A sharp Trudinger - Moser type inequality 
for unbounded domains in $\mathbb{R}^2$

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Abstract

The classical Trudinger-Moser inequality says that for functions with Dirichlet norm smaller or equal to 1 in the Sobolev space $H^1_0(\Omega)$ (with $\Omega \subset \mathbb{R}^2$ a bounded domain), the integral $\int_{\Omega} e^{4\pi u^2} dx$ is uniformly bounded by a constant depending only on $\Omega$. If the volume $|\Omega|$ becomes unbounded then this bound tends to infinity, and hence the Trudinger-Moser inequality is not available for such domains (and in particular for $\mathbb{R}^2$).

In this paper we show that if the Dirichlet norm is replaced by the standard Sobolev norm, then the supremum of $\int_{\Omega} e^{4\pi u^2} dx$ over all such functions is uniformly bounded, independently of the domain $\Omega$. Furthermore, a sharp upper bound for the limits of Sobolev normalized concentrating sequences is proved for $\Omega = B_R$, the ball or radius $R$, and for $\Omega = \mathbb{R}^2$. Finally, the explicit construction of optimal concentrating sequences allows to prove that the above supremum is attained on balls $B_R \subset \mathbb{R}^2$ and on $\mathbb{R}^2$.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ denote a bounded domain. The Sobolev imbedding theorem states that $H^1_0(\Omega) \subset L^p(\Omega)$, for $1 \leq p \leq 2^* = \frac{2N}{N-2}$, or equivalently, using the Dirichlet norm $\|u\|_D = (\int_{\Omega} |\nabla u|^2 dx)^{1/2}$ on $H^1_0(\Omega)$,

$$\sup_{\|u\|_D \leq 1} \int_{\Omega} |u|^p dx < +\infty, \quad \text{for } 1 \leq p \leq 2^*,$$

while this supremum is infinite for $p > 2^*$. The maximal growth $|u|^{2^*}$ is called “critical” Sobolev growth. In the case $N = 2$, every polynomial growth is admitted, but one knows by easy examples that $H^1_0(\Omega) \not\subset L^\infty(\Omega)$. Hence, one is led to look for a function $g(s) : \mathbb{R} \to \mathbb{R}^+$ with maximal growth such that

$$\sup_{\|u\|_D \leq 1} \int_{\Omega} g(u) dx < +\infty.$$

It was shown by Pohozaev [12], Trudinger [14] and Moser [11] that the maximal growth is of exponential type. More precisely, the Trudinger-Moser inequality states that for $\Omega \subset \mathbb{R}^2$ bounded

$$\sup_{\|u\|_D \leq 1} \int_{\Omega} (e^{\alpha u^2} - 1) dx = c(\Omega) < +\infty \quad \text{for } \alpha \leq 4\pi,$$

The inequality is optimal: for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the corresponding supremum is $+\infty$.

The supremum (1.1) becomes infinite for domains $\Omega$ with $|\Omega| = \infty$, and therefore the Trudinger-Moser inequality is not available for unbounded domains. Related inequalities for
unbounded domains have been proposed by Cao [5] and Tanaka [2], however they assume a growth $e^{\alpha u^2}$ with $\alpha < 4\pi$, i.e. with subcritical growth.

In this paper we show that replacing the Dirichlet norm $\|u\|_D = (\int_{\Omega} |\nabla u|^2 dx)^{1/2}$ by the standard Sobolev norm on $H^1_0(\Omega)$, namely

$$\|u\|_S = \left( \|u\|_D^2 + \|u\|_{L^2}^2 \right)^{1/2} = \left( \int_{\Omega} (|\nabla u|^2 + |u|^2) dx \right)^{1/2}$$

yields a bound independent of $\Omega$. More precisely, we prove

**Theorem 1.1** There exists a constant $d > 0$ such that for any domain $\Omega \subset \mathbb{R}^2$

$$\sup_{\|u\|_S \leq 1} \int_{\Omega} (e^{4\pi u^2} - 1) dx \leq d$$

The inequality is sharp: for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the supremum is $+\infty$.

In an interesting paper, L. Carleson and A. Chang [6] proved that the supremum in (1.1) is attained if $\Omega = B_1(0)$, the unit ball in $\mathbb{R}^2$. This result was extended to arbitrary bounded domains in $\mathbb{R}^2$ by M. Flucher [9]. In their proof, Carleson and Chang used a "concentration-compactness" argument. They consider "normalized concentrating sequences", i.e. normalized (in the Dirichlet norm) sequences which converge weakly to 0 and (being radial) blow up at the origin. They showed that for any such sequence $\{u_n\}$ one has

$$\lim_{n \to \infty} \int_{B_1(0)} (e^{4\pi u_n^2} - 1) dx \leq e |B_1|$$

Hence, one may say that $e |B_1|$ is the highest possible "concentration" or "non-compactness" level (see also P.L. Lions [10], and H. Brezis - L. Nirenberg [3] for the related situation for Sobolev embeddings). Carleson and Chang went on to show that

$$\sup_{\|u\|_D \leq 1} \int_{B_1} (e^{4\pi u^2} - 1) dx > e |B_1|$$

and hence, since no concentration can happen at a level above $e |B_1|$, they concluded that the supremum in (1.1) is attained.

Let us call the maximal limit in (1.4) the Carleson-Chang limit, in symbol: cc-lim. In [7] an explicit normalized concentrating sequence $\{y_n\}$ with

$$\lim_{n \to -\infty} \int_{B_1} (e^{4\pi y_n^2} - 1) dx = \text{cc-lim} \sup_{\|u\|_D \leq 1} \int_{B_1} (e^{4\pi u^2} - 1) dx = e |B_1|$$

was constructed.

In this paper we analyze the corresponding Carleson-Chang limit for concentrating sequences which are normalized in the Sobolev norm. We will show
Theorem 1.2
1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let $R > 0$ such that $|\Omega| = |B_R|$. Then

\begin{equation}
\underset{\|u_n\|_{S} \leq 1}{\text{cc-lim}} \int_{\Omega} (e^{4\pi u_n^2} - 1) dx \leq \pi e^{1-D(R)} ,
\end{equation}

where

\[ D(R) = 2K_0(R)[2RK_1(R) - 1/I_0(R)] > 0 \, , \text{ with } \lim_{R \to +\infty} D(R) = 0 . \]

Here, $I_k(x)$ and $K_k(x)$ denote the $k$-th modified Bessel functions of the first and second kind, i.e. the solutions of the equation

\[-x^2u''(x) - xu'(x) + (x^2 + k^2)u(x) = 0 \, , \, k = 0, 1, 2, ...\]

2. Let $\Omega \subseteq \mathbb{R}^2$ be an arbitrary domain. Then

\begin{equation}
\underset{\|u_n\|_{S} \leq 1}{\text{cc-lim}} \int_{\Omega} (e^{4\pi u_n^2} - 1) dx \leq \pi e .
\end{equation}

3. The bound in (1.7) is sharp for $\Omega = B_R(0)$, and the bound in (1.8) is sharp for $\Omega = \mathbb{R}^2$.

It is remarkable that for $\Omega = B_1(0)$ with Dirichlet normalization and for $\Omega = \mathbb{R}^2$ with Sobolev normalization the corresponding Carleson-Chang limits coincide, that is

\[
\underset{\|u_n\|_{D} \leq 1}{\text{cc-lim}} \int_{B_1} (e^{4\pi u_n^2} - 1) dx = \underset{\|u_n\|_{S} \leq 1}{\text{cc-lim}} \int_{\mathbb{R}^2} (e^{4\pi u_n^2} - 1) dx = e \pi .
\]

In the final result of the paper we prove

**Theorem 1.3** For any ball $\Omega = B_R(0)$ and for $\Omega = \mathbb{R}^2$ holds

\begin{equation}
\sup_{\|u\|_{S} \leq 1} \int_{\Omega} (e^{4\pi u^2} - 1) dx > e^{1-D(R)} \pi .
\end{equation}

This implies in particular that the supremum (1.9) is attained in the cases of $\Omega = B_R(0)$ and $\Omega = \mathbb{R}^2$. 

