KOSZUL COMPLEXES AND SYMMETRIC FORMS
OVER THE PUNCTURED AFFINE SPACE

PAUL BALMER AND STEFAN GILLE

Abstract. Let $X$ be a scheme which is not of equicharacteristic 2 and let $U^n_X \subset A^n_X$ be the punctured affine $n$-space over $X$. If $n \equiv \pm 1$ modulo 4, we show that there exists a ±1-symmetric bilinear space $(E^n_X, \varphi^n_X)$ over $U^n_X$ which cannot be extended to the whole affine space $A^n_X$ and which is locally metabolic for $n \geq 2$. If $X$ is regular, contains $\frac{1}{2}$ and has finite Krull dimension, we show that the total Witt ring $W_{\text{tot}}(U^n_X)$ of $U^n_X$ is a free $W_{\text{tot}}(X)$-module with two generators: the Witt classes of $<1>$ and of the above $(E^n_X, \varphi^n_X)$. We describe $W_{\text{tot}}(U^n_X)$ similarly when $n$ is even.

Introduction

Let $X$ be scheme. We are studying the (total) graded Witt ring

$$W_{\text{tot}}(X) := \bigoplus_{i \in \mathbb{Z}} W^i(X)$$

where the groups $W^i$ are the derived Witt groups of Balmer [2, 3] and where the multiplicative structure is the one of Gille-Nenashev [10]. See more in Section 2.

We fix an integer $n \geq 1$ for the entire article. Consider the following open subset $U^n_X \subset A^n_X$ of the affine space $A^n_X = \text{Spec}(\mathbb{Z}[T_1, \ldots, T_n])$:

$$U^n_X := \bigcup_{j=1}^n \text{Spec}(\mathbb{Z}[T_1, \ldots, T_n, T_j^{-1}]) \subset A^n_X.$$

For any scheme $X$, define by base-change the open subscheme $U^n_X \subset A^n_X$, called the punctured affine space over $X$, i.e. define $U^n_X$ by the following pull-back square:

$$\begin{array}{ccc}
U^n_X & \xrightarrow{\sigma_X} & X \\
\downarrow^{\nu_X} & & \downarrow \\
U^n_Z & \xrightarrow{} & \text{Spec}(\mathbb{Z}).
\end{array}$$

(1)

Our main result is Theorem 7.14 below, which says in particular:

**Theorem.** If the scheme $X$ is regular, contains $\frac{1}{2}$ and has finite Krull dimension, there is a decomposition $W_{\text{tot}}(U^n_X) = W_{\text{tot}}(X) \oplus W_{\text{tot}}(X) \cdot \varepsilon$ for some Witt class $\varepsilon = \varepsilon_X^{(n)}$ in $W^{n-1}(U^n_X)$. If $n=1$, we have $\varepsilon^2 = 1$. If $n \geq 2$, we have $\varepsilon^2 = 0$ and an isomorphism

$$W_{\text{tot}}(U^n_X) \cong \frac{W_{\text{tot}}(X)[\varepsilon]}{\varepsilon^2}$$

of graded rings, with the generator $\varepsilon$ in degree $n - 1$. 

*Date:* October 30, 2003.
Our second goal is a “classical” description of the generator $\varepsilon_X^{(0)} \in W^{n-1}(U_X^n)$. Recall a few facts. First, the derived Witt groups are 4-periodic: $W^i = W^{i+4}$. Secondly, $W^0$ and $W^2$ are naturally isomorphic to the usual Witt groups $W^0_{\text{sym}}$ and $W^2_{\text{sym}}$ of symmetric and skew-symmetric vector bundles respectively, as defined by Knebusch [12]. Thirdly, $W^1$ and $W^3 = W^{-1}$ are groups of formations, see Walter [15]. Therefore, if we want to describe in classical terms our generator $\varepsilon_X^{(0)}$ in $W^{n-1}$, we are bound to produce an explicit element of the above nature, i.e. a $\pm 1$-symmetric form or formation, depending on the congruence of $n$ modulo 4.

In this introduction, we focus on the case where $n$ is odd and we write $n - 1 = 2\ell$. In this case, we have to describe a $(-1)^\ell$-symmetric bundle

$$(E_X^{(0)}, \varphi_X^{(0)})$$

over $U_X^n$ whose class in $W^0_{\text{sym}}$ will be our generator $\varepsilon_X^{(0)}$. Let us stress that this will be a (skew-)symmetric space of classical nature, which does not involve triangulated categories. By the very naturality of the original problem, it suffices to construct this (skew-)symmetric bundle when $X = \text{Spec}(\mathbb{Z})$ and then to pull it back over an arbitrary scheme $X$. Therefore, we start with a description of $(E_X^{(0)}, \varphi_X^{(0)})$.

Let us denote by $A := \mathbb{Z}[T_1, \ldots, T_n]$ the polynomial ring in $n$ variables and by $K_* = K_*(A, T)$ the (homological) Koszul complex over $A$ for the $A$-sequence $T := (T_1, \ldots, T_n)$. There is a well-known isomorphism of complexes $\Theta_* : K_* \cong \text{Hom}_A(K_*, A)[n]$, see [7, Chap. 1.6] for instance. Since $K_*|_{U_2^n}$ is locally split, the $\mathcal{O}_{U_2^n}$-module

$$E_Z^{(0)} := \text{Coker} \left( K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1} \right)|_{U_2^n} \cong \text{Ker} d_{\ell}|_{U_2^n}$$

is locally free. From $n = 2\ell + 1,$ one easily sees that $\Theta_{\ell} \circ d_{\ell+1}$ induces an isomorphism

$$\varphi_Z^{(0)} : E_Z^{(0)} \cong \text{Hom}_{\mathcal{O}_{U_2^n}}(E_Z^{(0)}, \mathcal{O}_{U_2^n})$$

which is $(-1)^\ell$-symmetric. For a scheme $X$, with the base-change morphism $\nu_X : U_X^n \rightarrow U_2^n$ as in diagram (1), we define $(E_X^{(0)}, \varphi_X^{(0)}) := \nu_X(E_Z^{(0)}, \varphi_Z^{(0)})$.

If $n$ is even, we can construct similarly a complex of length 1, with a suitable (skew-)symmetric form, i.e. a formation, whose class in the Witt group $W^{n-1}(X)$ is our wanted $\varepsilon_X^{(0)}$. This complex is also obtained by chopping off some parts of the above Koszul complex.

Putting things together, if we define the integer $-1 \leq r \leq 2$ by the equation $n - 1 \equiv r \pmod{4}$, we produce “short symmetric r-spaces” $(F_X^{(0)}, \phi_X^{(0)})$ (if $n$ is odd this complex is concentrated in one degree and corresponds to $(E_X^{(0)}, \varphi_X^{(0)})$ above via the natural embedding $\text{VB}_X \hookrightarrow \text{B}^n(\text{VB}_X)$), having the following properties (Theorems 7.13 and 8.2):

**Theorem.** Let $X$ be a scheme, not of equicharacteristic 2 (in particular $2 \neq 0$).

(i) The symmetric r-space $(F_X^{(0)}, \phi_X^{(0)})$ can not be extended to $\mathbb{A}_X^n$, i.e. there does not exist a symmetric r-space $(P, \phi)$ over $\mathbb{A}_X^n$ whose restriction $(P, \phi)|_{U_2^n}$ is isometric (nor Witt equivalent) to $(F_X^{(0)}, \phi_X^{(0)})$. In particular, $(F_X^{(0)}, \phi_X^{(0)})$ is not extended from $X$ either. Hence, if $n$ is odd, the same is true for the classical (skew-)symmetric space $(E_X^{(0)}, \varphi_X^{(0)})$. 

