NONPARAMETRIC RENEWAL FUNCTION ESTIMATION
AND SMOOTHING BY EMPIRICAL DATA

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Abstract. We consider an estimate of the renewal function (rf) using a limited number of
independent observations of the interarrival times for an unknown interarrival-time distribution
(itd). The nonparametric estimate is derived from the rf-representation as series of distribution
functions (dfs) of consecutive arrival times using a finite summation and approximations of the
latter by empirical dfs. Due to the limited number of observed interarrival times the estimate
is accurate just for closed time intervals \([0, t]\). An important aspect is the selection of an optimal
number of terms \(k\) of the finite sum. Here two methods are proposed: (1) an a priori choice of
\(k\) as function of the sample size \(l\) which provides almost surely (a.s.) the uniform convergence
of the estimate to the rf for light- and heavy-tailed itds if the time interval is not too large, and
(2) a data-dependent selection of \(k\) by the plot of the proposed estimate against \(k\) for a fixed
time \(t\). To evaluate both the efficiency of the estimate and the selection method of \(k\), a Monte
Carlo study is performed.

Keywords: Renewal function, nonparametric estimate, heavy-tailed distribution.

1. INTRODUCTION

Renewal processes have a wide range of applications in the warranty control, in the reliability
analysis of technical systems and particularly of telecommunication networks such as high-
speed packet-switched networks like the Internet. Normally, measurement facilities count
the events of interest, e.g. the number of requested and transferred Web pages, incoming
or out-coming calls, frames, packets or cells in consecutive time intervals of fixed length.
It is important for planning and control purposes to estimate the related traffic load in
terms of the mean numbers of counted events and their variances in these intervals. In such
applications the renewal function (rf) constitutes the basic characteristic of an underlying
renewal process since by means of this function the expectation and variance of the number
of arrivals of the relevant events before a fixed time instant can be calculated (cf. [11]).
To estimate the rf, several realizations of the counting process may be required, e.g. the
observations of the number of calls within several days. We estimate here the rf using
inter-arrival times between events of only one realization of the process.

Let \(F(t) = \text{IP} \{\tau_n < t\}\) denote the common distribution function (df) of the i.i.d. interarrival
times \(\{\tau_n, n = 1, 2, \ldots\}\) of these events with \(F(0+) = 0\). The renewal counting process
\(\{N_t, t \geq 0\}\) denotes the number of events before time \(t\), \(N_t = \text{max}\{n : t_n < t\}\) for \(t \geq 0\),
where \(t_n = \sum_{i=1}^{n} \tau_i, t_0 = 0\) are the arrival times.
The rf \(H(t)\) is expressed by

\[
H(t) = \mathbb{E}(N_t) = \sum_{n=1}^{\infty} \text{IP} \{t_n < t\} = \sum_{n=1}^{\infty} F^{*n}(t)
\]  

(1)
for $t \geq 0$ where $F^{*n}$ denotes the $n$-fold recursive Stieltjes convolution of $F$.

Several rf-estimation methods have been developed for a known interarrival-time distribution. Unfortunately, explicit forms of the rf are obtained only in rare cases, for example, if the interarrival times have a uniform distribution, or for the wide class of matrix-exponential distributions (exponential and Erlang distributions are from this class) (cf. [1]). Therefore, several attempts have been made to evaluate the rf computationally (cf. [3], [5], [8], [16], [23]).

If the mean $\mu$ and variance $\sigma^2$ of $F$ are finite, then the rf $H(t)$ may be approximated for large $t$ by an expression

$$H(t) = \frac{t}{\mu} + \frac{\sigma^2}{2\mu^2} - \frac{1}{2} + o(1)$$

widely used in the literature (cf. [11]). In [3] an alternative asymptotic expression is stated for the case that the Laplace-Stieltjes transform (LST) of $F(t)$ is a rational function. Such estimates do not perform well for small times $t$ relative to $\mu$, which is especially important for the warranty control of devices (cf. [9]).

In practice, it is a more realistic situation that the distribution is unknown or that just general information describing it is available. The restoration of the df or the probability density function (pdf), if the latter exists, may become complicated if the distributions of the random variables (rvs) are heavy-tailed. This means that those distributions have heavier tails than an exponential one (see [6], [7], [12], [17], [18]). Weibull distributions with a shape parameter less than one and Pareto distributions provide examples of such pdfs. Heavy-tailed distributions often arise in practice, for example, in insurance and queueing or in the characterization of World Wide Web (WWW) traffic (cf. [19]).