2 A uniform bound

In this section we prove Theorem 1.1. We begin with

**Proposition 2.1** Let $\Omega \subset \mathbb{R}^2$ denote a domain in $\mathbb{R}^2$, and let $H^1_0(\Omega)$ denote the standard Sobolev space equipped with the norm

\[ \|u\|_{S} = \left( \int_{\Omega} (|\nabla u|^2 + |u|^2) dx \right)^{1/2} \]

Then there exists a constant $d$ (independent of $\Omega$) such that

\begin{equation}
\sup_{\|u\|_{S} \leq 1} \int_{\Omega} (e^{4\pi u^2} - 1) dx \leq d .
\end{equation}
Proof. It is clear that

\[ \sup_{\| u \| \leq 1} \int_{\Omega} (e^{4\pi u^2} - 1) \, dx \leq \sup_{\| u \| \leq 1} \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1) \, dx \]

since any function \( u \in H^1_0(\Omega) \) can be extended by zero outside of \( \Omega \), obtaining a function in \( (H^1(\mathbb{R}^2), \| \cdot \|_{S}) \). Hence, it is sufficient to show that

\[ \sup_{\| u \| \leq 1} \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1) \, dx \leq d \]

(2.3)

We use symmetrization (see e.g. J. Moser [11]) by defining the radially symmetric function \( u^* \) as follows:

for every \( \rho > 0 \) let

\[ m(\{ x \in \mathbb{R}^2 ; \ u^*(x) > \rho \}) = m(\{ x \in \mathbb{R}^2 ; \ u(x) > \rho \}) . \]

Then \( u^* \) is a non-increasing function in \( |x| \). By construction

\[ \int_{\mathbb{R}^2} (e^{4\pi |u^*|^2} - 1) \, dx = \int_{|x| \leq r_0} (e^{4\pi u^2} - 1) \, dx \quad \text{and} \quad \int_{\mathbb{R}^2} |u^*|^2 \, dx = \int_{\mathbb{R}^2} |u|^2 \, dx \]

and it is known that

\[ \int_{\mathbb{R}^2} |\nabla u^*|^2 \leq \int_{\mathbb{R}^2} |\nabla u|^2 \, dx . \]

It is therefore sufficient to prove (2.3) for radially symmetric functions \( u(x) = u(|x|) \).

Thus, we may assume that \( u \) in (2.3) is radially symmetric and non-increasing. We divide the integral (2.3) into two parts, with \( r_0 > 0 \) to be chosen:

\[ \int_{|x| \geq r_0} (e^{4\pi u^2} - 1) = \int_{|x| \leq r_0} (e^{4\pi u^2} - 1) + \int_{|x| \geq r_0} (e^{4\pi u^2} - 1) \]

(2.4)

We write the second integral as

\[ \int_{|x| \geq r_0} (e^{4\pi u^2} - 1) = \sum_{k=1}^{\infty} \int_{|x| \geq r_0} (4\pi)^k |u|^{2k} \frac{k!}{k} \]

(2.5)

We estimate the single terms by the following ”radial lemma” (see Berestycki - Lions, [4], Lemma A.IV):

\[ |u(r)| \leq \frac{1}{\sqrt{\pi}} \| u \|_{L^2} \frac{1}{r} , \quad \text{for all } r > 0 , \]

(2.6)

Hence we obtain for \( k \geq 2 \):

\[ \int_{|x| \geq r_0} |u|^{2k} \leq \| u \|_{L^2}^{2k} \frac{2}{\pi^{k-1}} \int_{r_0}^{\infty} \frac{1}{r^{2k}} \, r \, dr = \frac{1}{k-1} \| u \|_{L^2}^{2k} \left( \frac{\| u \|_{L^2}^2}{\pi r_0^2} \right)^{k-1} . \]

(2.7)

This yields

\[ \int_{|x| \geq r_0} (e^{4\pi u^2} - 1) \leq 4\pi \| u \|_{L^2}^2 + 4\pi \| u \|_{L^2}^2 \sum_{k=2}^{\infty} \frac{k!}{k} \left( \frac{4\| u \|_{L^2}^2}{r_0^2} \right)^{k-1} \leq c(r_0) , \]

(2.8)

since \( \| u \|_{L^2} \leq 1 \).
To estimate the first integral in (2.4), let

$$ v(r) = \begin{cases} 
  u(r) - u(r_0), & 0 \leq r \leq r_0 \\
  0, & r \geq r_0 
\end{cases} $$

Then, by (2.6)

$$ u^2(r) = v^2(r) + 2v(r)u(r_0) + u^2(r_0) $$

(2.9)

$$ \leq v^2(r) + v^2(r) \frac{1}{\pi r_0^2} \|u\|_{L^2}^2 + 1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2 $$

$$ \leq v^2(r) \left[ 1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2 \right] + d(r_0) $$

hence

$$ u(r) \leq v(r) \left( 1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2 \right)^{1/2} + d^{1/2}(r_0) =: w(r) + d^{1/2}(r_0) $$

By assumption

$$ \int_{B_{r_0}} |\nabla v|^2 dx = \int_{B_{r_0}} |\nabla u|^2 dx \leq 1 - \|u\|_{L^2}^2 $$

and hence

$$ \int_{B_{r_0}} |\nabla w|^2 dx = \int_{B_{r_0}} |\nabla v(1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2)^{1/2}|^2 dx $$

(2.10)

$$ = (1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2) \int_{B_{r_0}} |\nabla u|^2 dx $$

$$ \leq (1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2)(1 - \|u\|_{L^2}^2) $$

$$ = 1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2 - \|u\|_{L^2}^2 \leq 1 $$

provided that $r_0^2 \geq \frac{1}{\pi}$. Since by (2.9) $u^2(r) \leq w^2(r) + d$ we get

$$ \int_{|x| \leq r_0} (e^{4\pi u^2} - 1) dx \leq e^{4\pi d} \int_{B_{r_0}} e^{4\pi u^2} dx $$

The result follows by the Trudinger-Moser inequality, since $w \in H^1_0(B_{r_0})$ with $\|w\|^2_D = \int_{B_{r_0}} |\nabla w|^2 dx \leq 1$.

In the next proposition we show that the result is optimal (as in the Dirichlet-norm case), namely that the supremum in (2.1) becomes infinite if the exponent $4\pi$ is replaced by a number $\alpha > 4\pi$.

**Proposition 2.2** Suppose that $\alpha > 4\pi$. Then, for any domain $\Omega \subseteq \mathbb{R}^2$

(2.11)

$$ \sup_{\|u\| \leq 1} \int_{\Omega} (e^{\alpha u^2} - 1) dx = +\infty. $$

**Proof.**
We may suppose that $0 \in \Omega$, and that for some $\rho > 0$ the ball $B_\rho(0) \subset \Omega$. We use a modified "Moser-sequence", see [11], defined in $B_\rho(0)$ and continued by zero in $\Omega \setminus B_\rho(0)$, and with Sobolev-norm $\leq 1$:

$$m_n(x) = \frac{1}{\sqrt{2\pi}} \left\{ \begin{array}{ll}
\frac{\log(\rho/|x|)}{\log n} (1 - \frac{\rho^2}{4\log n})^{1/2}, & \frac{\rho}{n} \leq |x| \leq \rho \\
\log n (1 - \frac{\rho^2}{4\log n})^{1/2}, & 0 \leq |x| \leq \rho/n
\end{array} \right. \quad (2.12)$$

One checks that $\|m_n\|_{H^1_0(\Omega)}^2 \leq 1$, for $n$ large. Hence one has

$$\sup_{\|u\|_S \leq 1} \int_\Omega (e^{4\pi u^2_n} - 1) \, dx \geq \lim_{n \to \infty} \int_{B_\rho} (e^{\alpha m^2_n} - 1) \, dx \geq 2\pi \int_0^{\rho/n} \left( e^{\frac{\alpha}{2\pi} \log n [1 - \frac{\rho^2}{4\log n}]} - 1 \right) \, r \, dr = 2\pi \left( n \frac{\alpha}{2\pi} e^{\frac{\alpha}{2\pi} - 1} \right) \frac{\rho^2}{2} \to +\infty, \text{ as } n \to \infty \quad (2.13)$$

3 Critical growth and concentration

Numerous studies in recent years have shown the close connection of critical growth with concentration phenomena, see e.g. the pioneering work of H. Brezis - L. Nirenberg [3].