(ii) Assume moreover that 2 is invertible in $X$ and that $n \geq 2$. Then the space $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ is locally hyperbolic, i.e. for any $x \in U^n_X$ we have $[(\mathcal{F}_X^{(n)}, \phi_X^{(n)})_x] = 0$ in $W^r(O_{U^n_X,x})$, and moreover its square is Witt trivial, i.e. $[(\mathcal{F}_X^{(n)}, \phi_X^{(n)})^2]$ is zero in $W^{2r}(U^n_X)$.

This theorem says that the spaces $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ are quite specific to $U^n_X$. They cannot be extended to $A^n_X$, not even up to Witt equivalence. In particular, these spaces are not metabolic on $U^n_X$. On the other hand, they do become metabolic as soon as we localize them to some principal open given by $T_i \neq 0$, see 7.11.

There are two appendices. In the first one, for the sake of completeness, we show that when $n \geq 3$ our locally free $O_{U^n_X}$-module $E_X^{(n)}$ cannot be extended to a locally free $O_{A^n_X}$-module and in particular $E_X^{(n)}$ is not free. The second appendix contains the compatibility between product and 4-periodicity, a fact which we use several times in this work.

Acknowledgment. We would like to thank Manuel Ojanguren for useful references. The first author is supported by the Swiss National Science Foundation, grant 620-066065.01. The second author would like to thank the FIM and the ETH-Zürich for hospitality and financial support.

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1. Conventions and notations

We collect here the notations which are kept unchanged in all sections.

First of all, recall that we have fixed an integer $n \geq 1$. We decompose it as

$$n = 4q + r + 1$$  (2)

where $q \in \mathbb{N}$ and $r \in \{-1, 0, 1, 2\}$. Note that $n - 1 \equiv r \mod 4$. We also baptize

$$\left\lfloor \frac{n}{2} \right\rfloor =: \ell.$$  (3)

Convention 1.1. Unless mentioned, a ring means a commutative ring with unit.
Convention 1.2. As always, when using a notation defined for schemes $X$ in the affine case, $X = \text{Spec}(R)$, we shall drop “Spec” as for instance: $\text{VB}_{\mathcal{O}(R)}$, $\mathbb{D}^b(\text{VB}_{\mathcal{O}(R)})$, $W^i(\text{VB}_{\text{Spec}(R)})$, $\mathbb{D}^b(\text{VB}_{\text{Spec}(R)})$, $W^i(\text{Spec}(R))$, and so on. See 2.11.

Convention 1.3. We shall say that a scheme is regular if it is noetherian and separated and if all its local rings are regular.

Notation 1.4. Let $X$ be a scheme. We denote by $\mathbb{A}^n_X$ the affine $n$-space and by $\mathbb{U}^n_X$ the punctured affine $n$-space over $X$. The obvious structure morphisms and base-change morphisms will be denoted as follows:

for any morphism of schemes $f : Y \to X$.

2. Recalling derived Witt groups

This section is a quick course on triangular Witt groups over schemes, included only for the reader’s convenience. Here, $X$ is a scheme with structure bundle $\mathcal{O}_X$.

2.1. Categories and dualities.

We denote by the symbol $\text{VB}_X$ the exact category of locally free $\mathcal{O}_X$-modules of finite rank, i.e. vector bundles. The usual duality on $\text{VB}_X$ is abbreviated

$$(-) \vee := \text{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X).$$

$\mathbb{D}^b(\text{VB}_X)$ stands for the bounded derived category of $\text{VB}_X$. We use homological notations for complexes. The translation functor $\Sigma : \mathbb{D}^b(\text{VB}_X) \to \mathbb{D}^b(\text{VB}_X)$, also written $P \mapsto [-1]$, is given by $(P[1])_j = P_{j-1}$; as usual, $\Sigma$ changes the sign of all differentials: $d_{P[-1]}^j = -d_{P}^{j-1}$.

Let $P = (P_\bullet, d_\bullet^P)$ be a complex in $\mathbb{D}^b(\text{VB}_X)$. Its dual $\mathcal{D}_X(P_\bullet)$ is the complex

$$\mathcal{D}_X(P_\bullet) := \cdots \to P_{-j} \xrightarrow{d_{P,-j}^\vee} P_{-(j-1)} \xrightarrow{d_{P,-j+1}^\vee} \cdots$$

and similarly for morphisms of complexes. In other words, $\mathcal{D}_X$ is the derived functor of $(-) \vee = \text{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$. This defines a duality on $\mathbb{D}^b(\text{VB}_X)$ turning it into a triangulated category with duality in the sense of [2]. The isomorphism between the identity and the double dual, $\varpi : \text{id}_{\mathbb{D}^b(\text{VB}_X)} \xrightarrow{\sim} \mathcal{D}_X \mathcal{D}_X$, is given in each degree $j$ by the canonical (evaluation) isomorphism $\text{can}_{P_j} : P_j \to P_j^{\vee \vee}$. We consider $\text{VB}_X$...
as a subcategory of $\mathbb{D}^b(VB_X)$ via the natural embedding $c_0: VB_X \rightarrow \mathbb{D}^b(VB_X)$, which maps a vector bundle $P \in VB_X$ to the complex

$$
\cdots \rightarrow 0 \rightarrow 0 \rightarrow P \rightarrow 0 \rightarrow 0 \rightarrow \cdots
$$

and does the same on morphisms. The restriction of the duality $D_X$ to this subcategory is the original duality of $VB_X$ and the restriction of $\varpi$ is the above can.

**Definition 2.2.** Let $P_\bullet$ be a complex in $\mathbb{D}^b(VB_X)$. Let $i \in \mathbb{Z}$, and $\phi : P_\bullet \rightarrow D_X(P_\bullet)[i]$ be a morphism in $\mathbb{D}^b(VB_X)$. We say that $\phi$ is an *symmetric i-form* on the complex $P_\bullet$ if

$$
D_X(\phi)[i] \cdot \varpi_{P_\bullet} = (-1)^{\frac{i(i+1)}{2}} \phi.
$$

We then say that $(P_\bullet, \phi)$ is a symmetric *i-pair*. If $\phi$ is moreover an isomorphism we say that $(P_\bullet, \phi)$ is a symmetric *i-space* over $X$. Two symmetric i-pairs $(P_\bullet, \phi)$ and $(Q_\bullet, \psi)$ are called isometric if there exists in $\mathbb{D}^b(VB_X)$ an isometry between them, that is, an isomorphism $h : P_\bullet \cong Q_\bullet$ such that $\phi = D_X(h)[i] \cdot \psi \cdot h$.

**Remark 2.3.** Note that if $(P_\bullet, \phi)$ is a symmetric i-pair then $(P_\bullet[2], \phi[2])$ is a symmetric $(i + 4)$-pair because $D_X(P_\bullet)[1] = D_X(P_\bullet[-1])$ for all $P_\bullet \in \mathbb{D}^b(VB_X)$.