In this paper we propose to estimate the rf without any information about the form of the underlying distribution and we use only an empirical sample $T_l = f_{n}; n = 1, 2, \ldots, l$ of the nonnegative i.i.d. interarrival times between events of the size $l$. The stated nonparametric estimate is related to a histogram-type estimate where the unknown probabilities $P\{t_n < t\}$ in (1) are replaced by the corresponding empirical dfs and a limited number of terms $k$ is used in the summation. A similar nonparametric estimate

$$H_l(t, k) = \sum_{n=1}^{k} F_l^{(n)}(t)$$

was proposed by Frees [9], [10] and further investigated in [22]. These authors have used

$$F_l^{(n)}(t) = \left( \frac{l}{n} \right)^{-1} \sum_{c} \theta(t - (\tau_{i_1} + \cdots + \tau_{i_n}))$$

as estimate of the arrival-time distribution. Here $\theta(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$ and $\sum_{c}$ denotes the sum over all $\left( \frac{l}{n} \right)$ distinct index combinations $\{i_1, i_2, \ldots, i_n\}$ of length $n$. The $U$-statistic $F_l^{(n)}(t)$ is a minimum-variance unbiased estimator of $F^{*n}(t)$. In contrast to our estimate (4), which uses just one combination of adjacent interarrival times, the computation of the $H_l(t, k)$ is awkward.

The accuracy of such types of estimates depends on $k$. Frees has obtained the uniform consistency of $H_l(t, k)$ a.s. on compact intervals $[0, t], 0 \leq t < \infty$, under the assumptions
that $k = l$ and $F(t)$ has a positive mean and finite variance, and the asymptotic normality of $H_i(t, k)$ for each fixed point $t$ under some moment conditions (cf. [10]). In [13] the convergence of a non-computational empirical rf

$$H_{IP}(t) = \sum_{n=1}^{\infty} \hat{F}_t^{*n}(t)$$

(3)
on $R$ as $l \to \infty$ has been proved. Here, $\hat{F}_t^{*n}(t)$ is the $n$-fold convolution of the empirical df $F_i(t)$ based on the sample $T_i$. However, the data-dependent selection of $k$ (which is important for moderate samples) was not considered in [9], [10], [13], and [22].

In our consideration, we use an unbiased estimate of $F^{*n}(t)$, but its variance is not minimal. We compensate this inaccuracy by the data-dependent selection of $k$ and the use of larger samples. An a priori choice of $k$ as a function of the sample size $l$ is proposed to obtain a.s. the uniform convergence of the estimate to the rf $t \to \infty$ for light- and heavy-tailed pdfs. As data-dependent choice of $k$ the plot of the estimate (4) against $k$ for a fixed time $t$ is applied.

The paper is organized as follows. In Section 2 the histogram-type estimate of the rf is investigated. Theorems 1-4 related to the a.s. uniform convergence of this estimate to the rf are stated. The selection of the parameter $k$ by the plot is described. In Section 3 a comparison of (4) with Frees’ estimate is given. Finally, the findings are summarized in the Conclusion. In Appendix the proofs of the Theorems of Section 2 are presented.

2. HISTOGRAM-TYPE ESTIMATE OF THE RENEWAL FUNCTION

We consider the estimate of the rf $H(t)$ which was first introduced in [15]. Let $[r]$ denote the integer part of a number $r$. Inserting the empirical mean for $\mathbb{E} (N_i)$, we replace the df $\mathbb{P} \{ t_n < t \}$ by the empirical df $F_i(t) = \frac{1}{n} \sum_{i=1}^{n} \theta(t - t_i^n)$.

It is its unbiased estimate and $t_i^n = \sum_{q=1+n(i-1)}^{n} \tau_q, i = 1, \ldots, l_n, l_n = \lceil \frac{1}{n} \rceil, n = 1, \ldots, k$, are the observations of the rv $t_n$. Then one can estimate the renewal function $H(t)$ based on the samples of independent renewal-time observations $t_1 = \{ t_1^1, \ldots, t_1^l \}, \ldots, t_k = \{ t_k^1, \ldots, t_k^l \}$ by:

$$\tilde{H}(t, k, l) = \frac{1}{l} \sum_{n=1}^{l} \sum_{i=1}^{t_n} \theta(t - t_i^n)$$

(4)

Note that $\tilde{H}(t, k, l) = k$ holds for $t \in [t_{\max}(k), \infty)$ where $t_{\max}(k) = \max_{1 \leq n \leq k} \max_{1 \leq i \leq t_n} t_i^n$ and $k$ is some fixed number.

The errors of the estimation arise both from the approximation of $H(t)$ in (4) by a finite sum and the approximation of $\mathbb{P} \{ t_n < t \}$ by the empirical df $F_i(t)$:

$$\| H(t) - \tilde{H}(t, k, l) \| = \| \sum_{n=k+1}^{\infty} \mathbb{P} \{ t_n < t \} + \sum_{n=1}^{k} (\mathbb{P} \{ t_n < t \} - F_i(t)) \|$$

(5)

By this formula one can see that $\tilde{H}(t, k, l)$ as well as the estimator (2) are biased since $k$ is limited.

