Social Choice and Topology[1]

A Case of Pure and Applied Mathematics

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I have never seen much difference between pure and applied mathematics. Either mathematical problems and theories are suggested by applications, and then they may be applied in concrete cases – sometimes in a quite different disguise and framework. Or there is "pure" mathematical research, usually done for different objectives – or for no reason at all? – which later is being applied in unforeseen contexts.

What follows in this note is a case of quite recent and both simple and unexpected application of algebraic topology to a different field of intellectuel enterprise. To my surprise it makes use of the results in an old paper of mine.

The title "Social choice and topology" may be misleading; this is not about the sociology of topologists! The term Social Choice appears in models of mathematical economy for a long time already (cf Remark 1.3.). Here we refer to a new version, only a few years old. It is a model for decision making in economic, social, political etc contexts. It turns out to lead to topological, even homotopical problems.

1. The new social choice model

1.1. One considers a set P of elements called preferences. It has the structure of a topological space (normally given by a metric). This is natural, because the preferences are numbers, real or complex, point sets in number spaces, vectors, configurations etc. One further considers a society of n agents numbered $1, 2, ..., n$. If $p_j \in P$ is the preference of the agent j the element $(p_1, ..., p_n)$ of the topological product P^n of n copies of P is the "profile" of a society. The *Social Choice* is a function

$$F : P^n \longrightarrow P$$

which associates to each profile $\in P^n$ a social preference $\in P$. It has to fulfill the following properties

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(a) Continuity
(b) Unanimity
(c) Anonymity

(a) is a natural requirement: small changes of a profile should lead to small changes of the outcome.

(b) means

$$F(p, p, ..., p) = p$$

for all $p \in P$. In other words, if Δ is the map given by $\Delta(p) = (p, p, ...p)$ then

$$\Delta \longrightarrow P^n \longrightarrow P$$

is the identity of P. The subspace $\Delta(P)$ of P^n is called the diagonal.

(c) means that F is invariant under all permutations of the indices $1, 2, ..., n$, i.e.under the symmetric group Σ_n (all agents are equal).

1.2. The existence of such a map F according to properties of the space P is clearly a topological problem. It is even a homotopy problem: One actually is looking for a map of the symmetric product P^n/Σ_n (one identifies points of P^n equivalent under all permutations of the indices) to P. On the diagonal, which survives in P^n/Σ_n, the map is given and it has to be extended to all of P/Σ_n. But the extension property of a continuous map only depends on the homotopy type of the spaces involved – we avoid pathologies and assume throughout that P is a "polyhedron", i.e. a cell complex; and that it is connected.

The existence problem of such a function F from P^n to P with properties (a), (b), (c) has been treated 50 years ago in one of my papers [E], of course under different terminology. I did not know the term social choice. On the other hand it is clear that the economists' group investigating the new social choice concept, G.Chichilnisky and her collaborators (cf [C]) did not know of my old paper. I was informed of this new work through a paper by Ch.Horvath [H] published in 2001; there it is mentioned that "the fundamental result ... was established by B.Eckmann in 1954".

1.3. Remark. In the famous work of Arrow (see e.g. [K-S]) the term social choice is used in a different sense (this is why I say "the new social choice model"). There the social choice function σ from P^n to P must fulfill (b), but (c) is not required. From Arrow's axioms it follows that σ is necessarily a projection onto one copy, say number k, in P^n, i.e $\sigma(p_1, ..., p_n) = p_k$ for all elements of P^n: agent number k is a dictator. This cannot happen in the "new " model because of (c). Moreover in [K-S] the preference set is not a topological space, but consists of preorders of a certain set X.

2. Spaces with means

2.1. The paper I published in 1954 is on "Spaces with means" (Räume mit Mittelbildungen). A mean, or an n-mean, or a generalized mean, in a connected space P is exactly the same as a social choice function F above.

Elementary examples of n-means are the arithmaric mean of n numbers in an interval of the real line, and the geometric mean.

Generalized means have a long history. They started in a paper by Kolmogorov in 1930 on quasi-arithmetic means and were continued by various authors in connection with analytic or arithmetic questions. An important step was the thesis of G.Aumann (1933, under Carathodory). In a further paper (1935) Aumann formulated the conditions (a), (b), (c) above, mainly for subsets of number spaces. In 1943 he states that the existence is a topological problem for arbitrary spaces; he solves it, however, for a few very special cases only. Thus the problem was open and well-known at the time I decided to look at it from the viewpoint of algebraic topology, with emphasis on algebraic.

2.2. At this point a historical remark is in order. Let me recall that the period 1950 to 1954 was very significant not only for algebraic topology and algebra, but also for other fields of mathematics. It was the time when categories and functors became mathematical tools; they had been, in the original version of Eilenberg and MacLane, rather a language to formulate naturality. But now functors from modules to modules or Abelian groups were used by Cartan and Eilenberg (their book appeared in 1956, more general categories are mentioned in an appendix by D.Buchsbaum). The book "Foundations of Algebraic Topology" by Eilenberg and Steenrod (1952) is about functors from spaces to Abelian groups, but categories and functors are mentioned rather reluctantly in Chap.IV.