Let $f : Y \rightarrow X$ be a morphism of schemes. There is a natural isomorphism of functors $f^*D_X \cong D_Y f^*$ which is induced by the natural isomorphism of locally free $O_Y$-modules $f^*\text{Hom}_{O_X}(P, O_X) \cong \text{Hom}_{O_Y}(f^*P, O_Y)$. If now $(P_\bullet, \phi)$ is a symmetric i-space over $X$ then the isomorphism

$$
f^*(P_\bullet) \xrightarrow{f^*\phi} f^*(D_X(P_\bullet)[i]) = f^*(D_X(P_\bullet))[i] \xrightarrow{\eta_f, [i]} D_Y(f^*P_\bullet)[i]
$$

is a symmetric i-form and so $f^*(P_\bullet, \phi) := (f^*(P_\bullet), \eta_f, [i] \cdot f^*(\phi))$ a symmetric i-space over $Y$.

**2.4.** “Short” i-forms: Forms and formations.

We present examples of symmetric i-pairs $(P_\bullet, \phi)$ in four cases $i = -1, 0, 1, 2$.

\[\begin{array}{c|c|c|c|c|c}
\hline
\text{i} & & & & & \\
\hline
\text{deg 0} & \text{deg 1} & \text{deg 0} & \text{deg 0} & \text{deg 0} & \\
\hline
\text{i} = 0 & P_0 & \cdots & 0 & \cdots & 0 & \cdots \\
\hline
\phi_0 \rightarrow & \phi_0' \rightarrow & \phi_1 \rightarrow & \phi_1' \rightarrow & \phi_1'' \rightarrow & \phi_2 \rightarrow & \phi_2'' \rightarrow \\
\text{i} = 1 & P_1 & \cdots & 0 & \cdots & 0 & \cdots \\
\hline
\text{i} = 2 & \text{deg 1} & \text{deg 0} & \text{deg -1} & \\
\hline
\text{i} = -1 & \text{deg 0} & \text{deg 0} & \text{deg -1} & \\
\hline
\end{array}\]

In each case, the complexes $P_\bullet$ and $P_\bullet'$ are depicted horizontally and the symmetric i-form $\phi : P_\bullet \rightarrow D_X(P_\bullet)[i]$ vertically. The symmetric pairs of the left-hand column
are classical symmetric and skew-symmetric forms embedded in $\mathbb{D}^b(VB_X)$ via the functor $c_0$ (slightly pushed to the left for $i = 2$). These symmetric $i$-pairs are $i$-spaces exactly when $\phi_0$ and $\phi_1$ is an isomorphism. The symmetric $i$-pairs of the right-hand column are $i$-spaces when $\phi$ is a quasi-isomorphism, i.e. when its cone is an exact complex; these are formations; we call them symmetric if $i = -1$ and skew-symmetric if $i = 1$. The four types of $i$-form presented above will be called short, for the obvious reasons.

2.5. Product of symmetric spaces.

The precise definition of this product is given in [10], where the reader will also find an explanation for the existence of two different products – the left and the right one – which differ by signs. To fix the ideas, we will use here the left product. Let $(P_\bullet, \phi)$ be a symmetric $i$-form and $(Q_\bullet, \psi)$ a symmetric $j$-form. The product

$$(P_\bullet, \phi) \ast (Q_\bullet, \psi)$$

is then a symmetric $(i + j)$-form on the tensor product (of complexes) $P_\bullet \otimes_{\mathcal{O}_X} Q_\bullet$ and we denote it by $(P_\bullet \otimes_{\mathcal{O}_X} Q_\bullet, \phi \ast \psi)$. Up to signs and identifications like for instance $P_\bullet \otimes_{\mathcal{O}_X} (Q_\bullet[j]) \simeq (P_\bullet \otimes_{\mathcal{O}_X} Q_\bullet')[j]$, the morphism of complexes $\phi \ast \psi$ is equal to the tensor product $\phi \otimes \psi$. Via $c_0$, this product coincides on short 0-spaces with the usual tensor product of symmetric spaces as defined in Knebusch [12].

2.6. Symmetric cones.

We now recall the important cone construction. Let $\phi : P_\bullet \rightarrow \mathcal{D}_X(P_\bullet)[i]$ be a symmetric $i$-form (maybe not an isomorphism). Let $Q_\bullet$ be the mapping cone of $\phi$. Then, there exists an isomorphism $\psi$ such that the following diagram commutes:

$$
\begin{array}{ccc}
P_\bullet & \xrightarrow{\phi} & \mathcal{D}_X(P_\bullet)[i] \\
\cong & \xrightarrow{(1-(\cdot)+1)} & \mathcal{D}_X(P_\bullet)[i] \\
\mathcal{D}_X(P_\bullet) & \xrightarrow{\mathcal{D}_X(\phi)[i]} & \mathcal{D}_X(P_\bullet)[i] \\
\mathcal{D}_X(\phi)[i] & \cong & \mathcal{D}_X(\phi)[i] \\
\end{array}
$$

If the isomorphism $\psi$ is moreover a symmetric $(i+1)$-form, we call such a diagram a cone diagram (over $\phi$) and we say that $(Q_\bullet, \psi)$ is a symmetric cone of the pair $(P_\bullet, \phi)$.

Note that both rows of the diagram are exact triangles in $\mathbb{D}^b(VB_X)$: The upper one by definition and the lower one is the dual of the upper row, shifted $i$ times. Assume for a moment that 2 is invertible over our scheme $X$. Then we can always choose the isomorphism $\psi$ to be a symmetric $(i+1)$-form, see [2]. Moreover, if $(Q'_\bullet, \psi')$ is another symmetric cone of $\phi$, then there exists an isometry $(Q_\bullet, \psi) \simeq (Q'_\bullet, \psi')$. We say then that $(Q_\bullet, \psi)$ is the symmetric cone of $\phi$, in symbols:

$$(Q_\bullet, \psi) = \text{cone } \phi = \text{cone}(P_\bullet, \phi).$$

2.7. Witt groups.

The usual Witt group of symmetric (respectively skew-symmetric) spaces $W_{\text{us}}(X)$ (respectively $W_{\text{us}}^-(X)$) classifies these spaces up to isometry and modulo metabolic ones. More information about these Witt groups can be found in the fundamental paper of Knebusch [12]. The $i$-th derived Witt group $W^i(X)$ classifies symmetric
\section{Periodicity.}

The Witt groups are contravariant functors. If \( f : Y \longrightarrow X \) is a morphism of schemes then the assignment \( [P, \phi] \mapsto [f^*(P), \phi] \) defines a homomorphism \( f^* : \mathcal{W}^i(X) \longrightarrow \mathcal{W}^i(Y) \) for all \( i \in \mathbb{Z} \).

