of (2.15), we find that for large \(N, m, k \geq 1\):

\[
P_{X_{R_k}}[N^x_I > m] = P_{X_{R_k}}[D_{m+1}^x < D_1] \leq (1 - c) P_{X_{R_k}}[D_{m}^x < D_1] \leq (1 - c)^m P_{X_{R_k}}[N^x_I > 0]. \tag{2.17}
\]

Coming back to (2.14), we see that when \(e^\lambda (1 - c) < 1\), for large \(N\), any \(x \in E\), and \(k \geq 1\):

\[
P^N\text{-a.s.}, E_{X_{R_k}}[e^{\lambda N_k^x}] \leq 1 + \frac{(e^\lambda - 1)}{1 - e^\lambda (1 - c)} P_{X_{R_k}}[N^x_I > 0]. \tag{2.18}
\]

Note that when \(k \geq 2\), \(P^N\)-a.s., \(X_{R_k} \in \partial (B(x)^c)\), and

\[
P_{X_{R_k}}[N^x_I > 0] \leq \sup_{y \in \partial (B(x)^c)} P_y[H_C(x) < T_{B(x)}] \leq \tag{2.19}
\]

\[
sup_{y \in \partial B^c} E_y \left[ \sum_{n=0}^{T_{B-1}} \mathbb{1}\{X_n \in C(0)\} \right] \inf_{z \in C(0)} E_z \left[ \sum_{n=0}^{T_{B-1}} \mathbb{1}\{X_n \in C(0)\} \right].
\]

Using now estimates on the Green function of simple random walk in a strip \(V = \mathbb{Z}^d \times \{-2N+1, \ldots, 2N-1\}\), cf. (2.13) of [15], to bound the numerator from above, cf. the term before the multiplication sign in (2.20) below, and for the denominator a similar
inequality as (1.11) to bound \( g_N(\cdot, \cdot) \) from below by \( c g(\cdot, \cdot) \) on \( B_\infty(0, [N^\gamma]) \times B_\infty(0, [N^\gamma]) \), cf. the term after the multiplication sign in (2.20) below, we see that for large \( N \), for \( k \geq 2 \), and any \( x \in E \):

\[
P^N\text{-a.s., } P_{X_{R_k}}[N^x_k > 0] \leq c \frac{N^N(\cdot + 1)(\cdot + 1)}{N^{(d-1)/2}} \times N^{-2\gamma} = c N^{-(d-1)(1-\gamma)}.
\]

Inserting this bound in (2.18), the claim (2.13) i) readily follows. As for (2.13) ii), noting that \( X_0 \) is uniformly distributed over \( B \) under \( P^N \), we see that

\[
P^N[X^0_0 \notin B(x), H_{C(x)} \circ \theta_{R_k} < T_{\tilde{B}(x)} \circ \theta_{R_k}]
\]

\[
P^N[X_0 \in B(x), H_{C(x)} < T_{\tilde{B}(x)}] \leq c N^{-(d-1)(1-\gamma)} +
\]

\[
|B|^{-1} \sum_{y \in B(0) \cap B(x)} E_y \left[ \sum_{n=0}^{T_{\tilde{B}(x)}-1} \mathbb{1}\{X_n \in C(x)\} \right] /c N^{2\gamma},
\]

where we have used once again the same lower bound on the denominator of the last expression in (2.19), as explained above (2.20). From the reversibility of the walk on \( E \) with respect to the counting measure, the Green function of the walk killed outside \( \tilde{B}(x) \), (that is defined analogously to (1.9)), is symmetric in its arguments. We hence find that

\[
P^N[N^x_k > 0] \leq c N^{-(d-1)(1-\gamma)} +
\]

\[
N^{-(d+1+2\gamma)} \sum_{z \in C(x)} E_z \left[ \sum_{n=0}^{T_{\tilde{B}(x)}-1} \mathbb{1}\{X_n \in B(0) \cap B(x)\} \right] \leq c N^{-(d-1)(1-\gamma)} + c N^{-(d+1+2\gamma)} N^{(d+1)\gamma} \sup_{z \in C(x)} E_z[T_{\tilde{B}(x)}] \leq c N^{-(d-1)(1-\gamma)}.
\]

Coming back to (2.18), with \( k = 1 \), and integrating over the distribution of \( X_{R_1} \), we readily obtain (2.13) ii).