A rough upper bound of the bias is given by

$$\text{bias}(t, k, l) = H(t) - \mathbb{E} \tilde{H}(t, k, l) = \sum_{n=k+1}^{\infty} \mathbb{P} \{ t_n < t \} \leq \sum_{n=k+1}^{\infty} (F(t))^n = \frac{(F(t))^{k+1}}{1 - F(t)}$$

(6)
For small $t \ F(t)$ is generally small and $F(t) < 1$, thus, this error is small. To provide a good approximation of $\mathbb{P} \{ t_n < t \}$ by the empirical df, according to the Glivenko-Cantelli theorem sufficiently large values $l_n$ should be used, i.e. $k < l$. Note that $l_k = 1$ for $\frac{1}{2} < k \leq l$, i.e. the sample $t_k$ contains only one point. Therefore, it is reasonable to take $k \leq \frac{l}{2}$. In the following, we provide an optimized estimate of $k$. On the other hand, to provide a good approximation of $H(t)$ by means of $\tilde{H}(t, k, l)$ in general, the value of $k$ should be large enough. Therefore, the estimate $\tilde{H}(t, k, l)$ is sensitive to the choice of $k$ and the length of the estimation interval $[0, t]$. Obviously, the estimate $\tilde{H}(t, k, l)$ may only be accurate within the interval $[0, t_{max}(k)]$, since the sample size $l$ is limited.

### 2.1. Convergence of the Histogram-Type Estimate

Now we investigate the convergence of estimate (4) to the rf in the metric of the space $C$ of continuous functions. To estimate the risk (5), one is interested in the $t$-regions $[0, t]$ $\subset [0, t_{max}(k)]$. The main problem is to estimate the systematic error $\sum_{n=k+1}^{\infty} \mathbb{P} \{ t_n < t \}$. To estimate it, one needs some information about the df $F(t)$ of the rv $\tau$ and then one can use precise large deviation results for $\mathbb{P} \{ t_n < t \}$. Such a principal information may be the existence of the moment generating function (Cramer’s condition (C.c.)) (cf. [20]). The rv $\tau$ satisfies C.c. if there exists $\theta > 0$ such that $\mathbb{E} (e^{\theta \tau}) < \infty$. The C.c. is equivalent to an exponential decay rate of $1 - F(t)$ and it is satisfied for light-tailed distributions. The C.c. provides the existence of all moments of the rv $\tau$.

**Theorem 1.** Let $\{ \tau_1, \ldots, \tau_l \}$ be a sequence of i.i.d. rvs and $t \in [0, t_{max}(k)]$. We suppose that $\mathbb{E}[\tau]^m < \infty$ for some integer $m \geq 3$, $\mathbb{E}\tau = \mu$, $\text{var}(\tau) = \sigma^2$, and that the parameter $k$ obeys

$$k = o(l^p) \quad (\text{as } l \to \infty), \quad 0 < p < 1/3. \quad (7)$$

Then

$$\mathbb{P} \left\{ \omega : \lim_{l \to \infty} \sup_t |H(t) - \tilde{H}(t, k, l)| = 0 \right\} = 1,$$

holds.

The rate of this uniform convergence may be proved for the class $\tilde{S}$ of itds such that

$$1 - F(t) \geq \exp(-\nu t)$$

for any $t \in [0, T]$ and some $\nu > 0$. We assume, without loss of generality, that $[0, T] = [0, 1]$. The class $\tilde{S}$ includes, for example, the exponential distribution and the Weibull distribution with a shape parameter larger than one. Hence, it follows for the estimate of the right-hand side of (6):

$$\frac{(F(t))^{k+1}}{1 - F(t)} \leq \frac{(1 - \exp(-\nu t))^{k+1}}{\exp(-\nu t)}$$

Then, for $F(t) \in \tilde{S}$ the error of an approximation by (4) in the metric of $C$ is estimated by

$$\sup_t |H(t) - \tilde{H}(t, k, l)| \leq \sup_t \left( \frac{(1 - \exp(-\nu t))^{k+1}}{\exp(-\nu t)} + \sum_{n=1}^{k} (\mathbb{P} \{ t_n < t \} - F_{l_n}(t)) \right) \quad (8)$$
Theorem 2. If \( \{\tau_1, \ldots, \tau_l\} \) is an i.i.d. sample with the df \( F \in \mathcal{S} \), \( t \in [0,1] \) and the parameter \( k = c \cdot l^\rho \) \( (c = c(\nu, \alpha, \rho) \geq -l^{-\rho} \left( \frac{\nu + \alpha \ln l}{\ln(1 - \exp(-\nu))} + 1 \right) > 0) \), \( 0 < \rho < 1/3 - (2/3)\alpha \), \( 0 < \alpha < 1/2 \), then the asymptotic rate of convergence of the estimate \( \tilde{H}(t,k,l) \) to \( H(t) \) is given by the expression

\[
P\left\{ \omega : \lim_{l \to \infty} \sup_t |P^t[H(t) - \tilde{H}(t,k,l)]| \leq c_1 \right\} = 1,
\]

where \( c_1 \) is a constant that is independent of \( l \).

Then, the following confidence interval is derived for the rf.