\subsection{Agreement and localization (with \( \frac{1}{2} \)).}

We assume now that “\( X \) contains \( \frac{1}{2} \)”, i.e. that \( X \) is a \( \mathbb{Z}[1/2] \)-scheme, i.e. \( 2 \) is invertible in the ring \( \Gamma(X, \mathcal{O}_X) \). The main result of [3] is that the functor \( c_0 : \mathcal{V}B_X \longrightarrow \mathcal{D}^b(\mathcal{V}B_X) \) induces isomorphisms:

\[
\mathcal{W}(X) = \mathcal{W}_{us}(X) \cong \mathcal{W}^0(X) \quad [P, \phi] \mapsto [c_0(P), c_0(\phi)]
\]

and

\[
\mathcal{W}^-(X) = \mathcal{W}_{us}^-(X) \cong \mathcal{W}^2(X) \quad [Q, \psi] \mapsto [c_0(Q)[1], c_0(\psi)[1]].
\]

Other Witt groups appearing in this work are the Witt groups with support. For a complex \( P_* \in \mathcal{D}^b(\mathcal{V}B_X) \) let

\[
supp P_* := \{ x \in X \mid H_j(P_*)_x \neq 0 \text{ for at least one } j \},
\]

be its (homological) support. Let \( Z \) be a closed subscheme of \( X \) with open complement \( U \). The full triangulated subcategory of \( \mathcal{D}^b(\mathcal{V}B_X) \) which consists of complexes with support contained in \( Z \) is denoted \( \mathcal{D}^b_Z(\mathcal{V}B_X) \). The restriction of the duality \( \mathcal{D}_X \) to \( \mathcal{D}^b_Z(\mathcal{V}B_X) \) is again a duality, turning \( \mathcal{D}^b_Z(\mathcal{V}B_X) \) into a triangulated category with duality. The corresponding triangular Witt groups \( \mathcal{W}_Z^i(X) \) \((i \in \mathbb{Z})\) are called the derived Witt groups of \( X \) with support in \( Z \). They appear in the localization sequence of Balmer [2]. If \( X \) is a regular scheme then there is an exact sequence

\[
\cdots \longrightarrow \mathcal{W}^i(X) \longrightarrow \mathcal{W}^i(U) \longrightarrow \mathcal{W}_Z^{i+1}(X) \longrightarrow \mathcal{W}^{i+1}(X) \longrightarrow \cdots.
\]

The connecting morphism \( \partial \) comes from the cone construction 2.6 as follows. Let \( w \in \mathcal{W}^i(U) \). Then \( \partial(w) = [\text{cone}(P_*, \phi)] \), where \( (P_*, \phi) \) is any symmetric \( i \)-pair over \( X \) with \([[(P_*, \phi), w] = w \) (the existence of \((P_*, \phi)\) is guaranteed by regularity of \( X \)). The Witt groups with support are natural and so is the localization sequence.

\subsection{The graded Witt ring.}

The (left) product of symmetric spaces of 2.5 yields a product structure:

\[
\star : \mathcal{W}^i(X) \times \mathcal{W}^j_Z(X) \longrightarrow \mathcal{W}^{i+j}_Z(X) \quad ([P_*, \phi], [Q_*, \psi]) \mapsto [(P_*, \phi) \star (Q_*, \psi)]
\]

for any \( i, j \in \mathbb{Z} \), any scheme \( X \) and closed subset \( Z \subseteq X \). Via this pairing, \( \mathcal{W}^{\text{tot}}(X) := \bigoplus_{i \in \mathbb{Z}} \mathcal{W}^i(X) \) is a graded skew-commutative associative \( \mathcal{W}^0(X) \)-algebra, the graded Witt ring of \( X \) and \( \mathcal{W}^{\text{tot}}_Z(X) := \bigoplus_{i \in \mathbb{Z}} \mathcal{W}^i_Z(X) \) is a graded \( \mathcal{W}^{\text{tot}}(X) \)-module.
This product is compatible with the connecting homomorphism $\partial$ in the localization sequence by [10, Thm. 2.11], i.e. we have a commutative diagram

$$
\begin{array}{ccc}
W^i(X) \times W^j(U) & \overset{\ast}{\longrightarrow} & W^{i+j}(U) \\
id \times \partial & & \partial \\
W^i(X) \times W^{j+1}_Z(X) & \overset{\ast}{\longrightarrow} & W^{i+j+1}_Z(X)
\end{array}
$$

(5)

In words, the connecting homomorphism is a left $W^{\text{tot}}(X)$-linear map. But note that it is not right $W^{\text{tot}}(X)$-linear:

$$
\partial(y \ast x) = (-1)^{ij} \partial(x \ast y) \overset{(5)}{=} (-1)^{ij} x \ast \partial(y) = (-1)^{ij} \partial(y) \ast x
$$

(6)

(the first and last equation by skew-commutativity), where $x \in W^i(X)$ and $y \in W^j(U)$.

**Remark 2.11.** Of course, Convention 1.2 applies here as well. For instance, if $X = \text{Spec}(R)$ is affine and $Z \subset X$ is defined by the ideal $I$ we might say that a complex ‘has support in the ideal $I$’ and we shall write $W^i_I(R)$ instead of $W^i_Z(X)$.

### 3. Basic facts about Koszul complexes

In this section, $A$ is a ring and $\underline{T} = (T_1, \ldots, T_n)$ is any sequence in $A$. As before, we write the dual as $M^\vee := \text{Hom}_A(M, A)$, for any $A$-module $M$.

We first recall the definition of the Koszul complex

$$
K_\bullet(A, T) := (K_\bullet, d_\bullet) .
$$

Let $e_1, e_2, \ldots, e_n$ be a basis of the free $A$-module $A^n = \bigoplus_{i=1}^n A \cdot e_i$. The $A$-module $K_i = K_i(A, T) := \bigwedge^i A^n$ is by definition the $i$-th exterior power of $A^n$. As is well-known, the module $K_i$ is free with basis $\{e_{j_1} \wedge \cdots \wedge e_{j_i} \mid 1 \leq j_1 < \cdots < j_i \leq n\}$. The differential $d_i = d_i(A, T) : K_i \longrightarrow K_{i-1}$ is given by

$$
e_{j_1} \wedge \cdots \wedge e_{j_i} \longmapsto \sum_{k=1}^{i} (-1)^{k-1} T_{j_k} \cdot e_{j_1} \wedge \ldots \hat{e}_{j_k} \ldots \wedge e_{j_i},$$

where the symbol $\hat{e}_{j_k}$ indicates that $e_{j_k}$ has been omitted. We consider this (homological) Koszul complex $K_\bullet(A, \underline{T})$:

$$
\cdots 0 \longrightarrow K_n(A, \underline{T}) \overset{d_n(A, \underline{T})}{\longrightarrow} K_{n-1}(A, \underline{T}) \longrightarrow \cdots \overset{d_1(A, \underline{T})}{\longrightarrow} K_0(A, \underline{T}) \longrightarrow 0 \cdots
$$

as an element of $\mathbb{D}^b(\text{VB}_A)$ with $K_j(A, \underline{T})$ in degree $j$.

There is a structure of symmetric $n$-space on $K_\bullet(A, \underline{T})$, that we now give in an economic way; see more details in Remark 3.3. For each $i = 1, \ldots, n$, let $K_i(A, T_i) \in \mathbb{D}^b(\text{VB}_A)$ be the short Koszul complex for the one-element sequence $(T_i)$, i.e.

$$
K_i(A, T_i) = \cdots \longrightarrow 0 \longrightarrow A \overset{T_i}{\longrightarrow} A \longrightarrow 0 \longrightarrow \cdots
$$

$\deg 1 \quad \deg 0$. 