As pointed out in the introduction, it is of particular interest to study the "highest level of noncompactness" for the functional $\int_\Omega (e^{4\pi u^2_n} - 1) \, dx$, under the restriction $\|u\|_S \leq 1$. In view of this, we make the following definition:

**Definition 3.1** A sequence $\{u_n\} \subset H^1_0(\Omega)$ is a Sobolev-normalized concentrating sequence (for short, SNC-sequence), if

a) $\|u_n\|_S = 1$

b) $u_n \rightharpoonup 0$, weakly in $H^1_0(\Omega)$

c) $\exists \, x_0 \in \Omega$ such that $\forall \rho > 0 : \int_{\Omega \setminus B_\rho(x_0)} (|\nabla u_n|^2 + |u_n|^2) \, dx \to 0$

Next, we define the Carleson-Chang limit as the maximal limit of SNS-sequences:

**Definition 3.2** Let

$$\Sigma := \{ \{u_n\} \subset H^1_0(\Omega) \mid \{u_n\} \text{ is a SNC-sequence} \} ,$$

and define the Carleson-Chang limit as

$$\text{cc-lim} \sup_{\|u_n\|_S \leq 1} \int_\Omega (e^{4\pi u^2_n} - 1) \, dx := \sup_{\Sigma} \limsup_{n \to \infty} \int_\Omega (e^{4\pi u^2_n} - 1) \, dx .$$

The following "concentration-compactness alternative" by P.L. Lions (restated in our notation) is relevant for our purposes:

**Proposition** (P.L. Lions, [10], Theorem I.6). Let $\{u_n\} \subset H^1_0(\Omega)$ satisfy $\|u_n\|_S \leq 1$; we may assume that $u_n \rightharpoonup u$. Then either
\{u_n\} is a SNC-sequence or
\[ \int_{\Omega}(e^{4\pi u_n^2} - 1)dx \to \int_{\Omega}(e^{4\pi u^2} - 1)dx; \text{ this holds in particular if } u \neq 0. \]

Then one has

**Proposition 3.3** Suppose that
\[
S := \sup_{\|u\|_S \leq 1} \int_{\Omega}(e^{4\pi u^2} - 1)dx > \lim_{\|u_n\|_S \to 1} \int_{\Omega}(e^{4\pi u_n^2} - 1)dx.
\]

Then the supremum \(S\) is attained.

**Proof.** Let \(\{y_n\}\) denote a maximizing sequence for \(S\), and assume that \(S\) is not attained. We may assume that \(y_n \rightharpoonup y\). By the alternative of P.L. Lions we get \(y = 0\), and \(\{y_n\}\) is a SNC-sequence. Hence
\[
S = \lim_{n \to \infty} \int_{\Omega}(e^{4\pi y_n^2} - 1)dx \leq \lim_{\|u_n\|_S \to 1} \int_{\Omega}(e^{4\pi u_n^2} - 1)dx < S
\]
Contradiction! □

### 4 Upper bound for the Carleson-Chang limit

In this section we prove an explicit upper bound for the Carleson-Chang limit. In particular, we prove the estimates (1.7) and (1.8) of Theorem 1.2. In section 7 we will show that the bound in (1.7) is sharp for \(\Omega = B_R\), with any radius \(R > 0\), and the bound in (1.8) is sharp for \(\Omega = \mathbb{R}^2\).

**Proof.**

1. Using symmetrization as in section 2, we see that it is sufficient to prove (1.7) for radial functions in \(B_R(0)\). Following J. Moser [11] we perform the change of variables

\[
r = e^{-t/2}, \quad \text{and setting } w_n(t) = (4\pi)^{1/2} y_n(r),
\]

we transform the radial integrals on \([0,R]\) into integrals on the half-line \([-2 \log R, +\infty)\). We will write throughout the paper: \(\alpha_R = -2 \log R\), with \(\alpha_R = -\infty\) if \(R = +\infty\). One checks that
\[
\int_{B_R} |\nabla y_n(x)|^2dx = 2\pi \int_0^R \left| \frac{d}{dr} y_n(r) \right|^2 r dr = \int_{\alpha_R}^{\infty} \left| w_n'(t) \right|^2 dt
\]
and
\[
\int_{B_R} (e^{4\pi y_n^2(x)} - 1)dx = 2\pi \int_0^R (e^{4\pi y_n^2(r)} - 1)r dr = \pi \int_{\alpha_R}^{\infty} (e^{w_n^2(t)} - 1)e^{-t}dt
\]
and similarly
\[
\int_{B_R} |y_n(x)|^2dx = 2\pi \int_0^R |y_n(r)|^2 r dr = \frac{1}{4} \int_{\alpha_R}^{\infty} \left| w_n(t) \right|^2 e^{-t} dt.
\]

The SNC-sequences in this new setting are characterized by:

a) \(\|w_n\|_S^2 := \int_{\alpha_R}^{\infty} (|w_n'|^2 + \frac{1}{4}|w_n|^2 e^{-t})dt = 1\), \(w_n(\alpha_R) = 0\)

b) \(w_n \rightharpoonup 0\), weakly in \(H^1([\alpha_R, +\infty))\)
and the estimate (1.7) (which we seek to prove) becomes

\[ \text{cc-\textit{lim}} \pi \int_{\alpha_R}^{\infty} (e^{w_n^2(t)} - 1) e^{-t} dt \leq e^{1-D(R)} \]

for SNC-sequences \( \{w_n\} \subset H^1([\alpha_R, +\infty)) \).

Let now denote \( \{w_n\} \) a maximizing SNC-sequence for the Carleson-Chang limit (1.7). We may assume that the sequence \( \{w_n\} \) satisfies

\[ \text{lim}_{n \to \infty} \pi \int_{\alpha_R}^{\infty} (e^{w_n^2(t)} - 1) e^{-t} dt > 2 \pi e^{-D(R)} , \]

since otherwise the theorem is proved. Note that we may assume that \( w_n(t) \) is an increasing function on \( [\alpha_R, +\infty) \). Fix \( A_R \geq 1 \) such that

\[ t - 2 \log t - D(R) > 1 , \forall t \geq A_R . \]

\textbf{Claim 1:} There exists a number \( n_1 \) such that

\[ w_n(t) < 1 , \forall t \leq A_R , \forall n \geq n_1 \]

Indeed, for \( 0 < R < +\infty \) we can estimate

\[ w_n(t) \leq (A_R + 2 \log R)^{1/2} \left( \int_{\alpha_R}^{A_R} |w_n'|^2 dt \right)^{1/2} \]

\[ =: (A_R + 2 \log R)^{1/2} \delta_n , \text{ for } t \leq A_R , \]

with \( \delta_n \to 0 \) as \( n \to 0 \), by c).