Corollary 1. If assumptions of Theorem 2 hold, then the following inequalities hold with a probability of at least \( 1 - \xi \), \( 0 < \xi < 1 \):

\[
\tilde{H}(t,k,l) - D \leq H(t) \leq \tilde{H}(t,k,l) + D,
\]

where

\[
D = t^{-\alpha} + k \sqrt{-\frac{k \ln(\xi/2)}{2t}}.
\]

In practice, interarrival times are often described by distributions with heavy tails (cf. [4], [12]). Two classes of heavy-tailed distributions are well known: the distributions with regularly varying tails where \( 1 - F(t) = t^{-\alpha}L(t) \), \( t > 0, \alpha > 0 \) and \( L \) is a slowly varying function, and the subexponential distributions, i.e. the distributions with the property: for any \( \varepsilon > 0 \) there exists \( T = T(\varepsilon, F) \) such that for any \( t > T, 1 - F(t) > \exp(-\varepsilon t) \). It is the specific feature of heavy-tailed distributions that they do not satisfy C.c.. If \( t \) is not too large, an approximation of \( P\{t_n > t\} \) by the tail of the standard normal distribution \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy \) is used for heavy-tailed distributions, namely,

\[
P\{t_n > t\} \sim \Phi\left( \frac{t}{\sqrt{n}} \right)
\]

(this means that \( \lim_{n \to \infty} \sup_{0 < t < \frac{t_n}{\sqrt{n}}} \left| \frac{P\{t_n > t\}}{\Phi\left( \frac{t}{\sqrt{n}} \right)} - 1 \right| = 0 \) if \( t \in (0, c_0 / h_n) \) for any choice of the sequence \( h_n \to \infty \) as \( n \to \infty \) (cf. [17]).

Several threshold sequences \( c_0 \) are proposed by different authors. For example, for a Weibull distribution with a shape parameter \( 0 < \alpha \leq 0.5 \), \( c_0 / h_n \sim n^{1/(2-\alpha)} \) and for \( 0.5 < \alpha < 1 \), \( c_0 / h_n \sim n^{2/3} \), for distributions with regularly varying tails and \( \alpha > 2 \), \( c_0 / h_n \) may be \( \sim n^{0.5} \ln^{0.5} n \) (cf. [18]). Hence, the following Theorem can be proved.

Theorem 3. If \( \{\tau_1, \ldots, \tau_l\} \) is a sequence of i.i.d. rvs with the heavy-tailed df \( F(t) \), \( t \in (0, \min(t_{\max}(k), \frac{1}{h_k})) \), and the parameter \( k \) obeys (7), then

\[
P\left\{ \omega : \lim_{l \to \infty} \sup_t |H(t) - \tilde{H}(t,k,l)| = 0 \right\} = 1,
\]

holds.
We consider now the rate of uniform convergence for distributions with regularly varying tails.

By the representation theorem [7], [18] the slowly varying function \( c(x) \) can be rewritten in the form

\[
L(x) = c(x) \exp \left\{ \int_{x_0}^{x} \frac{\varepsilon(y)}{y} \, dy \right\}, \quad x \geq x_0
\]

for some \( x_0 > 0 \), where \( c(\cdot) \) is a measurable non-negative function such that \( \lim_{x \to \infty} c(x) = c_0 \in (0, \infty) \) and \( \varepsilon(x) \) is continuous function, \( \varepsilon(x) \to 0 \) as \( x \to \infty \).

In the following theorem, we will assume that \( c(x) \) is a monotone decreasing or increasing function and \( \varepsilon(x) \) is a non-positive function.

**Theorem 4.** Let \( \{\tau_1, \ldots, \tau_l\} \) be i.i.d. non-negative regularly varying rv's with the tail \( F(x) = L(x)x^{-\alpha}, x > 0, \alpha > 0 \), the parameter \( k = d \cdot l^\rho, d = -A \),

\[
0 \leq \rho < \frac{1}{3} - \frac{2}{3} \beta,
\]

\[
A = A(\beta) = \frac{(\beta + (\alpha - \varepsilon(\theta))(1 + \eta)) \ln l - \ln e^* + \varepsilon(\theta) \ln x_0}{\ln \left(1 - c^*(1+\eta)(-\alpha+\varepsilon(\theta))^x_0^{-\varepsilon(\theta)}\right)} + 1,
\]

\( c^* = \min(c_0, c(a)) \), \( \eta > 1/(\alpha - \varepsilon^*) \), \( 0 < \varepsilon^* < \alpha, 0 < \beta < 1/2, \theta \in [x_0, t_{\max}(k)], x_0 > 0, t \in [a, t_{\max}(k)], a > 0 \), then

\[
\mathbb{P} \left\{ \omega : \lim_{l \to \infty} \sup_{t} l^\beta |H(t) - \tilde{H}(t, k, l)| \leq c_1 \right\} = 1,
\]

where \( c_1 \) is a constant that is independent of \( l \).