This complex can be equipped with the following symmetric 1-form (see 2.2): \[ K_n(A, T_i) = \begin{array}{ccc} \cdots & 0 & \rightarrow A & \rightarrow 0 \cdots \\ \Theta(A, T_i) & = & \begin{array}{ccc} \cdots & 0 & \rightarrow A & \rightarrow 0 \cdots \\ \rightarrow (-T_i) & \rightarrow & A & \rightarrow 0 \cdots \\ \deg 1 & \rightarrow & \deg 0, \\ \end{array} \end{array} \]

where we identify \( A = \text{Hom}_A(A, A) \) as usual. This is of course a symmetric 1-space since \( \Theta(A, T_i) \) is an isomorphism of complexes, hence a quasi-isomorphism. It is easily checked that the tensor product of complexes \( K_n(A, T_1) \otimes_A \cdots \otimes_A K_n(A, T_n) \) is equal to the Koszul complex \( K_n(A, \mathcal{T}) \) of the sequence \( \mathcal{T} = (T_1, \ldots, T_n) \) and therefore we can give the following:

**Definition 3.1.** With the above notations, we define a symmetric \( n \)-form \[ \Theta(A, \mathcal{T}) : K_n(A, \mathcal{T}) \longrightarrow D(A(K_n(A, \mathcal{T}))[n] \]

as the product (see 2.5) \[ (K_n(A, \mathcal{T}), \Theta(A, \mathcal{T})) := (K_n(A, T_1), \Theta(A, T_1)) \ast \cdots \ast (K_n(A, T_n), \Theta(A, T_n)). \]

This defines a symmetric \( n \)-space \( (K_n(A, \mathcal{T}), \Theta(A, \mathcal{T})) \) which we call the canonical space on the Koszul complex \( K_n(A, \mathcal{T}) \).

We have the following functorial property.

**Lemma 3.2.** Let \( f : A \longrightarrow B \) be a morphism of rings. Then, there is a natural isometry \[ f^*(K_n(A, \mathcal{T}), \Theta(A, \mathcal{T})) \cong (K_n(B, f(\mathcal{T})), \Theta(B, f(\mathcal{T}))). \]

**Proof.** The natural isomorphism \( K_n(A, \mathcal{T}) \otimes_A B \cong K_n(B, f(\mathcal{T})) \) is an isometry. \( \square \)

**Remark 3.3.** To define this canonical space on \( K_n(A, \mathcal{T}) \), it is not necessary to use the product structure of the derived Witt groups. The advantage of this approach is that we see at once that the canonical \( n \)-space is a symmetric \( n \)-space, but for calculations in the sequel it might be useful to have a good description of the symmetric \( n \)-form \( \Theta(A, \mathcal{T}) \). We define an isomorphism \[ \rho : K_n(A, \mathcal{T}) \cong D(A(K_n(A, \mathcal{T}))[n] \]

following [7], Sect. 1.6. We fix for this an isomorphism \( \omega : \wedge^n(A^n) \cong A \), and define an \( A \)-bilinear pairing \[ b_i : K_i(A, \mathcal{T}) \times K_{n-i}(A, \mathcal{T}) \longrightarrow A \]

by \( (x, y) \mapsto \omega(x \wedge y) \) for all \( 0 \leq i \leq n \). This \( b_i \) induces a homomorphism \( \rho_i : K_i(A, \mathcal{T}) \longrightarrow \text{Hom}_A(K_{n-i}(A, \mathcal{T}), A) = K_{n-i}(A, \mathcal{T})^\vee \) which is an isomorphism for all \( 0 \leq i \leq n \). It is straightforward (although a little cumbersome) to check that \[ d_{n-(i-1)}(A, \mathcal{T})^\vee \cdot \rho_i = (-1)^{i-1} \cdot d_i(A, \mathcal{T}). \]

Consider the family of morphisms \( \rho_i \epsilon \epsilon \mathbb{Z} \) defined by \( \rho_i := (-1)^{(i+1) + \frac{n(n-1)}{2}} \cdot \rho_i \) for \( 0 \leq i \leq n \) and by \( \rho_i = 0 \) otherwise. This defines an isomorphism of complexes \( \rho = \rho_n : K_n(A, \mathcal{T}) \longrightarrow D(A(K_n(A, \mathcal{T}))[n] \), which coincides with the morphism of complexes \( \Theta(A, \mathcal{T}) \) as a thrilling calculation using [10, Ex. 1.4, Rem. 1.9] shows.
By the following lemma this is easier to see if $\mathcal{T}$ is a regular sequence, which is the only interesting case for us here.

**Lemma 3.4.** Assume that $\mathcal{T}$ is a regular sequence, i.e. $I := \sum_{i=1}^n A T_i \neq A$ and $T_i$ is not a zero divisor in $A / \sum_{j=1}^{i-1} A T_j$ for all $i = 1, \ldots, n$. Identify $A \cong \text{Hom}_A(A, A)$ as usual. Then the following properties hold:

(i) The Koszul complex $K_*(A, \mathcal{T})$ and its $n$-dual $\mathcal{D}_A(K_*(A, \mathcal{T}))[n]$ are $A$-free resolutions of $A / I$.  

(ii) We have $B_0(\Theta(A, \mathcal{T})) = (-1)^{\frac{n(n-1)}{2}} \text{id}_{A/I}$.  

(iii) For any morphism in $\mathbb{D}^b(VB_A)$ between the Koszul complex and its $n$-dual $\varsigma : K_*(A, \mathcal{T}) \longrightarrow \mathcal{D}_A(K_*(A, \mathcal{T}))[n]$, there exists an $s \in A$ such that $\varsigma = s \cdot \Theta(A, \mathcal{T})$ in $\mathbb{D}^b(VB_A)$.  

(iv) In (iii), $\varsigma$ is an isomorphism in $\mathbb{D}^b(VB_A)$ if and only if $s + I$ is a unit in the quotient ring $A / I$.

**Proof.** For (i), $\mathcal{T} = (T_1, \ldots, T_n)$ is a regular sequence by assumption and so the complex $K_*(A, \mathcal{T})$ is an $A$-free resolution of $A / I$ by [7, Cor. 1.6.14]. For (ii), see Remark 3.3 above. Point (iii) follows from (i) and (ii). Point (iv) is immediate from (iii). \qed

**Remark 3.5.** It is clear that the restriction of $K_*(A, \mathcal{T})$ becomes zero in the derived category $\mathbb{D}^b(VB_{A[T_i^{-1}]})$ for all $1 \leq j \leq n$. Hence the complex $K_*(A, \mathcal{T})$ has support in the closed subset of Spec($A$) defined by the ideal $I := \sum_{j=1}^n A T_j$. Therefore the symmetric $n$-space $(K_*(A, \mathcal{T}), \Theta(A, \mathcal{T}))$ defines an element in 

$$[K_*(A, \mathcal{T}), \Theta(A, \mathcal{T})] \in W^n_1(A).$$

**Proposition 3.6.** Let $1 \leq i \leq n$. Define the ideal $I_i := \sum_{k \neq i} A T_k$ of $A$. Then $[K_*(A, \mathcal{T}), \Theta(A, \mathcal{T})] = 0$ in $W^n_1(A)$.