For \( R = +\infty \) and \( 0 < t \leq A_R \) we estimate

\[ w_n(t) = w_n(0) + \int_0^t w'(t) dt \leq w_n(0) + t^{1/2} \left( \int_0^t |w_n'|^2 \right)^{1/2} dt \]

The second term goes to zero, as above. For the estimate of \( w_n(0) \) we use the following Radial Lemma (see W. Strauss, [13]), valid for radial functions \( v(r) \) in \( H^1(\mathbb{R}^2) \) and for \( r \geq 1 \):

\[ (r + \frac{1}{2}) v^2(r) \leq \frac{5}{4} \int_r^\infty (|v'|^2 + |v|^2) \rho d\rho \]

We transform this inequality (as before) by the change of variables \( r = e^{-t/2} \) and \( w(t) = (4\pi)^{1/2} v(r) \) and get, for \( t \leq 0 \):

\[ (e^{-t/2} + \frac{1}{2}) w^2(t) \leq \frac{5}{2} \int_{-\infty}^{-t/2} (|w'(t)|^2 + \frac{1}{4} |w(t)|^2 e^{-t} ) dt . \]

Hence, we get for \( w_n(0) \), using the concentration property of \( w_n \)

\[ w_n^2(0) \leq \frac{5}{3} \int_{-\infty}^{0} (|w'(t)|^2 + \frac{1}{4} |w(t)|^2 e^{-t} ) dt =: \sigma_n^2 \to 0 , \text{ as } n \to \infty . \]
Thus the claim is proved.

By claim 1 we conclude that for \( n \) sufficiently large (\( 0 < R \leq +\infty \))

\[
w_n^2(t) < 1 < A_R - 2\log A_R - D(R), \quad \alpha_R \leq t \leq A_R.
\]

Let now \( a_n > A_R \) denote the first \( t > A_R \) with

\[
w_n^2(a_n) = a_n - 2\log a_n - D(R).
\]

Such an \( a_n \) exists (for \( n \) sufficiently large), since otherwise

\[
w_n^2(t) < t - 2\log t - D(R), \quad \forall t \geq A_R \geq 1, \quad \text{as } n \to \infty,
\]

and thus

\[
\pi \int_{\alpha_R}^{\infty} (e^{w_n^2} - 1)e^{-t} dt \leq \pi \int_{\alpha_R}^{A_R} (e^{w_n^2} - 1)e^{-t} + \pi \int_{A_R}^{\infty} e^{t-2\log t-D(R)-t}
\]

The second term on the right is bounded by \( \pi e^{-D(R)} \), and in the following claim 2 we prove that the first term goes to 0, for \( n \to \infty \), and thus we have a contradiction to assumption (4.5).

**Claim 2:**

\[
\pi \int_{\alpha_R}^{A_R} (e^{w_n^2} - 1)e^{-t} \to 0 \text{ as } n \to \infty.
\]

This is immediate for \( 0 < R < +\infty \), since then this term can be estimated, using (4.7), by

\[
\pi (R^2 - e^{-A_R})(e^{R^2(A+\alpha R)} - 1) \to 0 \text{ as } n \to \infty.
\]

If \( R = +\infty \) we write

\[
\int_{-\infty}^{0} (e^{w_n^2} - 1)e^{-t} dt + \int_{0}^{A_R} (e^{w_n^2} - 1)e^{-t} dt
\]

The second term is now estimated as before, while for the first term we use a series expansion:

\[
\int_{-\infty}^{0} (e^{w_n^2} - 1)e^{-t} dt = \int_{-\infty}^{0} \sum_{k=1}^{\infty} \frac{|w_n(t)|^{2k}}{k!} e^{-t} dt
\]

\[
= \int_{-\infty}^{0} |w_n(t)|^2 e^{-t} dt + \int_{-\infty}^{0} \frac{1}{2} |w_n(t)|^4 e^{-t} dt + \sum_{k=3}^{\infty} \int_{-\infty}^{0} \frac{|w_n(t)|^{2k}}{k!} e^{-t} dt
\]

The first term goes to zero by concentration, the second term can be estimated by Sobolev (by returning to the variable \( r \) and back to \( t \))

\[
\int_{-\infty}^{0} w_n^4 e^{-t} dt \leq c_0 \left( \int_{-\infty}^{0} (|w_n|^2 + \frac{1}{4} |w_n|^2 e^{-t}) dt \right)^2
\]

and hence also goes to zero by concentration. For the third term, observe that by (4.8) we get for \( t \leq 0 \)

\[
w_n^2(t) \leq \frac{5}{4} e^{-t/2 + 1/2} \sigma_n^2 \leq c e^{t/2} \sigma_n^2
\]

Hence we can estimate the series as

\[
\sum_{k=3}^{\infty} \int_{-\infty}^{0} \frac{c^k}{k!} \sigma_n^{2k} e^{k t/2} e^{-t} dt \leq \sum_{k=3}^{\infty} c^k \sigma_n^{2k} \int_{-\infty}^{0} e^{t/2} dt \leq c_1 \sigma_n^6 2,
\]
and thus claim 2 is proved.

Thus we have proved the existence of a number \( a_n > A_R \) as claimed in (4.9).

We now prove, for \( 0 < R \leq +\infty \)

i) \[ \pi \int_{a_R}^{a_n} (e^{w_n^2} - 1)e^{-t}dt \to 0, \text{ as } n \to \infty. \]

ii) \[ \lim_{n \to \infty} \pi \int_{a_n}^{\infty} (e^{w_n^2} - 1)e^{-t}dt \leq \pi e^{1-D(R)} \]

Proof of i): Note that the argument above shows that \( a_n \to +\infty \) as \( n \to \infty \), since for an arbitrarily large number \( A_R \) there exists \( n_0(A_R) \) such that \( a_n > A_R \) for \( n \geq n_0 \). By (4.9) we have

\[
\pi \int_{a_R}^{a_n} (e^{w_n^2} - 1)e^{-t}dt \leq \int_{a_R}^{A} (e^{w_n^2} - 1)e^{-t}dt + \pi \int_{A}^{a_n} e^{-2\log t-D(R)}dt.
\]

Let \( \epsilon > 0 \): for the second term we get \( \pi e^{-(D(R)(1-1/A))} < \epsilon/2 \), for \( A \) sufficiently large, and then the first term becomes \( \leq \epsilon/2 \), for \( n \geq n_0(A, \epsilon) \), proceeding as in Claim 2.

Proof of ii): We apply the following basic estimate which was proved in [6] (we cite it here in the form given in [7], Proposition 2.2):

**Lemma** (Carleson-Chang): For \( a > 0 \) and \( \delta > 0 \) given, suppose that

\[
\int_{a}^{\infty} |w'(t)|^2dt \leq \delta.
\]

Then

\[
\int_{a}^{\infty} e^{w^2-t}dt \leq e^{1-\delta}e^K, \quad \text{with} \quad K = w^2(a)(1 + \frac{\delta}{1-\delta}) - a.
\]

We apply this Lemma to our sequence \( \{ w_n \} \), with \( a = a_n \) given in (4.9), and \( \delta = \delta_n = \int_{a_n}^{\infty} (|w_n'|^2 + \frac{1}{4}|w_n|^2 e^{-t})dt \). Furthermore, in the following section 5, (5.1) and section 6, Proposition 6.4, it is shown that:

For \( a > 0 \) and \( b > 0 \) given, let

\[
S_{a,b} = \{ u \in H^1(\alpha_R, a), \ u(\alpha_R) = 0, \ \int_{\alpha_R}^{a} (|u'|^2 + \frac{1}{4}|u|^2 e^{-t})dt = b \}.
\]

Then the supremum

\[
\sup\{ \|u\|^2_{\infty} : u \in S_{a,b} \}
\]

is attained by a function \( y \), with

\[
\|y\|_{\infty}^2 = y^2(a) = b(a - D(R)) + O\left(\frac{1}{a}\right).
\]