**Corollary 2.** If assumptions of Theorem 4 hold, \( \eta > (1 - \ln \nu/\ln l)/(\alpha - \varepsilon^*) \), then with probability at least \( 1 - \nu, 0 < \nu < 1 \), the inequality (9) holds, where

\[
D = l^{-\beta} + k \sqrt{\frac{k}{2l} \ln \left(\frac{\nu - l^1 - \eta(\alpha - \varepsilon^*)}{2}\right)}.
\]

The Theorems determine the values of \( k \) as functions of the sample size \( l \). These values \( k \) are given only up to a rough asymptotic equivalence. For instance, \( k \) can be multiplied by any positive constant and the Theorems remain valid. In practice, one needs exact optimal values of \( k \) which are adapted to the empirical data. Therefore, we subsequently consider a data-dependent selection of \( k \).

### 2.2. Selection of \( k \) by a plot.

As a practical tool of the visual selection of \( k \) the plot of the histogram-type estimate against \( k \) may be used. The example of similar approach is the Hill’s plot for the tail index selection.

The idea of the plot is based on uniform convergence of estimate (4) to the true rf when \( k \) increases as \( l \to \infty \). Then one can select for the fixed \( t \) the minimal \( k \) corresponded to the stable interval of the plot, i.e.

\[
k^* = \arg \min \{k : \tilde{H}(t, k, l) = \tilde{H}(t, k + 1, l), k = 1, ..., l - 1.\}
\]
In Figure 1 one can see the plots of the histogram-type estimate (4) against $k$ for different fixed time intervals $[0, t]$, $t \in \{1, 3, 5, 10\}$. The Weibull distribution with the shape parameter $s = 3$ and the sample size $l = 50$ are considered.

$H(t)$ is the true rf. For the fixed number of points $T$ and for the mentioned distributions the latter was taken from those tables presented in [2].

The results of the calculation are presented in the Tables 1 to 4. Frees’ results are included here, where $H_{3n}(t)$ denotes the estimate (2). Since (2) requires much computational effort, only $k \in \{5, 10\}$ and $l \leq 30$ were considered. For (4) the parameter $k$ is calculated by the plot method, i.e. by (13). The tables 1 to 4 show that for all estimates
Fig. 2. Histogram-type estimate of the renewal function against \( k \) for a Weibull(\( s = 3 \)) (top plot) and Gamma(\( s = 0.55, \lambda = 1 \)) (bottom plot) distributions and the corresponded rfs. \( k \) is selected by the plot. The values of rfs were taken from the tables, [2].

(a) the mean squared error increases as \( T \) increases;
(b) for any fixed \( T \) the mean squared error decreases as \( l \) becomes larger;
(c) the bias does not exhibit a stable behavior.

Comparing \( \hat{H}(t, k) \) with \( H_{3n}(t) \) one may conclude that

(a) the mean-squared error is less for \( \hat{H}(t, k) \) for sample size equals to 100.
(b) the bias of \( H_{3n}(t) \) is less than that of \( \hat{H}(t, k) \); irrespectively, the mean squared error of the latter tends to be less, especially for the smallest \( T \).

4. CONCLUSIONS AND DISCUSSION

In this paper we have developed and investigated a nonparametric histogram-type estimate of the rf which does not require any knowledge about the form of the inter-arrival time distribution.

Due to the limited number of empirical data the histogram-type estimate (as well as Frees’ estimate (2)) can be applied for closed time intervals \([0, t]\) with a relatively small \( t \). Compared to Frees’ estimate \( F_l^{(n)}(t) \) of the arrival-time distribution \( F^{*n}(t) \) in (2), the estimate (4) proposed here uses a simpler and rougher estimate of \( F^{*n}(t) \). The rf \( H(t) \) is approximated by a finite sum of estimates of the arrival-time distributions with \( k \) terms. The parameter \( k \) is selected to compensate the error of the risk function. The estimate (4)
Table 1. Part I: Gamma ($s = 0.55, \lambda = 1, \mathbb{E}\tau = 0.55$)

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<th>Size</th>
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<th>$H_{3n}(t)$ BIAS $10^4$</th>
<th>$H_{3n}(t)$ MSE $10^4$</th>
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Table 2. Part II: Gamma ($s = 0.55, \lambda = 1, \mathbb{E}\tau = 0.55$)

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### Table 3. Part I: Weibull ($s = 3$, $E_T = 0.89$)

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may be computed for sufficiently large \( l \) and \( k \) which is not realistic for (2). The Theorems 1 and 3 state both for heavy- and light-tailed inter-arrival time distributions (itds) those values of the parameter \( k \) as functions of the sample size which provide a.s. the uniform convergence of the histogram-type estimate (4) to the true rf for sufficiently small \( t \). It is proved that a smaller value of \( k \) \((k < l)\) than in [10] is sufficient to get a reliable estimate of the rf. In Theorems 2 and 4 the rates of the uniform convergence and confidence intervals of the rf for the classes of itds with the exponential and regularly varying tails are presented. But these Theorems determine \( k \) only up to a rough asymptotic equivalence. Such a value \( k \) does not depend on the empirical data. This feature may influence the accuracy of the estimation.