We will now prove (2.12) and thus conclude the proof of Proposition 2.2. To this end we choose \( 0 < \lambda \leq c_1 \), cf. (2.13), and using the strong Markov property at \( R_k^x \) we find that for large \( N \), any \( x \in E \),

\[
E^N \left[ \exp \left\{ \lambda \sum_{1 \leq k \leq N(\cdot + 1)(\cdot + 1)} N^x_k \right\} \right] \leq \left( 1 + \frac{c_2 \lambda}{N(\cdot + 1)(\cdot + 1)} \right)^{N^{(d-1)(1-\gamma)}} \leq \exp\{c_2 \lambda\}.
\]

Hence for large \( N \), the probability in (2.12) is smaller than

\[
c N^{2d+1} \exp\{-\lambda c_0 (\log N) + c_2 \lambda\},
\]

and choosing \( \lambda = c_1, c_0 \) large enough we obtain (2.12).

We now come back to the proof of Theorem 2.1. We define for \( x \in E \), the successive returns to \( C(x) \) and departures from \( \tilde{C}(x) \) of the walk:

\[
\begin{align*}
\tilde{R}^x_1 &= H_{C(x)}, \tilde{D}^x_1 = T_{\tilde{C}(x)} \circ \theta_{\tilde{R}^x_1} + \tilde{R}^x_1, \text{ and for } m \geq 1, \\
\tilde{R}^x_{m+1} &= H_{C(x)} \circ \theta_{\tilde{R}^x_m} + \tilde{D}^x_m, \tilde{D}^x_{m+1} = T_{\tilde{C}(x)} \circ \theta_{\tilde{R}^x_{m+1}} + \tilde{R}^x_{m+1}.
\end{align*}
\]

\]
Again for simplicity, cf. below (2.5), we write $\widetilde{R}_m, \widetilde{D}_m$ in place of $\widetilde{R}_m, \widetilde{D}_m$ when this causes no confusion. With (2.6) we also find:

\begin{equation}
(2.24) \quad \text{for large } N, \text{ } P^N\text{-a.s., for all } x \in E, \sup \{ m \geq 1; \widetilde{R}_m \leq N^{2d-\delta} \} \leq \sum_{k \geq 1} N^x_k 1 \{ R_k^x \leq N^{2d-\delta} \}.
\end{equation}

From now on we assume that $\gamma$, cf. (2.3), satisfies

\begin{equation}
(2.25) \quad 0 < \gamma \leq \frac{\delta}{(d-1)}.
\end{equation}

From (2.10), using the fact visits to $C(x)$ only occur during the time intervals $[\widetilde{R}_m, \widetilde{D}_m - 1]$, we find that:

\begin{equation}
(2.26) \quad \lim_{N \to \infty} P^N \left[ \forall x \in E, \sum_{n=0}^{N^{2d-\delta}} 1 \{ X_n \in C(x) \} \leq \frac{c_0(\log N)}{\sum_{m=1}^{\theta_{\widetilde{R}_m}} T_{C(x)} \circ \theta_{\widetilde{R}_m}} = 1. \right.
\end{equation}

Analogously to (1.19), we also have for $N \geq 1$:

\begin{equation}
(2.27) \quad \sup_{x,y \in E} E_x \left[ \exp \left\{ \frac{c}{N^{2\gamma}} T_{C(x)} \right\} \right] \leq c'.
\end{equation}

Note that for large $N$, $P^N$-a.s., the sum in the probability in (2.26) vanishes for all $x \in E$ with $|x^{d+1}| \geq N^{2d}$. We hence find that:

\begin{equation}
(2.28) \quad \lim_{N \to \infty} P^N \left[ \text{for some } x \in E, \sum_{n=0}^{N^{2d-\delta}} 1 \{ X_n \in C(x) \} \geq c_3(\log N) N^{2\gamma} \right] \leq \lim_{N \to \infty} cN^{2d+d} \sup_{x \in E} \left[ \sum_{1 \leq m \leq c_0(\log N)} T_{C(x)} \circ \theta_{\widetilde{R}_m} \geq c_3(\log N) N^{2\gamma} \right] \leq \lim_{N \to \infty} cN^{2d} \exp \left\{ -c_3(\log N) \right\} c^{\frac{1}{\log N}} = 0,
\end{equation}

if $c_3$ is chosen large enough. In other words, when $\gamma$ fulfills (2.25), we see that:

\begin{equation}
(2.29) \quad \lim_{N \to \infty} P^N \left[ \text{for all } x \in E, \sum_{n=0}^{N^{2d-\delta}} 1 \{ X_n \in C(x) \} \leq c_3(\log N) N^{2\gamma} \right] = 1.
\end{equation}