**Proof.** The group $W^{n-1}_{I_i}(A)$ obviously contains the element

$$y := [K_*(A, T_i), \Theta(A, T_1)] \ast \ast \ast [K_*(A, T_{i-1}), \Theta(A, T_{i-1})]$$

$$\ast \ast \ast [K_*(A, T_{i+1}), \Theta(A, T_{i+1})] \ast \ast \ast [K_*(A, T_n), \Theta(A, T_n)].$$

Recall from 2.10 that the product also gives

$$\ast : W^1(A) \times W^{n-1}_{I_i}(A) \longrightarrow W^n_1(A).$$

Since the product is skew-commutative, we have

$$[K_*(A, T_i), \Theta(A, T_i)] \ast y = (-1)^{i-1}[K_*(A, \mathcal{T}), \Theta(A, \mathcal{T})],$$

where we consider $[K_*(A, T_i), \Theta(A, T_i)]$ as an element of $W^1(A)$. Therefore the result follows from the observation that this element is indeed zero in $W^1(A)$. In fact, the complex $c_0(A) \in \mathbb{D}^b(VB_A)$ is a Lagrangian (cf. [2], Sect. 2) of the symmetric 1-space $(K_*(A, T_i), \Theta(A, T_i))$:

$$
\begin{array}{cccccccc}
K_*(A, T_i) &=& \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow\quad & & & & & & & & & & \\
c_0(A) &=& \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \\
\end{array}
$$

\[c_0(A) = \cdots \longrightarrow 0 \longrightarrow A \overset{T_i}{\longrightarrow} A \overset{id}{\longrightarrow} 0 \longrightarrow \cdots\]
and so $[K_\ast(A, T), \Theta(A, T)] = 0$ in $W^1(A)$. \hfill \square

In the above proof, note that the class $y \in W^n_{q-1}(A)$ does not belong to $W^n_{q-1}(A)$ and thus the argument can not be used to deduce that $[K_\ast(A, T), \Theta(A, T)] = 0$ in $W^n(A)$. On the contrary, see Theorem 7.2.

**Corollary 3.7.** $[K_\ast(A, T), \Theta(A, T)] = 0$ in $W^n(A)$.

**Proof.** Clear since $W^n(A) \longrightarrow W^n(A)$ factors via $W^n(A)$ for instance. \hfill \square

4. Koszul cut in two

We want to “split” the Koszul complex of Section 3 into two pieces, dual to each other. This is easy to understand but a little tricky to write. Recall our running conventions of Section 1 that $r + 4q = n - 1$, see (2), and that $\ell := \lceil \frac{n}{2} \rceil$, see (3). Now, more precisely, we want to define a symmetric $r$-pair $(C_\ast(A, T), \Xi(A, T))$, such that there is an isometry

$$\text{cone}(C_\ast(A, T), \Xi(A, T)) \simeq (K_\ast(A, T), \Theta(A, T)) [-2q].$$

We abbreviate the canonical form on $K_\ast(A, T) =: K_\ast$ by

$$\Theta := \Theta(A, T),$$

and set

$$E = E(A, T) := \text{Coker} \left( K_{\ell+2}(A, T) \xrightarrow{d_{\ell+2}(A, T)} K_{\ell+1}(A, T) \right). \tag{7}$$

Let $\text{pr}_E = \text{pr}_{E(A, T)} : K_{\ell+1} \longrightarrow E = \text{Coker} \ d_{\ell+2}$ be the projection. Since $d_{\ell+1}d_{\ell+2} = 0$ there exists a unique morphism $d_{\ell+1} = d_{\ell+1}(A, T) : E \longrightarrow K_\ell$, such that

$$d_{\ell+1}(A, T) = d_{\ell+1}(A, T) \cdot \text{pr}_E. \tag{8}$$

For each $j = 0, \ldots, n$, we have $\text{rank}_A (K_j) = \binom{n}{j}$. In particular, if $n = 2\ell + 1$ is odd, we have $\text{rank}_A K_\ell = \text{rank}_A K_{\ell+1}$ and life will be easy. When $n = 2\ell$ is even, $K_\ell$ has maximal (even) rank $\binom{n}{\ell}$ and we need some preparatory considerations. In this case, the symmetric $n$-form $\Theta_\ell : K_\ast \xrightarrow{\sim} \mathcal{DA}(K_\ell)[n]$ gives an isomorphism

$$\Theta_\ell : K_\ell \longrightarrow K_\ell^\vee = \text{Hom}_A(K_\ell, A)$$

which is symmetric if $\ell$ is even and skew-symmetric otherwise.

**Lemma 4.1.** If $n = 2\ell$ is even, there exists two totally isotropic subspaces $L$ and $M$ of $(K_\ell, \Theta_\ell)$, of same rank $\frac{1}{2}(\binom{n}{\ell})$, such that $K_\ell = L \oplus M$ and such that $\Theta_\ell$ becomes

$$\Theta_\ell = \begin{pmatrix} 0 & (\lambda^\vee \cdot \text{can}_M) \\ -1(\lambda) & 0 \end{pmatrix} : K_\ell = L \oplus M \longrightarrow L^\vee \oplus M^\vee = K_\ell^\vee,$$

where $\lambda : L \xrightarrow{\sim} M^\vee$ is an isomorphism. Moreover, we have

$$(-1)^\ell d_{\ell+1}^\vee \cdot (\text{pr}_L)^\vee \cdot \lambda^\vee \cdot \text{can}_M \cdot \text{pr}_M \cdot d_{\ell+1} + d_{\ell+1}^\vee \cdot (\text{pr}_M)^\vee \cdot \lambda \cdot \text{pr}_L \cdot d_{\ell+1} = 0, \tag{9}$$

where $\text{pr}_L : K_\ell \longrightarrow L$ and $\text{pr}_M : K_\ell \longrightarrow M$ denote the projections.
\textbf{Proof.} Let $e_1, \ldots, e_n$ be a basis of $A^n$. The complementary subspaces of $K_\ell = \bigwedge^\ell A^n$ are given by

\[ L := \bigoplus_{2 \leq i_2 < i_3 < \ldots < i_\ell \leq n} A \cdot e_{i_2} \wedge e_{i_3} \wedge \ldots \wedge e_{i_\ell}, \]

and

\[ M := \bigoplus_{2 \leq i_1 < i_2 < \ldots < i_\ell \leq n} A \cdot e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_\ell}, \]

both have rank \( \binom{2\ell-1}{\ell-1} = \binom{2\ell}{\ell} = \frac{1}{2} \binom{2\ell}{\ell} \). Now use the description of $\Theta_\ell$ given in Remark 3.3. Let $\omega : \bigwedge^n A^n \xrightarrow{\cong} A$ be the isomorphism which sends $e_1 \wedge \ldots \wedge e_n$ to $1 \in A$. Then $\Theta_\ell(x)(y) = \pm \omega(x \wedge y)$. From this we easily get that both subspaces are totally isotropic: for $L$ it is because $e_1 \wedge e_1 = 0$ and for $M$ it is because two subsets with $\ell$ elements in $\{2, \ldots, n\}$ must intersect. Since $\Theta_\ell$ is a $(-1)^\ell$-symmetric isomorphism, its decomposition in $L \oplus M$ must be as claimed in the lemma.