To estimate \( k \) by samples of a moderate size, the plot method is used. It has been shown that the histogram-type estimate (4) has a larger bias but a smaller mean squared error compared to Frees’ estimate (2).

In conclusion, a reliable computationally tractable histogram-type estimate of the renewal function has been investigated that is also applicable to heavy-tailed inter-arrival-time distributions.

Acknowledgements
The author is grateful to Prof. Marc Burger and Prof. Paul Embrechts for their kind assistance and invitation in ETH.

APPENDIX 1

Proof of Theorem 1:
By (5) we get for \( 0 \leq t \leq t_{\text{max}}(k) \):

\[
\sup_t |H(t) - \tilde{H}(t, k, l)| \leq \sup_t \sum_{n=k+1}^{\infty} \mathbb{P} \{ t_n < t \} + k \max_{1 \leq n \leq k} \sup_t \mathbb{P} \{ t_n < t \} - F_{l_n}(t) \]

Under the conditions of the Theorem it follows by well-known results that

\[
\mathbb{P} \left\{ \frac{t_n - n\mu}{\sigma \sqrt{n}} < t \right\} = \Phi(t) + \sum_{i=1}^{m-2} \frac{Q_i(t)}{n^{1/2}} + o \left( n^{-(m-2)/2} \right), \tag{14}
\]

holds uniformly in \( t \in (-\infty, \infty) \), where \( Q_i \) are expressions involving the density function \( \varphi(x) = (2\pi)^{-1/2} \exp\{-x^2/2\} \) of the standard normal df \( \Phi \), the Hermite polynomials and semi-invariants of \( \tau_i \) (see [7], Theorem 2.3.2, p.85; [20]). Particularly,

\[
Q_1(x) = \varphi(x) \frac{1-x^2}{6} \frac{\mathbb{E}(\tau_1 - \mu)^3}{\sigma^2}
\]

and for \( m = 3 \)

\[
\mathbb{P} \left\{ \frac{t_n - n\mu}{\sigma \sqrt{n}} < t \right\} = \Phi(t) + \frac{\mathbb{E}(\tau_1 - \mu)^3}{6\sigma^3 \sqrt{n}} (1 - t^2) \frac{1}{\sqrt{2\pi}} \exp\{-t^2/2\} + o \left( \frac{1}{\sqrt{n}} \right)
\]
holds. Hence, $P \{ t_n < t \}$ is defined like the right-hand side of (14) with the replacement of $t$ by $(t - n\mu)/(\sigma\sqrt{n})$.

Hence, $\sum_{n=1}^{\infty} P \{ t_n < t \}$ converges for $m \geq 3$ and

$$\sum_{n=k+1}^{\infty} P \{ t_n < t \} \leq c, \quad c > 0$$

holds. If

$$k \max_{1 \leq n \leq k} \sup_{t} |P \{ t_n < t \} - F_{i_n}(t)| \leq \eta$$

holds for any constant $\eta > 0$, then at $t \in [0, t_{\max}(k)]$,

$$\sup_{t} |H(t) - \tilde{H}(t, k, l)| \leq c + \eta$$

follows. Hence:

$$P \left\{ \sup_{t} |H(t) - \tilde{H}(t, k, l)| > c + \eta \right\} \leq P \left\{ k \max_{1 \leq n \leq k} \sup_{t} |P \{ t_n < t \} - F_{i_n}(t)| > \eta \right\}$$

The right-hand side may be estimated using the asymptotical estimate of the convergence rate of the empirical df to the true df [21]

$$P \left\{ \sup_{t} |P \{ t_n < t \} - F_{i_n}(t)| > \eta \right\} \leq 2 \exp \left( -2l_{n} \eta^{2} \right), \quad (16)$$

which is satisfied for sufficiently large $l$. Then it follows

$$P \left\{ \sup_{t} |H(t) - \tilde{H}(t, k, l)| > c + \eta \right\} \leq 2 \exp \left( -2l_{n} \eta^{2} \right) = P(\eta, l, k)$$

Since $k = o(l^p)$ holds, $0 < \rho < 1/3$ the series $\sum_{i=1}^{\infty} P(\eta, l, k)$ converges at least for one $\eta > 0$, and according to the Borel-Cantelli lemma the assertion of the Theorem follows.