2.3. In order to transform the n-mean in a space into an n-mean in a group I applied furctors from spaces to groups (continuous maps to homomorphisms). This clearly works if the functor preserves products.

Each homotopy group π_i is such a functor: π_i of a topological product is the direct product of the π_i of the factors. π_1, the fundamental group is in general not Abelian, the π_i, $i \geq 2$ are. This yields groups with means as discussed in the next section. The passage can be done in one stroke using the functor concept. In my old paper I preferred, however, to establish everything by explicit computation – it seems that the time was not ripe to use the functorial property in a publication!

Proposition 1. An n-mean in the space P induces a homomorphic n-mean in each homotopy group $\pi_i(P)$, $i = 1, 2, ...$

3. Groups with means

3.1. We consider an arbitrary group G, Abelian or not, but use additive notation, so that the neutral element is 0. An n-mean in G is a function

$$f : G^n \longrightarrow G$$

with properties (a) homomorphism, (b) and (c) as before for some $n \geq 2$. It implies strong restrictions on G.

First we note that for any $x \in G$

$$f(x, 0, ..., 0) = f(0, x, 0, ..., 0) = .. = f(0, ..., 0, x)$$

written $g(x)$. Now $g(x + y) = g(x) + g(y)$ and

$$g(x) + g(y) = f(x, 0, 0, ..., 0) + f(0, y, ..., 0) = f(x, y, 0, ..., 0)$$

whence $g(x) + g(y) = g(y) + g(x)$. Further

$$ng(x) = g(nx) = f(x, x, ..., x) = x.$$

Thus $x + y = y + x$, i.e. G is Abelian. Multiplication in G by n is therefore an endomorphism of G; since by the above g is its inverse it follows that multiplication by n is an automorphism of G.

Theorem 2. If the group G admits an n-mean for some $n \geq 2$ then it is Abelian and multiplication by n is an automorphism of G, and G is uniquely divisible by n.

3.2. We can write n^{-1} for the automorphism g above. Since

$$nf(x_1, ...x_n) = ng(x_1 + ... + x_n) = x_1 + ... + x_n$$

it follows that

$$f(x_1, ..., x_n) = n^{-1}(x_1+, ... + x_n),$$

i.e. there is only one n-mean on G, the "arithmetic mean".

Conversely, if the Abelian group G is divisible by n then it admits an n-mean (and only one).

. **3.3.** From Theorem 2 it follows that if G is finitely generated and admits an n-mean then the order of any element must be prime to n; in particular there cannot be elements of infinite order. If G is finitely generated and admits n-means for all n then G is $= 0$.

4. Application to spaces

4.1. Applying the results on groups to spaces with means it follows that a space P with an n-mean for $n \geq 2$ has Abelian fundamental group and all its homotopy groups $\pi_i(P)$ are uniquely divisible by n.

Example. $P = S^k$, $k \geq 1$. Since $\pi_k(S^k) = \mathbb{Z}$, the sphere S^k does not admit any n-mean, $n \geq 2$.

4.2. Let P be a polyhedron (cell-complex) with finitely generated integral homology groups. We assume that it admits n-means for all n. Since $\pi_1(P)$ is Abelian, it is isomorphic to $H_1(P)$ whence finitely generated and divisible by all n, and thus $= 0$. By the Hurewicz theorem $\pi_2 = H_2$, finitely generated and divisible by all n, whence $= 0$, etc. Thus all $\pi_i(P)$ are zero, and the theorem of J.H.C.Whitehead tells that P is contractible.

Theorem 3. If a polyhedron with finitely generated homology groups admits n-means for all n then it is contractible.

This applies in particular to a finite polyhedron (consisting of finitely many cells).

Note that homotopy invariance tells that any contractible space P admits n-means for all n.

4.3. Serre's generalization of the Hurewicz theorem modulo a class $\mathcal{C}$ of abelian groups tells, in our case, that all homotopy groups being uniquely divisible by n is equivalent to this holding for all homology groups (with integer coefficients). I knew already Serre's theorem but proved directly that homology groups are divisible by n; I probably thought again that using new generalizations would make the paper too difficult.

Thus, for example, a closed orientable manifold cannot admit any n-mean for $n \geq 2$ since in the top dimension the homology group is infinite cyclic.

5. Non-contractible spaces with means, H-spaces

5.1. The discussion of groups with means in Section 3 yields further results. They are mostly contained in a later paper (1962) by Ganea, Hilton and myself [EGH].