Note that $\Theta : K_* \rightarrow D_A(K_*)[n]$ is a morphism of complexes and we have

\[ d_{\ell+1}^\vee \cdot \Theta_\ell \cdot d_{\ell+1} = 0 : \]

\[ K_{\ell+1} \xrightarrow{d_{\ell+1}} K_\ell \xrightarrow{\Theta_\ell} K_\ell^\vee \xrightarrow{d_{\ell+1}^\vee} K_{\ell+1}^\vee. \]

In the decomposition $K_\ell = L \oplus M$, the morphism $d_{\ell+1} : K_{\ell+1} \rightarrow K_\ell$ becomes $\begin{pmatrix} \text{pr}_L & d_{\ell+1} \\ \text{pr}_M & d_{\ell+1} \end{pmatrix}$. Replacing this in $d_{\ell+1}^\vee \cdot \Theta_\ell \cdot d_{\ell+1} = 0$ gives equation (9) by a direct matrix multiplication. \qed

\textbf{Definition 4.2.} As the above discussion shows, we will have to distinguish the cases where $n$ is odd from those where $n$ is even and the definition extends over 4.3-4.6 below. We shall consider a sign

\[ \epsilon_n \in \{-1, 1\} \]

which will be fixed later on, see 6.3.

We start with $n = 2\ell + 1$ odd.

\textbf{4.3. Case $r = 0$.}

Then $\ell = 2q$ is even and $(C_*(A, T), \Xi(A, T))$ is defined to be the following symmetric 0-pair:

\[ \begin{array}{c}
0 \rightarrow K_n \xrightarrow{d_n} \cdots \rightarrow K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1} \xrightarrow{\epsilon_n \Theta_\ell \cdot d_{\ell+1}} K_{\ell+1}^{\vee} \xrightarrow{d_{\ell+2}^\vee} K_{\ell+2}^{\vee} \rightarrow \cdots \rightarrow 0 \rightarrow 0 \\
0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow K_{\ell+1}^{\vee} \xrightarrow{d_{\ell+2}^\vee} K_{\ell+2}^{\vee} \rightarrow \cdots \rightarrow d_n^\vee \rightarrow K_n^{\vee} \rightarrow 0
\end{array} \]

\[ \begin{array}{c}
\deg \ell \\
\deg 0.
\end{array} \]
If $T$ is a regular sequence then the Koszul complex $K_\bullet$ is exact and so we have the following quasi-isomorphism
\[
C_\ast(A, T) = \begin{array}{c}
0 \\
\vdots \\
K_n \\
\vdots \\
K_{\ell+2} \\
\vdots \\
K_{\ell+1} \\
0 \\
\end{array} 
\xrightarrow{p=p(A, T)} \begin{array}{c}
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
E \\
\end{array} 
\xrightarrow{c_0(E)} \begin{array}{c}
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
0 \\
\end{array}.
\]
\[
\deg \ell \quad \deg 0.
\]

4.4. Case $r = 2$.
Then $\ell = 2q + 1$ is odd and $(C_\ast(A, T), \Xi(A, T))$ is defined to be the following symmetric 2-pair:
\[
0 \xrightarrow{0} K_n \xrightarrow{d_n} \cdots \xrightarrow{d_{\ell+2}} K_{\ell+1} \xrightarrow{0} \cdots \xrightarrow{0} 0 \\
0 \xrightarrow{0} 0 \xrightarrow{0} \cdots \xrightarrow{\epsilon_n \Theta_{\ell+1} d_{\ell+1}} K^\vee_{\ell+1} \xrightarrow{d'_{\ell+2}} K^\vee_{\ell+2} \xrightarrow{\cdots d''_{\ell}} K^\vee_n \xrightarrow{0} \\
\deg \ell + 1 \quad \deg 1.
\]
As above if $T$ is a regular sequence the projection $\text{pr}_E : K_{\ell+1} \longrightarrow E$ induces a quasi-isomorphism of complexes $p = p(A, T) : C_\ast(A, T) \longrightarrow c_0(E)[1]$.

Now let $n = 2\ell$ be even.

We fix two totally isotropic subspaces $L$ and $M$ of $K_{\ell}$ and an isomorphism $\lambda : L \longrightarrow M^\vee$ as in Lemma 4.1 and keep notations as there. We set
\[
h := \lambda^\vee \cdot \text{can}_M \cdot \text{pr}_M \cdot d_{\ell+1} : K_{\ell+1} \longrightarrow L^\vee.
\]
We now define the space $(C_\ast(A, T), \Xi(A, T))$ for $n$ even. It follows from equation (9) in Lemma 4.1 that both squares in the middle of the two diagrams below commute and so the morphism $\Xi(A, T)$ is really a morphism of complexes.

4.5. Case $r = -1$.
Then $\ell = 2q$ is even and $(C_\ast(A, T), \Xi(A, T))$ is defined to be the following symmetric $(-1)$-pair:
\[
0 \xrightarrow{0} K_n \xrightarrow{d_n} \cdots \xrightarrow{d_{\ell+2}} K_{\ell+1} \xrightarrow{\text{pr}_L d_{\ell+1}} L \xrightarrow{0} \cdots \xrightarrow{0} 0 \\
0 \xrightarrow{0} 0 \xrightarrow{0} \cdots \xrightarrow{(\epsilon_n h') \cdot \text{can}_L} L^\vee \xrightarrow{K^\vee_{\ell+1}} \cdots \xrightarrow{-d''_{\ell}} K^\vee_n \xrightarrow{0} \\
\deg \ell - 1 \quad \deg 0 \quad \deg -1
\]
If the sequence $T$ is regular the homology of $C_\ast(A, T)$ is not concentrated in one degree (as in the case $n$ odd) but there exists a “short” complex $F_\ast(A, T)$ defined
as follows and which is quasi-isomorphic to $C_*(A, T)$:

\[
C_*(A, T) = \cdots \rightarrow K_{\ell+2} \overset{d_{\ell+2}}{\rightarrow} K_{\ell+1} \overset{pr_{L}.d_{\ell+1}}{\rightarrow} L \rightarrow 0 \rightarrow \cdots
\]

\[
F_*(A, T) := \cdots \rightarrow 0 \overset{E}{\rightarrow} L \rightarrow 0 \rightarrow \cdots
\]

\[
\begin{array}{c}
\text{deg 0} \\
\text{deg -1}
\end{array}
\]

4.6. Case $r = 1$.

Then $\ell = 2q + 1$ is odd and $(C_*(A, T), \Xi(A, T))$ is defined to be the following symmetric 1-pair:

\[
\begin{array}{c}
0 \rightarrow K_n \overset{d_n}{\rightarrow} \cdots \rightarrow K_{\ell+2} \overset{d_{\ell+2}}{\rightarrow} K_{\ell+1} \overset{pr_{L}.d_{\ell+1}}{\rightarrow} L \rightarrow 0 \rightarrow 0 \\
0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \overset{L'}{\rightarrow} K_{\ell+1}' \overset{-d_{\ell+2}'}{\rightarrow} K_n' \rightarrow 0
\end{array}
\]

\[
\begin{array}{c}
\text{deg $\ell$} \\
\text{deg 1} \\
\text{deg 0}
\end{array}
\]