Thus, choosing \( a = a_n \) and \( b = b_n = 1 - \delta_n \) we get for \( w_n \in S_{a_n,b_n} \)

\[
w_n^2(a_n) \leq a_n - a_n\delta_n - D(R) + O(\delta_n) + O\left(\frac{1}{a_n}\right),
\]

which implies together with (4.9)

\[
\delta_n \leq \frac{2\log a_n}{a_n} + O\left(\frac{\log a_n}{a_n^2}\right).
\]
Thus we have for $K = K_n$ in the Lemma of Carleson and Chang

$$K_n = w_n^2(a_n)(1 + \frac{\delta_n}{1 - \delta_n}) - a_n \leq \left( a_n - a_n\delta_n - D(R) + O\left(\frac{\log a_n}{a_n}\right) \right) \left(1 + \delta_n + O(\delta_n^2)\right) - a_n$$

(4.11)

$$= -D(R) - \delta_n D(R) + O\left(\frac{\log a_n}{a_n}\right) + a_n O(\delta_n^2)$$

$$= -D(R) + O\left(\frac{(\log a_n)^2}{a_n}\right)$$

Hence we obtain by the Lemma of Carleson and Chang for any maximizing SNC-sequence $\{w_n\}$

$$\lim_{n \to \infty} \pi \int_a^\infty e^{w_n^2 - 1} e^{-t} dt \leq \lim_{n \to \infty} \pi e^{1 - \frac{1}{\delta_n} D(R)};$$

thus ii) is proved.

With i) and ii) we now easily complete the proof of the first statement of Theorem 1.2

2. It is clear that for $\Omega_0 \subset \Omega_1$ the corresponding cc-limits are increasing. Thus, it is sufficient to prove 2) for $\Omega = \mathbb{R}^2$; this corresponds to setting $R = +\infty$, which was included in the proof of 1).

5 An auxiliary variational problem

In this section we consider the following variational problem: Determine

$$\sup \{ \|u\|_\infty \mid u \in S_{a,b}\},$$

where

$$S_{a,b} = \left\{ u \in H^1(\alpha R, a) \mid u(\alpha R) = 0, \int_0^a \left( |u'|^2 + \frac{R^2}{4} |u|^2 e^{-t} \right) dt = b > 0 \right\}$$

Note that $S_{a,b} \subset L^\infty(\alpha R, a)$, with compact embedding, and hence it is easily seen that the supremum in (5.1) is attained: let $y_a \in S_{a,b}$ such that

$$\|y_a\|_\infty^2 = \sup \{ \|u\|_\infty^2 \mid u \in S_{a,b}\}.$$  

(5.2)

In order to determine the value of (5.2) we need to identify the maximizing function $y_a \in S_{a,b}$. The natural way to do this consists in deriving the Euler-Lagrange equation associated to (5.1), but we encounter the difficulty that the functional $y \mapsto \|y\|_\infty^2$ is not differentiable. However, this functional is convex, and hence its subdifferential exists. We briefly recall this notion, and then derive the Euler-Lagrange equation for (5.1). For the proofs of some of the results we refer to [8].

**Definition 5.1** Let $E$ be a Banach space, and $\psi : E \to \mathbb{R}$ continuous and convex. Then we denote by $\partial \psi(u) \subset E'$ the subdifferential of $\psi$ in $u \in E$, given by

$$\mu_u \in \partial \psi(u) \iff \psi(u + v) - \psi(u) \geq \langle \mu_u, v \rangle, \forall v \in E;$$

here $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $E$ and $E'$. An element $\mu_u \in \partial \psi(u)$ is called a subgradient of $\psi$ at $u$. 

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In [8], Lemma 2.2, it is proved that

**Lemma:** If \( \psi \) satisfies in addition

\[
\psi(x) \geq 0, \quad \forall x \in E, \text{ and } \psi(tx) = t^2 \psi(x), \quad \forall t \geq 0,
\]
then

\[
\mu \in \partial \psi(u) \iff \begin{cases} 
\langle \mu, u \rangle = 2 \psi(u) \\
\langle \mu, x \rangle \leq \langle \mu, u \rangle, \quad \forall x \in \psi^u = \{ x \in E; \psi(x) \leq \psi(u) \}
\end{cases}.
\]

Furthermore, by an easy variation of [8], Lemma 2.3 and Corollary 2.4, one has:

**Lemma 5.2** Suppose that \( \psi : E \to \mathbb{R} \) satisfies (5.3), and \( \phi \in C^1(E, \mathbb{R}) \) satisfies \( \langle \phi'(x), x \rangle = 2 \phi(x) \), \( \forall x \in E \). If \( y \in E \) is such that

\[
\psi(y) = \sup_{\{u \in E, \phi(u) = b\}} \psi(u),
\]
then

\[
\phi'(u) \in \frac{b}{\psi(u)} \partial \psi(u)
\]

**Proof.** The Euler-Lagrange equation

\[
\phi'(u) \in \lambda \partial \psi(u) \quad \text{for some } \lambda > 0
\]
is obtained as in [8], Lemma 2.3 and Corollary 2.4. The value

\[
\lambda = \frac{b}{\psi(u)}
\]
is found by testing (5.4) with \( u \):

\[
2b = 2\phi(u) = \langle \phi'(u), u \rangle = \lambda \langle \mu_u, u \rangle = \lambda 2\psi(u).
\]

We now apply Lemma 5.2 to our situation, and obtain

**Theorem 5.3** Let \( E = \{ v \in H^1(\alpha_R, a); v(\alpha_R) = 0 \} \), and consider

\[
\psi(u) = \| u \|_\infty^2 : E \to \mathbb{R}
\]
and

\[
\phi(u) = \int_{\alpha_R}^a (|u'(x)|^2 + \frac{1}{4} |u(x)|^2 e^{-x}) dx.
\]

Suppose that \( y \in E \) satisfies

\[
\psi(y) = \sup \{ \psi(u) \mid u \in E, \phi(u) = b \};
\]
then \( y \) satisfies (weakly) the equation

\[
-y''(x) + \frac{1}{4} y(x) e^{-x} = \frac{b}{\|y\|_\infty^2} \mu_y, \quad \text{where } \mu_y \in \partial \psi(y) \subset E'.
\]
The auxiliary Euler-Lagrange equation

It remains to determine the subgradient \( \mu_y \) in equation (5.5). Again following [8], Lemma 2.6, 2.7 and 2.8 we find:

**Proposition 6.1** Let \( K_y = \{ x \in [\alpha R, a] ; |y(x) = \|y\|_\infty \} \). Then

i) \( \text{supp } \mu_y \subset K_y \)

ii) \( K_y = \{ a \} \)

iii) \( \mu_y = \|y\|_\infty \delta_a \), the Dirac delta-function concentrated in the point \( a \).

Thus, equation (5.5) becomes

\[
\begin{cases}
-y'' + \frac{1}{4} ye^{-t} = \frac{b}{\|y\|_\infty} \delta_a , & \alpha_R \leq t \leq a \\
y(\alpha_R) = 0
\end{cases}
\]

From this one now concludes easily that equation (5.5) is equivalent to solving the equation

\[
\begin{cases}
-w'' + \frac{1}{4} we^{-t} = 0 \\
w(\alpha_R) = 0
\end{cases}
\]

with the condition that

\[
\int_{\alpha R}^{a} (|w'(t)|^2 + \frac{1}{4} |w(t)|^2 e^{-t}) dt = b
\]

the last condition is obtained by multiplying equation (6.1) by \( y \) and integrating.

We now determine the explicit solution of equation (6.2).