To conclude the proof of Theorem 2.1, we will use the next geometric lemma that holds true for $d \geq 1$ and general $0 < \gamma < 1$. We refer to the end of the Introduction for our convention concerning constants.

**Lemma 2.4.** $(d \geq 1, 0 < \gamma < 1)$

*There is a positive constant $c(\gamma)$ such that for $N \geq c(\gamma)$, whenever $S \subseteq E$ disconnects $E$, there is an $x \in E$ such that*

\begin{equation}
(2.30) \quad |C(x) \cap S| \geq cN^{d\gamma}, \text{ (cf. (2.5) for the notation)}.
\end{equation}

**Proof.** Assume $S$ disconnects $E$, and denote with $T_{Top}$ the connected component of $E \setminus S$ containing $(\mathbb{Z}/N\mathbb{Z})^d \times [M, \infty)$, when $M$ is large. We can define the function:

\begin{equation}
(2.31) \quad t(x) = \frac{1}{|C(x)|} \sum_{y \in C(x)} 1 \{ y \in T_{Top} \} = \frac{|T_{Top} \cap C(x)|}{|C(x)|}, \text{ } x \in E.
\end{equation}
Note that
\[(2.32) \quad t(x) = 1, \text{ for large } x^{d+1}, \quad t(x) = 0, \text{ for large negative } x^{d+1}.
\]
Moreover when \(|x - x'| = 1\), (with \(\Delta\) standing for the symmetric difference)
\[(2.33) \quad |t(x) - t(x')| \leq \frac{|C(x)\Delta C(x')|}{|C(0)|} \leq \frac{c}{N^\gamma}.
\]
Thus for \(N \geq c(\gamma)\), there is at least one \(x_* \in E\) such that:
\[(2.34) \quad \left|t(x_*) - \frac{1}{2}\right| \leq \frac{c}{N^\gamma} < \frac{1}{4}.
\]
Then for \(A \subseteq C(x_*)\) we define the relative boundary of \(A\):
\[(2.35) \quad \partial_{C(x_*)} A = \{y \in C(x_*) \setminus A; \exists x \in A \text{ such that } |y - x| = 1\}.
\]
Observe that:
\[(2.36) \quad \partial_{C(x_*)} (\text{Top} \cap C(x_*)) \subseteq S \cap C(x_*),
\]
indeed any point in \(C(x_*)\) neighbor of a point in \(\text{Top} \cap C(x_*)\) has to belong to \(S\) if it is not in \(\text{Top} \cap C(x_*)\). Moreover from the isoperimetric controls in (A.3), p. 480 of [7],
\[(2.37) \quad |\partial_{C(x_*)} (\text{Top} \cap C(x_*))| \geq c |\text{Top} \cap C(x_*)|^\frac{d}{d+1} \geq c |C(x_*)|^\frac{d}{d+1} \geq c N^{\delta y}.
\]
This and (2.36) proves (2.30).

We can now conclude the proof of Theorem 2.1. Assuming \(d \geq 3\), and (2.25), we see that (2.29) and (2.30) imply that
\[(2.38) \quad P_N^N [X_{[0, N^{\delta_d}] \text{ disconnects } E}] \xrightarrow{N \to \infty} 0.
\]
The combination of Theorem 2 and Theorem 2.1 clearly yield the proof of Theorem 1 of the Introduction.

\[\square\]

**Remark 2.5.**

When \(d = 1\), it is immediate to prove that (2.1) holds. This together with Remark 1.4 shows that Theorem 1 holds when \(d = 1\) as well. However as already mentioned in the discussion below (0.6) when \(d \geq 3\), there is a substantial discrepancy between the disconnection time \(T_N\) and \(\tilde{C}_N\) the cover time of the box by the projection of \(X_*\), whereas for \(d = 1\), both \(\log T_N / \log N\) and \(\log \tilde{C}_N / \log N\) tend to 2 in \(P\)-probability. The results of Section 3 will amplify the qualitative difference between the two cases.