A non-contractible (infinite) polyhedron admitting n-means for all n can be constructed as follows. We consider $P = K(\mathbb{Q}, k)$, the Eilenberg-MacLane space with $\pi_i(P) = 0$ for $i \neq k$ and $= \mathbb{Q}$ for $i = k$ where k is an arbitrary integer ≥ 1. Such a space can be constructed by means of cells (starting with a k-sphere and using mapping cylinders); it is finite dimensional if k is odd. Since the homotopy group of a topological product is the direct product of the homotopy groups of the factors we have $P^n = K(\mathbb{Q}^n, k)$. For any two spaces $K(G_1, k)$ and $K(G_2, k)$ the homotopy classes of maps are in bijective correspondence with the homomorphisms $G_1 \longrightarrow G_2$. The group $\mathbb{Q}$ is divisible by all n. Thus the n-mean $\mathbb{Q}^n \longrightarrow \mathbb{Q}$ defines a homotopy class of maps $P^n \longrightarrow P$. We can take a map which is the identity on the diagonal $\Delta(P)$; it is homotopically Σ_n-invariant. By a lemma of Grothendieck on cohomology with rational coefficients (the homotopy classes of maps from a space to $K\mathbb{Q}, k)$ constitute such a cohomology group) we can obtain a map which is strictly Σ_n-invariant.

Theorem 4. $K(\mathbb{Q}, k)$ admits n-means for all n.

5.2. This example turns out to be fundamental for any (non-contractible) space with n-means for all n. We first show ([EGH]) that such a space is an H-space – it suffices that there is an n-mean for a single $n \geq 2$.

We recall that an H-space is a space X admitting a continuous multiplication $\mu : X \times X \longrightarrow X$ with two-sided unit $e \in X$ up to homotopy. This means that the two maps $X \longrightarrow X$ defined by $\mu(p, e)$ and $\mu(e.p)$, $p \in X$ are homotopic to the identity.

Let $F : P^n \longrightarrow P$ be an n-mean for some $n \geq 2$, and e a point $\in P$. The map $\phi : P \longrightarrow P$ defined by $\phi(p) = F(p, e, ..., e)$ for all $p \in P$ induces in each homotopy group of P precisely the homomorphism called g in 3.1 which is an automorphism. Thus ϕ induces isomorphisms in all homotopy groups and is therefore, by the theorem of J.H.C.Whitehead, a homotopy equivalence. Let ψ be its inverse so that $\psi\phi$ and $\phi\psi$ are both homotopic to the identity of P.

We now put for $p, q \in P$

$$\mu(p,q) = \psi F(p,q,e,...,e).$$

Then $\mu(p,e) = \psi F(p,e,...,e) = \psi\phi(p)$ for all $p \in P$. This map is homotopic to the identity of P, and the same argument works for $\mu(e,p)$.

Theorem 5. If P admits an n-mean for some $n \geq 2$ then it is an H-space.

5.3. Using a theorem of W.Browder [B] we can draw a very strong coclusion from that result. If P is a finite polyhedron (even a little more generally) and an H-space then Browder's theorem tells that either P is contractible or fulfills Poincaré duality like an orientable manifold. Thus some homology group $H_k(P),\ k \geq 1$ is infinite cyclic so that no n-mean, $n \geq 2$ can exist.

Theorem 6. If a finite polyhedron admits an n-mean for some $n \geq 2$ then it is contractible – and thus all n-means exist.

Note that this is much stronger than the version of Theorem 3 for finite polyhedra.

5.4. For (infinite) non-contractible polyhedra P admitting n-means for all n the H-space poperty yields a complete desription. Using more elaborate techniques due essentially to Hopf and Serre, and the fact that all homotopy groups of P are $\mathbb{Q}$-vector spaces one can prove:

Theorem 7. If P (non-contractible) admits n-means for all n then it is of the homotopy type of a topological product of Eilenberg-MacLane spaces $K(\mathbb{Q}, k_\nu)$.

Here interesting consequences for the social choice model must be mentioned. Take for the preference space P a product as in Theorem 7 with social choice function F and an arbitrary number n of agents. Then for any constant $d > 0$ there are profiles $(p_1, ..., p_n)$ such that the distance between $F(p_1, ..., p_n)$ and p_j is $> c$ for all j. [We assume that the topology is given by a suitable metric.] This means that no agent gets approximately what he wants– in the contrary! Would the agents agree to a solution by mediation?

The fact above is due to the construction of P which starts with spheres, and these do not admit any n-mean with $n \geq 2$. There are more precise ways to express the result. I just wanted to show that except for the contractible case either no social choice function can exist on P, or if it exists for all n then unexpected properties turn up.

5.5. Remark. I would like to mention a recent paper by Shmuel Weinberger [W] on the topological aspects of the social choice model. Weinberger did not know of our papers [E] and [EGH] on generalized means; these contain all topological results of [W] – except Theorem 7 above: we knew it but did not consider it important enough! In [W] its significance is explained very nicely and there are some further interesting observations concerning various aspects of the social choice model.

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