**Theorem 6.2** The solution of equation (6.2) is given by

- for \( 0 < R < +\infty \):

\[
w(t) = \gamma \left( K_0(e^{-t/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-t/2}) \right) =: \gamma z(t)
\]

- for \( R = +\infty \):

\[
w(t) = \gamma K_0(e^{-t/2})
\]

with unique coefficients \( \gamma = \gamma(R, a, b) \in \mathbb{R}^+ \).

Here \( I_k(x) \) and \( K_k(x) \) are the \( k \)-th modified Bessel functions of first and second kind, i.e. the solutions of the equation

\[-x^2 u''(x) - xu'(x) + (x^2 + k^2) u(x) = 0 , \ k = 1, 2, ...\]

**Proof.** By inspection.

It is crucial to determine with precision the value of the coefficient \( \gamma = \gamma(R, a, b) \) of \( w(t) \). This requires some lengthy calculations.

We begin by recalling the following relations for the modified Bessel functions (see e.g. [1], 9.6.27,28):

\[
\frac{d}{dx} I_0(x) = I_1(x) , \quad \frac{d}{dx} K_0(x) = -K_1(x) , \quad \frac{d}{dx} (x K_1(x)) = -x K_0(x)
\]
and the following integral relations
\[
\int_a^b |K_0(r)|^2 r dr = \left[\frac{1}{2} r^2 (K_0^2(r) - K_1^2(r))\right]_a^b
\]
\[
\int_a^b |K_1(r)|^2 r dr = \left[\frac{1}{2} r^2 (K_1^2(r) - K_0(r) K_2(r))\right]_a^b
\]
(6.7)
\[
\int_a^b |I_0(r)|^2 r dr = \left[\frac{1}{2} r^2 (I_0^2(r) - I_1^2(r))\right]_a^b
\]
\[
\int_a^b |I_1(r)|^2 r dr = \left[\frac{1}{2} r^2 (I_1^2(r) - I_0(r) I_2(r))\right]_a^b
\]
\[
\int_a^b [I_1(r) K_1(r) - I_0(r) K_0(r)] r dr = [I_0(r) K_1(r)]_a^b
\]
see [1]; for the last relation use integration by parts and (6.6).

Using these relations we will prove:

**Theorem 6.3**
1) Condition (6.3) yields for the coefficient \(\gamma = \gamma(R, a, b)\) in (6.4)
\[
\gamma^2 = 4 \frac{b}{a} \left[ 1 - \frac{4}{a} C(R) \right] + O(\frac{1}{a^3})
\]
for a large, with
\[
C(R) = \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R) K_2(R) + K_0^2(R) (1 - \frac{I_2(R)}{I_0(R)}) \right)
\]
+ 2RK_0(R)K_1(R) − 2R\frac{K_0(R) K_1(R)}{J_0(R)}
(6.8)
and \(C(+\infty) = 0\).
2) The solution \(w(t), \alpha_R \leq t \leq a\), of equation (6.2) is given by
• for \(0 < R < +\infty\):
\[
w(t) = 2 \sqrt{\frac{b}{a}} \left( 1 - \frac{4}{a} C(R) + O(\frac{1}{a^2}) \right)^{1/2} \left( K_0(e^{-t/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-t/2}) \right)
\]
(6.9)
• for \(R = +\infty\):
\[
w(t) = 2 \sqrt{\frac{b}{a}} \left( 1 + O(\frac{1}{a^2}) \right)^{1/2} K_0(e^{-t/2})
\]
(6.10)
**Proof.** Recall the definition of \(w(t)\) given in (6.4). We begin by evaluating the expression
\[
W^2(a) := \int_{\alpha_R}^a \left( |w'(x)|^2 + \frac{1}{4} |w^2(x)|^2 e^{-x} \right) dx
\]
Using the explicit form of \(w(t)\) in (6.4), the change of variable \(r = e^{-x/2}\), and the relations (6.6), we get
Using the relations (6.7) we get

\[ W^2(a) = \frac{1}{4} \int_{a}^{\infty} \left[ K_0'(e^{-x/2}) - \frac{K_0(R)}{I_0(R)} I_0'(e^{-x/2}) \right]^2 + \left| K_0(e^{-x/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-x/2}) \right|^2 e^{-x} \, dx \]

\[ = \frac{1}{2} \int_{e^{-a/2}}^{R} \left\{ \left| -K_1(r) - \frac{K_0(R)}{I_0(R)} I_1(r) \right|^2 + \left| K_0(r) - \frac{K_0(R)}{I_0(R)} I_0(r) \right|^2 \right\} r \, dr \]

\[ = \frac{1}{2} \int_{e^{-a/2}}^{R} \left\{ |K_1(r)|^2 + \frac{K_0^2(R)}{I_0^2(R)} |I_1(r)|^2 + |K_0(r)|^2 + \frac{K_0^2(R)}{I_0^2(R)} |I_0(r)|^2 \right\} r \, dr \]

\[ + 2 \frac{K_0(R)}{I_0(R)} (K_1(r) I_1(r) - K_0(r) I_0(r)) \right\} r \, dr \]

(6.11)

Using the relations (6.7) we get

\[ \frac{1}{2} \left\{ \left[ \frac{1}{2} R^2 (K_0^2(r) - K_0(r) K_2(r)) \right]_{e^{-a/2}}^{R} + \frac{K_0^2(R)}{I_0^2(R)} \left[ \frac{1}{2} R^2 (I_0^2(r) - I_0(r) I_2(r)) \right]_{e^{-a/2}}^{R} \right\} \]

\[ + \left[ \frac{1}{2} R^2 (K_0^2(r) - K_0^2(r)) \right]_{e^{-a/2}}^{R} + \frac{K_0^2(R)}{I_0^2(R)} \left[ \frac{1}{2} R^2 (I_0^2(r) - I_1^2(r)) \right]_{e^{-a/2}}^{R} \]

\[ + 2 \frac{K_0(R)}{I_0(R)} [I_0(r) K_1(r) r]_{e^{-a/2}}^{R} \}

(6.12)

Evaluating at the boundaries we obtain

\[ \frac{1}{4} R^2 \left( \frac{K_0^2(R)}{I_0^2(R)} - K_0(R) K_2(R) + K_0^2(R) \left( 1 - \frac{I_2(R)}{I_0(R)} \right) \right) + 2 R K_0(R) K_1(R) \]

\[ - \frac{1}{4} e^{-a} \left\{ K_0^2(e^{-a/2}) - K_0(e^{-a/2}) K_2(e^{-a/2}) \right\} \]

\[ + \frac{K_0^2(R)}{I_0^2(R)} \left[ I_0^2(e^{-a/2}) - I_0(e^{-a/2}) I_2(e^{-a/2}) \right] \}

\[ - 2 e^{-a/2} \frac{K_0(R)}{I_0(R)} I_0(e^{-a/2}) K_1(e^{-a/2}) \]

(6.13)

For the terms with argument \( e^{-a/2}, a \) large, we now use the following behavior of the Bessel functions for \( x > 0 \) small, see [1],9.6.7-9: :

\[ K_0(x) \sim - \log x \]
\[ K_1(x) \sim \frac{1}{x} \]
\[ K_2(x) \sim \frac{2}{x^2} \]
\[ I_0(x) \sim 1 \]
\[ I_1(x) \sim \frac{1}{x} x \]
\[ I_2(x) \sim \frac{2}{x^2} x^2 \]