When \(d = 2\), as a result of (0.4), (0.5) and Theorem 1.1, we know that for any \(\delta > 0\),
\[(2.39) \quad \lim_{N \to 0} P_0 \left[ d - \delta \leq \frac{\log T_N}{\log N} \leq 2d + \delta \right] = 1,
\]
but the correct order of magnitude of \(T_N\) remains open. \[\square\]

11
3 Clogging at time $T_N$

The main objective of this section is to prove Theorem 2 of the Introduction, and thus show that when $d \geq 3$; for any $\epsilon > 0$, for large $N$, the truncated cylinder

$$B_N = \{x \in E; |x^{d+1}| \leq N^{d-\epsilon}\},$$

with high $P_0$-probability is pretty much “clogged” by the trajectory $X_{[0,T_N]}$. This effect ought to be contrasted with what happens when $d = 1$, cf. Remark 3.2.

Theorem 3.1. ($d \geq 3$)

$$\text{(3.2)} \quad \text{For } \epsilon, \eta \in (0,1), \max_{x \in B_N} \frac{d(x, X_{[0,T_N]}^N)}{N^\eta} \to 0, \text{ in } P_0\text{-probability},$$

(cf. above (0.7) for the notation).

Proof. We introduce the sequence $\tau_k, k \geq 0$, of $(\mathcal{F}_n)$-stopping times describing the successive displacements of the $\mathbb{Z}$-component $X_{n}^{d+1}$ of $X$, at distance $2N$:

$$\text{(3.3)} \quad \left\{ \begin{array}{l}
\tau_0 = 0, \tau_1 = \inf\{n \geq 0, |X_n^{d+1} - X_0^{d+1}| \geq 2N\}, \text{ and for } k \geq 1,
\tau_{k+1} = \tau_k + \theta_{\tau_k}.
\end{array} \right.$$ 

On an auxiliary probability space $(\sum, \mathcal{A}, P)$, we consider a simple random walk in continuous time $(\tilde{Z}_t)_{t \geq 0}$, on $\mathbb{Z}$, starting at 0, with exponential holding times of parameter 1, its discrete skeleton $(Z_k)_{k \geq 0}$, and its sequence of times of successive jumps $(\tilde{S}_k)_{k \geq 0}$, so that

$$\text{(3.4)} \quad \tilde{Z}_t = Z_k, \text{ for } S_k \leq t < S_{k+1}, \text{ and } S_0 = 0 < S_1 < \cdots < S_k < \cdots \text{ tends to } \infty.$$ 

From the strong Markov property applied at times $(\tau_k)_{k \geq 0}$, we find that

$$\text{(3.5)} \quad \text{under } P_0, \quad (Z_k^N)_{k \geq 0} \overset{\text{def}}{=} \left( \frac{1}{2N} X_{\tau_k}^{d+1} \right)_{k \geq 0} \text{ has same law as } (Z_k)_{k \geq 0} \text{ under } P.$$ 

We will be interested in various local time processes, namely:

$$\text{(3.6)} \quad L_N(z, k) = \sum_{m=0}^{k} 1\{z_{\tau_m}^N = z\}, \quad L(z, k) = \sum_{m=0}^{k} 1\{z_{m} = z\}$$

$$\tilde{L}(z, t) = \int_{0}^{t} 1\{z_{s} = z\} ds, \text{ with } t \geq 0, k \geq 0, z \in \mathbb{Z}.$$ 

We then choose:

$$\text{(3.7)} \quad \epsilon, \eta \in (0,1) \text{ and } 0 < \delta < \frac{1}{2} (\eta \wedge \epsilon).$$

We first observe that

$$\text{(3.8)} \quad \lim_{N \to \infty} P_0[\tau_{N^{d-\delta}}] \geq T_N] = 0.$$
Indeed the above probability is smaller than
\[ P_0[T_N < N^{2d-\delta/2} + P_0[\tau_{[N^{2d-\delta}]} \geq N^{2d-\delta/2}]. \]