**Proof of Theorem 2:**

Using (8) we have for $t \in [0, 1]$:

$$\sup_{t} |H(t) - \tilde{H}(t, k, l)| \leq (1 - \exp(-\nu))^{k+1} \exp(\nu) + k \max_{1 \leq n \leq k} \sup_{t} |P \{ t_n < t \} - F_{i_n}(t)|$$

and for $\alpha > 0$:

$$l^{\alpha} \sup_{t} |H(t) - \tilde{H}(t, k, l)| \leq l^{\alpha} (1 - \exp(-\nu))^{k+1} \exp(\nu) + l^{\alpha} k \max_{1 \leq n \leq k} \sup_{t} |P \{ t_n < t \} - F_{i_n}(t)|$$

Since $\rho < 1/3 - 2/3\alpha$, $0 < \alpha < 0.5$, $\rho > 0$, for sufficiently large $l$ and the corresponding $c(\nu)$ we get

$$l^{\alpha} (1 - \exp(-\nu))^{k+1} \exp(\nu) \leq 1,$$
therefore, if
\[ l^n k \max_{1 \leq n \leq k} \sup_t |\mathbb{P}\{t_n < t\} - F_{t_n}(t)| \leq \eta \]
for any constant $\eta > 0$, then it follows
\[ l^n \sup_t |H(t) - \tilde{H}(t, k, l)| \leq 1 + \eta. \]
Hence,
\[
\mathbb{P} \left\{ l^n \sup_t |H(t) - \tilde{H}(t, k, l)| > 1 + \eta \right\} < \mathbb{P} \left\{ l^n k \max_{1 \leq n \leq k} \sup_t |\mathbb{P}\{t_n < t\} - F_{t_n}(t)| > \eta \right\}.
\]
Using (16) we have:
\[
\mathbb{P} \left\{ l^n \sup_t |H(t) - \tilde{H}(t, k, l)| > 1 + \eta \right\} < 2 \exp \left( -2 \frac{l}{k^3} \left( \frac{\eta}{l^\alpha} \right)^2 \right) = P(\eta, l, k) \tag{17}
\]
Since $k = c(\nu) \cdot l^\rho$ and $\alpha + 1.5\rho < 0.5$ holds, the series $\sum_{i=1}^{\infty} P(\eta, l, k)$ converges at least for one $\eta > 0$, and according to the Borel-Cantelli lemma the assertion of the Theorem holds.

**Proof of Corollary 1:**
Let the right-hand side of (17) be equal to $0 < \chi < 1$:
\[ 2 \exp \left( -2 \frac{l}{k^3} \left( \frac{\eta}{l^\alpha} \right)^2 \right) = \chi \]
Hence, we have
\[ \eta = kl^n \sqrt{-\frac{k \ln(\chi/2)}{2l}}. \]
Then one gets the level of the confidence interval $D = (1 + \eta)l^{-\alpha}$.

**Proof of Theorem 3:**
Since for $t \in (0, \frac{a}{kn})$ expression (10) is valid for sufficiently large $n$,
\[
\sum_{n=k+1}^{\infty} \mathbb{P}\{t_n < t\} \sim \sum_{n=k+1}^{\infty} \Phi \left( \frac{t}{\sqrt{n}} \right)
\]
holds. The expansion at the right-hand side converges. Therefore, $\sum_{n=k+1}^{\infty} \mathbb{P}\{t_n < t\} < c$ follows, where $c$ is a constant. The rest of the proof is similar to the proof of Theorem 1.

**Proof of Theorem 4:** For $t \in [a, t_{\max}(k)]$, $a > 0$ we have from (11)
\[
\sup_t \frac{(F(t))^{k+1}}{1 - F(t)} = \sup_t \frac{(1 - L(t)t^{-\alpha})^{k+1}}{L(t)t^{-\alpha}} = \sup_t \frac{(1 - t^{-\alpha}c(t) \exp\{\int_t^{\infty} \frac{z(y)}{y} \, dy\})^{k+1}}{t^{-\alpha}c(t) \exp\{\int_t^{\infty} \frac{z(y)}{y} \, dy\}}.
\]
The mean value theorem implies

$$\exp\left\{ \int_{x_0}^{t} \frac{\varepsilon(y)}{y} dy \right\} = \exp\{\varepsilon(\theta) \ln \left( \frac{t}{x_0} \right) \} = \left( \frac{t}{x_0} \right)^{\varepsilon(\theta)},$$

for some $\theta \in [x_0, t]$. Hence,

$$\sup_t \frac{(F(t))^{k+1}}{1 - F(t)} = \sup_t \left( \frac{1 - c(t) t^{-\alpha + \varepsilon(\theta)x_0^{-\varepsilon(\theta)}}}{c(t) t^{-\alpha + \varepsilon(\theta)x_0^{-\varepsilon(\theta)}}} \right)^{k+1},$$

Then by $-\alpha + \varepsilon(\theta) < 0$ we have

$$\sup_t \frac{(F(t))^{k+1}}{1 - F(t)} = \left( \frac{1 - c_{inf}(t_{max}(k))^{-\alpha + \varepsilon(\theta)x_0^{-\varepsilon(\theta)}}}{c_{inf}(t_{max}(k))^{-\alpha + \varepsilon(\theta)x_0^{-\varepsilon(\theta)}}} \right)^{k+1},$$