(6.14)
We get
\[
\frac{1}{4} R^2 \left( K_0^2(R) - K_0(R) K_2(R) + K_0^2(R) (1 - \frac{I_2(R)}{I_0(R)}) \right) + 2 R K_0(R) K_1(R)
\]
\[
= \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R) K_2(R) + K_0^2(R) (1 - \frac{I_2(R)}{I_0(R)}) \right) + 2 R K_0(R) K_1(R)
\]
\[
= \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R) K_2(R) + K_0^2(R) (1 - \frac{I_2(R)}{I_0(R)}) \right) + 2 R K_0(R) K_1(R)
\]
(6.15)
\[
= \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R) K_2(R) + K_0^2(R) (1 - \frac{I_2(R)}{I_0(R)}) \right) + 2 R K_0(R) K_1(R)
\]
\[
+ \frac{1}{4} a - 2 \frac{K_0(R)}{I_0(R)} + O(a^2 e^{-a})
\]
We get
\[
\frac{1}{4} R^2 \left( K_0^2(R) - K_0(R) K_2(R) + K_0^2(R) (1 - \frac{I_2(R)}{I_0(R)}) \right) + 2 R K_0(R) K_1(R)
\]
(6.16)
\[
b = \gamma^2 W^2(a) = \gamma^2 \left( \frac{1}{4} a + C(R) + O(a^2 e^{-a}) \right)
\]
We rewrite (6.16) as
(6.17)
\[
\gamma^2 \frac{a}{4} \left( 1 + \frac{4}{a} C(R) + O(a e^{-a}) \right) = b
\]
which yields for \( \gamma = \gamma(a, b) \)
(6.18)
\[
\gamma^2 = \frac{4}{a} \left( \frac{b}{a} - 4 C(R) \right) + O(\frac{1}{a^3})
\]
This proves 1). Assertion 2) follows now from (6.4). Formula (6.10) follows from (6.9), noting that \( C(+\infty) = 0 \) and \( K_0(+\infty)/I_0(+\infty) = 0 \). \( \square \)

With this information we can now calculate the value \( \|w\|_\infty^2 = w^2(a) \):

**Proposition 6.4** Let \( w(t) \) denote the solution of (6.2), (6.3) and hence of (5.1). Then
\[
\|w\|_\infty^2 = w^2(a) = b \left[ a - D(R) \right] + O(\frac{1}{a})
\]

**Proof.** By (6.4) we have, using (6.14)
\[
w^2(a) = \gamma^2 \left( K_0(e^{-a/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-a/2}) \right)^2
\]
\[
= 4 \frac{b}{a} \left[ (1 - \frac{4}{a} C(R)) + O(\frac{1}{a^2}) \right] \left( K_0(e^{-a/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-a/2}) \right)^2
\]
(6.19)
\[
= 4 \frac{b}{a} \left[ (1 - \frac{4}{a} C(R)) \left( \frac{a}{2} - \frac{K_0(R)}{I_0(R)} \right)^2 + O(\frac{\log(a)}{a^4}) \right]
\]
\[
= b \left[ a - 4 C(R) - 4 \frac{K_0(R)}{I_0(R)} \right] + O(\frac{1}{a})
\]
Set
\[ D(R) = 4C(R) + 4 \frac{K_0(R)}{l_0(R)} ; \]
then (6.19) becomes
\[ w^2(a) = b \left[ a - D(R) \right] + O\left( \frac{1}{a} \right) \]
(6.21)

7 Construction of optimal concentrating sequences

In this section we show that the upper bounds for the Carleson-Chang limit
\[ \text{cc--lim} \| u_n \|_{S} \leq \int_{\Omega} (e^{4\pi u^2} - 1) dx \leq \pi e^{1-D(R)}, \]
given in Theorem 1.2 are sharp for $\Omega = B_R$ and $\Omega = \mathbb{R}^2$. We do this by constructing explicit optimal SNC-sequences $\{w_n\}$ for (7.1) for which the Carleson-Chang limit is equal to the bound on the right.

The construction of this sequence follows closely the proof of the upper bound for the Carleson-Chang limit, section 4, in combination with information on the optimal sequence for the corresponding Dirichlet-norm problem, see [7].

We begin by defining the sequence $\{w_n(t)\}$ on $[\alpha R, n]$: in Theorem 6.3, set $a = n$ and $b = 1 - \frac{2\log n}{n}$. Then, for $0 < R \leq +\infty$, let $w_n(t)$ be given by (6.9) or (6.10), respectively. Thus, $w_n(t)$ satisfies equation (6.2) with $a = n$, and condition (6.3) with $b = 1 - \frac{2\log n}{n}$. Furthermore, we have by Proposition 6.4
\[ w_n^2(n) = \sup \{ \| w_n \|_{S}^2 \mid w_n \in S_n \} = n - 2\log n - D(R) + O\left( \frac{1}{n} \right) , \]
(7.2)
where $S_n = \{ u \in H^1(\alpha R, n) \mid u(\alpha R) = 0, \int_{\alpha R}^{n} (|u'|^2 + \frac{1}{4}|u|^2 e^{-t}) dt = 1 - \frac{2\log n}{n} \}$. We remark that formula (7.2) constitutes a (late) motivation for the choice of $a_n$ in (4.9).

It remains to define $\{w_n(t)\}$ in $[n, +\infty)$. Here we can follow [7] where an optimal Dirichlet normalized concentrating sequence was constructed by analyzing carefully the proof of Carleson-Chang [6].

The complete definition of the optimal SNC-sequence $\{w_n(t)\}$ is:

**Definition 7.1** Let $w_n(t)$ be given by:
\[ w_n(t) = \begin{cases} w_n(t), & \text{given by (6.9) or (6.10), respectively,} \\ w_n(n) + \frac{1}{w_n(n)} \log \frac{1 + A_n}{A_n + e^{-(t-n)}} & t \geq n \end{cases} \]
where $A_n \in \mathbb{R}^+$ is such that
\[ \int_{\alpha R}^{\infty} (|w_n'(t)|^2 + \frac{1}{4} |w_n(t)|^2 e^{-t}) dt = 1 . \]
We show that $A_n \in \mathbb{R}^+$ can be chosen as in Definition 7.1, i.e. satisfying (7.4), with the estimate

**Lemma 7.2**

(7.5) \[ A_n = \frac{1}{n^2 e} + O\left(\frac{1}{n^4}\right) \]

**Proof.** First note that by condition (6.3)

(7.6) \[ \int_{nR}^{n} (|w'_n|^2 + \frac{1}{4} |w_n|^2 e^{-t}) dt = 1 - \frac{2 \log n}{n} \]

Thus, we look for a constant $A_n$ such that

(7.7) \[ \int_{n}^{\infty} (|w'_n|^2 + \frac{1}{4} |w_n|^2 e^{-t}) dt = \frac{2 \log n}{n} \]

Assume that $A_n \geq \frac{1}{3n^2}$, then one has

\[ \log\left(1 + \frac{A_n}{A_n + e^{-t-n}}\right) \leq \log\left(1 + \frac{1}{A_n}\right) \leq \log(1 + 3n^2) \]

and then by (7.3) and using that $w_n(n) = n + O(\log n)$ (by Proposition 6.4)

\[ w_n(t) \leq w_n(n) + \frac{1}{w_n(n)} \log(1 + 3n^2) \leq 2n, \text{ for } t \geq n, n \text{ large,} \]

and hence

\[ \int_{n}^{\infty} |w_n|^2 e^{-t} dt \leq 4n^2 e^{-n} \]

Therefore, condition (7.7) becomes

(7.8) \[ \int_{n}^{\infty} |w'_n|^2 = \frac{2 \log n}{n} + O(n^2 e^{-n}) \]

One proves as in [7] that this yields

\[ A_n = \frac{1}{n^2 e} + O\left(\frac{1}{n^4}\right) \]

We now give an asymptotic lower bound for $\pi \int_{\alpha R}^{\infty} (e^{w_n^2} - 1)e^{-t} dt$, as $n \to \infty$:

**Theorem 7.3** Let $\{w_n\}$ denote the sequence (7.3), and let $D(R)$ be given by (6.20). Then

\[ \pi \int_{\alpha R}^{\infty} (e^{w_n^2} - 1)e^{-t} \geq e^{-D(R)}(1 + 2D(R) \frac{\log n}{n}) + O\left(\frac{1}{n}\right). \]