In view of Theorem 2.1, the first term tends to 0 as \( N \) goes to infinity. As for the second term, it follows from (1.19) and the strong Markov property at \( \tau_k \) that
\[ P_0[\tau_{[N^{2d-\delta}]} \geq N^{2d-2\delta}] \leq \exp \left\{ -\frac{C}{N^2} N^{2d-2\delta} \right\} \frac{e^{N^{2d-2\delta}}}{N \to \infty}, \]
whence (3.8). As a result Theorem 3.1 will be proved once we show that:
\[
A \overset{\text{def}}{=} \lim_{N \to \infty} P_0\left[ \max_{x \in B_{e}} d(x, X_{0, \tau_{[N^{2d-2\delta}]}}) > N^\eta \right] = 0.
\]

To this end we observe that
\[
A \leq A_1 + A_2, \text{ where } \quad A_1 = P_0\left[ \max_{x \in B_{e}} d(x, X_{0, \tau_{[N^{2d-2\delta}]}}) > N^\eta, \text{ and} \right.
\]
\[
\inf_{|z| \leq N^{d-1-\epsilon}} L_N(z, [N^{2d-2\delta}]) \geq N^{d-1-2\delta} \right],
\]
\[
A_2 = P_0\left[ \inf_{|z| \leq N^{d-1-\epsilon}} L_N(z, [N^{2d-2\delta}]) < N^{d-1-2\delta} \right].
\]

We first bound \( A_1 \). For \( x \in E \), we denote with \( B_{x,\eta} \) the ball \( B_{\infty}(x, N^\eta) \subset E \), see the beginning of Section 1 for the notation, and note that standard Green function estimates imply that for large \( N \):
\[
\inf_{|y| \leq N^{d+1-2d+1}} P_y[H_{B_{x,\eta}} < \tau_1] \geq c N^{-(d-\eta)}.
\]

Now for \( x \in B_{e} \), denote with \( z \) some integer such that \( |2zN - x| \leq N \), and \( |z| \leq N^{d-1-\epsilon} \), (such a \( z \) exists for all \( x \in B_{e} \), when \( N \) is large). Let \( H^z_m, m \geq 1 \), stand for the successive times \( \tau_k, k \geq 0 \), when \( X_{\tau_k} \) has a \( Z \)-component equal to \( 2zN \). The strong Markov property of \( X \), at \( H^z_m \), shows that when \( N \) is large, we have:
\[
A_1 \leq |B_{e}| \max_{x \in B_{e}} P_0\left[ \text{for } 1 \leq m < N^{d-1-2\delta}, H_{B_{x,\eta}} \circ \theta_{H^z_m} > \tau_1 \circ \theta_{H^z_m} \right]
\]
\[
\overset{(3.11)}{\leq} |B_{e}| \left( 1 - c N^{-(d-1-\eta)} \right) N^{d-1-2\delta} \frac{(3.7)}{N \to \infty} 0.
\]

We now bound \( A_2 \) in (3.10). In view of (3.5), we can replace \( L_N \) with \( L \) and \( P_0 \) with \( P \) in the expression defining \( A_2 \). As a result we find
\[
A_2 \leq B_1 + B_2, \text{ with } \quad B_1 = P[L(0, [N^{2d-2\delta}]) < N^{d-1-\delta}], \text{ and } \]
\[
B_2 = P[L(0, [N^{2d-2\delta}]) \geq N^{d-1-\delta}, \text{ and } \inf_{|z| \leq N^{d-1-\epsilon}} L(z, [N^{2d-2\delta}]) < N^{d-1-2\delta}].
\]

Note that we have the simple random walk identity, cf. below (1.2) for the notation,
\[
P[L(0, k) < m] = P[m - 1 + H_{m-1} > k], \quad k \geq 0, m \geq 1.
\]
Hence for large $N$ we find:

\[
B_1 \leq P \left[ H_{\left[ N^{d-1-\delta} \right]} > \frac{1}{2} N^{2d-2-\delta} \right] \xrightarrow{N \to \infty} 0,
\]

using Example 6.6 in Chapter 7 of Durrett [8], p. 369. In order to bound $B_2$, we then write:

\[
B_2 \leq C_1 + C_2 + C_3, \quad \text{with}
\]