(18)

where

$$c_{inf} = \inf_t c(t) = \begin{cases} c(t_{max}(k)), & \text{if } c(t) \text{ monotone decreasing function} \\ c(a), & \text{if } c(t) \text{ monotone increasing function}. \end{cases}$$

Since $c(t_{max}(k)) > c_0$ than $c_{inf} \geq \min(c_0, c(a)) = c^*$ and the right-hand side of (18) is less than

$$\left( \frac{1 - c^*(t_{max}(k))^{-\alpha + \varepsilon(\theta)x_0^{-\varepsilon(\theta)}}}{c^*(t_{max}(k))^{-\alpha + \varepsilon(\theta)x_0^{-\varepsilon(\theta)}}} \right)^{k+1},$$

(19)

Assume, that

$$\max_{i} \tau_i \leq l^{\eta}$$

(20)

holds, where $\eta > 1/(\alpha - \varepsilon^*)$. Then

$$t_{max}(k) \leq l \max_{i=1,\ldots,l} \tau_i < l^{1+\eta}.$$ 

It implies, that (19) is less or equal than

$$\left( \frac{1 - c^*(l(1+\eta)(-\alpha + \varepsilon(\theta)))^{-\varepsilon(\theta)}}{c^*(l(1+\eta)(-\alpha + \varepsilon(\theta)))^{-\varepsilon(\theta)}} \right)^{k+1}.$$ 

Then from (18) we have

$$\sup_l \frac{l^\beta (F(t))^{k+1}}{1 - F(t)} \leq l^\beta \left( \frac{1 - c^*(l(1+\eta)(-\alpha + \varepsilon(\theta)))^{-\varepsilon(\theta)}}{c^*(l(1+\eta)(-\alpha + \varepsilon(\theta)))^{-\varepsilon(\theta)}} \right)^{k+1},$$

Since $(1 + \eta)(-\alpha + \varepsilon(\theta)) < 0$ then for sufficiently large $l$ the right-hand side of the latter inequality is less or equal to 1 for $k = -A \cdot l^\rho$ and $\rho \geq 0$. Note, that for $\beta > 0$ and sufficiently large $l$ $A < 0$ holds.

Therefore, if

$$l^\beta \max_{1 \leq n \leq k} \sup_t |\mathbb{P}(t_n < t) - F_{l_n}(t)| \leq \chi$$
for any constant $\chi > 0$, it follows from (5), (6)

$$l^3 \sup_t \|H(t) - \bar{H}(t, k, l)\| \leq 1 + \chi$$

Hence, from (20)

$$\mathbb{P}\{l^3 \sup_t \|H(t) - \bar{H}(t, k, l)\| > 1 + \chi\} < \mathbb{P}\{l^3 k \max_{1 \leq n \leq k} \mathbb{P}\{T_n < t\} - F_n(t)\| > \chi\} + \mathbb{P}\{\max_i \tau_i > l^\eta\}$$

holds. By the global property of the regularly varying rvs (see [7], p.38, [18]) we have

$$\mathbb{P}\{\max_i \tau_i > x\} \sim l\mathbb{P}\{\tau_i > x\} = l x^{-\alpha} L(x) \quad \text{as} \quad x \to \infty$$

The following property of slowly varying functions is often used: for all $\varepsilon^* > 0$ such $T$ exists, that for $x > T$

$$x^{-\varepsilon^*} \leq L(x) \leq x^{\varepsilon^*}$$

holds. Let $\varepsilon^* < \alpha$. Hence,

$$\mathbb{P}\{\max_i \tau_i > l^\eta\} \sim l^{1+\eta(\varepsilon^*-\alpha)} \quad \text{as} \quad l \to \infty.$$  

Using (16) we finally have

$$\mathbb{P}\{l^3 \sup_t \|H(t) - \bar{H}(t, k, l)\| > 1 + \chi\} < l^{1+\eta(\varepsilon^*-\alpha)} + 2 \exp\left(-2\chi^2 l^{1-2\beta}/k^3\right) = P(\eta, l, k)$$  

(21)

Since $k = dl^\beta$ and $0 < \rho < 1/3 - 2/3\beta$, $\eta > 1/((\alpha - \varepsilon^*)$ hold, the series $\sum_{l=1}^{\infty} P(\eta, l, k)$ converges at least for one $\eta > 0$, and according to the Borel-Cantelli lemma the assertion of the Theorem holds.

**Proof of Corollary 2:**

Let the right-hand side of (21) be equal to $0 < \nu < 1$:

$$l^{1-\eta(\alpha-\varepsilon^*)} + 2 \exp\left(-2\chi^2 l^{1-2\beta}/k^3\right) = \nu$$

Hence, we have

$$\chi = kl^\beta \frac{k}{2l} \ln \left( \frac{\nu - l^{1-\eta(\alpha-\varepsilon^*)}}{2} \right).$$

Then one gets the level of the confidence interval $D = (1 + \chi)l^{-\beta}$.

**REFERENCES**