**Proof.**

a) First note that

(7.9) \[ \pi \int_{\alpha R}^{n} (e^{w_n^2} - 1)e^{-t} dt \geq 0, \text{ for all } n \]
b) Consider now

\[ \pi \int_{n}^{\infty} (e^{w_{n}^2} - 1)e^{-t} = \pi \int_{n}^{\infty} e^{w_{n}^2 - t} + O(e^{-n}) . \]

Performing the change of variables \( s = t - n \), setting

\[ v_{n}(s) = \frac{1}{w_{n}(n)} \log \frac{A_{n} + 1}{A_{n} + e^{-s}} \]

and using that by Proposition 6.4

\[ w_{n}^2(n) = (1 - \frac{2 \log n}{n})[n - D(R)] + O(\frac{1}{n}) \]

we obtain

\[ \pi \int_{\alpha R}^{\infty} \exp \left( [w_{n}(n) + v_{n}(s)]^2 - s - n \right) ds \]

\[ \geq \pi \int_{\alpha R}^{\infty} \exp \left( w_{n}^2(n) + 2w_{n}(n)v_{n}(s) - s - n \right) ds \]

\[ \geq \pi \int_{\alpha R}^{\infty} \exp \left( n - 2 \log n - D(R) + 2D(R) \frac{\log n}{n} + O(\frac{1}{n}) + 2 \log \frac{A_{n} + 1}{A_{n} + e^{-s}} - s - n \right) \]

\[ \pi \int_{0}^{\infty} \exp(-2 \log n - D(R) + 2 \log \frac{A_{n} + 1}{A_{n} + e^{-s}} - s + 2D(R) \frac{\log n}{n} + O(\frac{1}{n})) \]

\[ = \pi e^{-D(R)} \frac{1}{n^2} \int_{0}^{\infty} \left( \frac{1 + A_{n}}{A_{n} + e^{-s}} \right)^2 e^{-s} ds \left( 1 + 2D(R) \frac{\log n}{n} + O(\frac{1}{n}) \right) \]

\[ = \pi e^{-D(R)} \frac{1}{n^2} \frac{1 + A_{n}}{A_{n}} \left( 1 + 2D(R) \frac{\log n}{n} + O(\frac{1}{n}) \right) \]

\[ = e \pi e^{-D(R)} \left( 1 + 2D(R) \frac{\log n}{n} \right) + O(\frac{1}{n}) \], as \( n \to \infty \).

Joining (7.9) and (7.10) we get

\[ \pi \int_{\alpha R}^{\infty} (e^{w_{n}^2} - 1)e^{-t} dt \geq e \pi e^{-D(R)}(1 + 2D(R) \frac{\log n}{n}) + O(\frac{1}{n}) , \]

and hence the theorem is proved.

We conclude this section by proving some properties of the function \( D(R) \):

**Lemma 7.4** Let \( D(R) \) given by (6.20). Then

\[ (7.11) \quad D(R) = 4R K_{0}(R)K_{1}(R) - 2 \frac{K_{0}(R)}{I_{0}(R)} . \]

Furthermore, \( D(R) > 0 \), for all \( R \in \mathbb{R}^{+} \), and

\[ D(R) \sim -2 \log R \], as \( R \to 0 \)

and

\[ D(R) \sim \frac{\pi}{R} e^{-2R} \], as \( R \to +\infty \).
Proof. The explicit form of \( D(R) \) is

\[
D(R) = 4C(R) + 4\frac{K_0(R)}{I_0(R)}
= R^2 \left( K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - \frac{I_1(R)}{I_0(R)}) \right) + 8RK_0(R)K_1(R) - 4\frac{K_0(R)}{I_0(R)}
\]

Using the relations (see [1], 9.6.26)

\[
K_2(x) - K_0(x) = \frac{2}{x}K_1(x) \quad \text{and} \quad I_0(x) - I_2(x) = \frac{2}{x}I_1(x)
\]

we get

(7.12)

\[
D(R) = 6RK_0(R)K_1(R) + (2RK_0(R)I_1(R) - 4)\frac{K_0(R)}{I_0(R)}
\]

which simplifies, using (see [1], 9.6.15)

(7.13)

\[
K_1(x)I_0(x) + K_0(x)I_1(x) = \frac{1}{x}
\]

to (7.11).

We prove that \( D(R) > 0 \), for all \( R > 0 \): by (7.11) we get, using again (7.13)

\[
D(R) = \frac{2K_0(R)}{I_0(R)} \left[ RK_1(R)I_0(R) - 1 + RK_1(R)I_0(R) \right]
= \frac{2K_0(R)}{I_0(R)} \left[ RK_1(R)I_0(R) - 1 + 1 - RK_0(R)I_1(R) \right] > 0,
\]

since \( K_1(x) > K_0(x) \) and \( I_0(x) > I_1(x) \), for all \( x > 0 \).

Next, using the behavior of the Bessel functions (6.14), for \( R > 0 \) small, we have

\[
D(R) \sim -4\log R - 2(-\log R) = -2\log R, \quad \text{for} \quad R > 0 \quad \text{small}.
\]

For the behavior of \( D(R) \) at \(+\infty\) we use the asymptotic behavior of the Bessel functions at \(+\infty\), see [1], 9.7.1-2:

\[
I_i(x) \sim \frac{1}{\sqrt{2\pi x}}e^x(1 - \frac{4i^2 - 1}{8x})
\]

(7.14)

\[
K_i(x) \sim \frac{\pi}{\sqrt{2\pi x}}e^{-x}(1 + \frac{4i^2 - 1}{8x})
\]

Hence, we obtain by (7.11)

(7.15)

\[
D(R) \sim 4R \frac{\pi}{\sqrt{2\pi R}} e^{-R} \left( 1 - \frac{1}{8R} \right) \frac{\pi}{\sqrt{2\pi R}} e^{-R} \left( 1 + \frac{3}{8R} \right)
- 2 \frac{\pi}{\sqrt{2\pi R}} e^{-R} \left( 1 + \frac{1}{8R} \right) \sqrt{2\pi R} e^{-R} \left( 1 - \frac{1}{8R} + O\left( \frac{1}{R^2} \right) \right)
\]

\[
\sim 2\pi e^{-2R} \left( 1 + \frac{1}{4R} \right) - 2\pi e^{-2R} \left( 1 - \frac{1}{4R} \right) = \frac{\pi}{R} e^{-2R}.
\]
8 The Supremum is attained

In this section we show that the supremum

\[
\sup_{\|u\|_S \leq 1} \int_{\Omega} (e^{4\pi a^2} - 1) dx
\]

is attained for any ball \( \Omega = B_R(0) \), as well as for \( \Omega = \mathbb{R}^2 \).

By Proposition 3.3 it suffices to prove

**Theorem 8.1** Let \( 0 < R \leq +\infty \). Then

\[
\sup_{\|u\|_S \leq 1} \int_{\alpha_R} (e^{u^2} - 1) e^{-t} dt > \lim_{\|u_n\|_S \leq 1} \pi \int_{\alpha_R} (e^{u^2_n} - 1) e^{-t} dt
\]

**Proof.** This follows immediately by Theorem 7.3: Choose an element of the maximizing sequence \( \{w_n\} \), with \( n \) sufficiently large. Then

\[
\sup_{\|u\|_S = 1} \pi \int_{\alpha_R} (e^{u^2} - 1) e^{-t} dt \geq \pi \int_{\alpha_R} (e^{w^2_n} - 1) e^{-t} dt > \pi e^{1-D(\mathbb{R})} = \lim_{\|u_n\|_S \leq 1} \int_{\alpha_R} (e^{u^2_n} - 1) dx .
\]

This completes the proof of Theorem 1.3.

References


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