\[
C_1 = P \left[ \tilde{L}(0, S_{\left[ N^{d-2-\delta} \right]}) > \frac{N^{d-1-\delta}}{2} \right], \quad \text{and}
\]

\[
\inf_{|z| \leq N^{d-1-\delta}} \tilde{L}(z, S_{\left[ N^{d-2-\delta} \right]}) < 2N^{d-1-2\delta},
\]

\[
C_2 = P \left[ \tilde{L}(0, S_{\left[ N^{d-2-\delta} \right]}) \leq \frac{N^{d-1-\delta}}{2} \right], \quad \text{and}
\]

\[
L(0, \left[ N^{2d-2-\delta} \right]) \geq N^{d-1-\delta},
\]

\[
C_3 = \sum_{|z| \leq N^{d-1-\delta}} P \left[ \tilde{L}(z, S_{\left[ N^{d-2-\delta} \right]}) \geq 2N^{d-1-2\delta} \right], \quad \text{and}
\]

\[
L(z, \left[ N^{2d-2-\delta} \right]) < N^{d-1-2\delta}.
\]

We first bound $C_2$. On the event that appears in $C_2$, the sum of the times spent at 0 by \( \tilde{Z} \), on the first \( \left[ N^{d-1-\delta} \right] - 1 \) visits is at most \( \frac{1}{2} N^{d-1-\delta} \), so that by an obvious distribution identity and the strong law of large numbers, we find:

\[
C_2 \leq P \left[ S_{\left[ N^{d-1-\delta} \right]} \leq \frac{1}{2} N^{d-1-\delta} \right] \xrightarrow{N \to \infty} 0.
\]

As for $C_3$, using Cramer-type bounds, we find that for large $N$:

\[
C_3 \leq c N^{d-1-\epsilon} P \left[ S_{\left[ N^{d-2-\delta} \right]} \geq 2N^{d-1-2\delta} \right] \xrightarrow{N \to \infty} 0.
\]

We now bound $C_1$. We denote with \((A_t)_{t \geq 0}\), the right-continuous inverse of \( \tilde{L}(0, s), s \geq 0 \). We see that

\[
C_1 \leq P \left[ \inf_{|z| \leq N^{d-1-\delta}} \tilde{L}(z, A_{\left[ N^{d-2-\delta} \right]}) \leq 2N^{d-1-2\delta} \right].
\]

For $t > 0$, we denote with \((W^t_u)_{u \geq 0}\), the unique strong solution of the stochastic differential equation

\[
W^t_u = t + 2 \int_0^u \sqrt{(W^t_v \vee 0)} \, d\beta_v, \quad u \geq 0,
\]

where \( \beta \) is a one-dimensional Brownian motion defined on some auxiliary probability space \((\tilde{\Sigma}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})\), cf. Ikeda-Watanabe [9], p. 238. It follows from (3.1), p. 21 of [13], that for $t > 0$, the following Ray-Knight theorem holds:

\[
(\tilde{L}(z, A_t))_{t \geq 0} \text{ and } (\tilde{L}(-z, A_t))_{t \geq 0} \text{ are independent under } P
\]

and distributed as \((W^t_z)_{z \geq 0}\), \( z \) integer.\]

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From (3.20) we see that $W^t_i$ satisfies the scaling identity

$$W^t_i \text{ has same distribution as } t W^{1/t}_i.$$  

Choosing $t = \frac{1}{2} N^{d-1-\delta}$, we obtain from (3.19), (3.21), (3.22) that

$$C_1 \leq 2 \tilde{P} \left[ \inf_{0 \leq u \leq 2N^{d-1}} W^1_u \leq 4N^{-\delta} \right] \xrightarrow{N \to \infty} 0,$$

since $W^1_0 = 1$, $W^1_1$ is continuous, and $\delta < \epsilon$, in view of (3.7). This proves (3.9) and thus concludes the proof of Theorem 3.1.

\[ \square \]

**Remark 3.2.**

When $d = 1$, the “clogging” effect mentioned in (3.2) does not take place, and with non-vanishing probability, as $N$ tends to infinity, there are points in $B_\epsilon$ at distance of order $N$ from $X_{[0, \tau_N]}$. This fact is a straightforward consequence of the invariance principle and the support theorem for Wiener measure.

\[ \square \]

**References**