As in the case $r = -1$, when $T$ is a regular sequence, we have a quasi-isomorphism $p = p(A, T) : C_*(A, T) \rightarrow F_*(A, T)$, where $F_*(A, T)$ is now the complex

\[
\cdots \rightarrow 0 \overset{E}{\rightarrow} K_{\ell+1} \overset{pr_{L}.d_{\ell+1}}{\rightarrow} L \rightarrow 0 \rightarrow \cdots
\]

\[
\begin{array}{c}
\text{deg 1} \\
\text{deg 0}
\end{array}
\]

Lemma 4.7. Let $f : A \rightarrow B$ be a morphism of rings. There is a natural isometry

\[
f^*(C_*(A, T), \Xi(A, T)) \cong (C_*(B, f(T)), \Xi(B, f(T))).
\]

Proof. Straightforward, cf. Lemma 3.2. \qed

Lemma 4.8. The mapping cone of the morphism $\Xi(A, T)$ is isomorphic (as a complex) to $K_*(A, T)[-2q]$.

Proof. This is an easy direct computation, which we leave to the reader. It is clear in the cases where $n$ is odd and it requires Lemma 4.1 for $n$ even. In all four cases, we use the isomorphism $\Theta$ to replace the $K_j^\vee$ by $K_{n-j}$ for $j \geq \ell + 1$. \qed

Remark 4.9. Note that we do not claim that the symmetric cone of the symmetric $r$-form $\Xi(A, T)$ is the Koszul complex with its canonical form $\Theta(A, T)$. This would be true, however, with a suitable choice of the sign $\epsilon_n$. Instead of going into these computations, we shall use a simplifying trick: see 6.3.
Let $R$ be a ring. We apply the constructions of Section 3 to $A := R[T_1, \ldots, T_n]$, the polynomial ring in $n$ variables over $R$, and to the sequence $T := (T_1, \ldots, T_n)$. The reader can think of $R = \mathbb{Z}$ or $R = \mathbb{Z}[1/2]$, since these are the important cases, from which the rest will be deduced.

**Definition 5.1.** The Koszul symmetric $n$-space $K^{(n)}_R$ over $A^n_R$ is the symmetric $n$-space where $K^{(n)}_R := K\bullet(A, T)$ will be called the Koszul complex over $A^n_R$ and where the symmetric $n$-form $\Theta^{(n)}_R := \Theta(A, T)$ is the one of Definition 3.1.

**Remark 5.2.** Pay attention: $K^{(n)}_R$ is a symmetric $n$-space defined over the ring $A = R[T_1, \ldots, T_n]$ and not over the ring $R$, as the notation might suggest.

It is clear that the Koszul symmetric $n$-space behaves well with respect to base-change. More precise, let $f : R \rightarrow S$ be a morphism of rings and let

$$\alpha_f : R[T_1, \ldots, T_n] \rightarrow S[T_1, \ldots, T_n]$$

be the obvious induced morphism. Then, by Lemma 3.2 there is a natural isometry

$$\alpha_f^*(K^{(n)}_R) \cong K^{(n)}_S.$$

In particular, $K^{(n)}_R$ is extended from $K^{(n)}_\mathbb{Z}$. This justifies the following extension of Definition 5.1.

**Definition 5.3.** Let $X$ be a scheme. We define the symmetric $n$-space

$$K^{(n)}_X := \alpha_X^*(K^{(n)}_\mathbb{Z})$$

where $\alpha_X : \mathbb{A}^n_X \rightarrow \mathbb{A}^n_\mathbb{Z}$ is the base-change morphism, see 1.4. We call $K^{(n)}_X$ the Koszul symmetric $n$-space over $\mathbb{A}^n_X$. Like before, we denote the underlying complex of free $\mathcal{O}_{\mathbb{A}^n_X}$-modules and its symmetric $n$-form by

$$K^{(n)}_X = \alpha_X^*(K^{(n)}_\mathbb{Z}) \quad \text{and} \quad \Theta^{(n)}_X = \alpha_X^*(\Theta^{(n)}_\mathbb{Z}).$$

**Remark 5.4.** It is obvious from the definition that for any morphism of schemes $f : Y \rightarrow X$ we have an isometry $\alpha_f^*(K^{(n)}_Y) \cong K^{(n)}_Y$ over $\mathbb{A}^n_Y$.

**Definition 5.5.** By Remark 3.5, the complex $K^{(n)}_X$ has support in the closed subset $\mathbb{A}^n_X \setminus U^n_X$ of $\mathbb{A}^n_X$. Therefore, the symmetric $n$-space $K^{(n)}_X$ represents a Witt class

$$\kappa^{(n)}_X := [K^{(n)}_X] \in W^n_{\mathbb{A}^n_X \setminus U^n_X}(\mathbb{A}^n_X).$$

**Remark 5.6.** Note that there are several choices of signs in the definition of the symmetric $n$-space $K^{(n)}_X$. Other sign conventions give the same space or its negative, but the results of this work are clearly independent of such choices.
6. The half-Koszul symmetric space $B_X^{00}$ over $U_X^n$

**Definition 6.1.** Let $R$ be a ring. We now apply the splitting of Section 4 to the space $K_R^{00}$ of Section 5. As above, we put $A := R[T_1, \ldots, T_n]$ and $\Sigma := (T_1, \ldots, T_n)$. We define

$$C_R^{00} := C_\Sigma(A, \Sigma) \quad \text{and} \quad \Xi_R^{00} := \Xi(A, \Sigma)$$

as defined in 4.3 to 4.6. For any scheme $X$ we define

$$C_X^{00} := \alpha_X^{00}(C_X^{00}) \quad \text{and} \quad \Xi_X^{00} := \alpha_X^{00}(\Xi_X^{00})$$

where $\alpha_X : \mathcal{A}_X^{00} \longrightarrow \mathcal{A}_X^{00}$ is the base-change morphism. This coincides with the above in the affine case by Lemma 4.7. For all $n \in \mathbb{N}$, the symmetric $r$-pair $(C_X^{00}, \Xi_X^{00})$ on $\mathcal{A}_X^{00}$ will be denoted by $C_X^{00}$.

**Remark 6.2.** We have the following facts:

1. The pair $C_X^{00}$ is a symmetric $r$-pair (see Definition 2.2) indeed.
2. If $f : Y \longrightarrow X$ is a morphism of schemes then there is a natural isometry

$$\alpha_f^{00}(C_X^{00}) \cong C_Y^{00}.$$ 

**6.3. The symmetric cone of $C_X^{00}$.**

Instead of calculating $\text{cone}(C_X^{00})$ directly (which is possible, but cumbersome) we take full advantage of Lemma 3.4. More precisely, we use the fact that any quasi-isomorphism $K,*(\mathbb{Z}, \Sigma) \longrightarrow D_{\mathbb{Z}[T_1, \ldots, T_n]}(K,*(\mathbb{Z}, \Sigma))$ is equal to the symmetric $n$-form $\pm \Theta^{00}_\Sigma$ in $\mathbb{D}^b(VB_{\mathbb{Z}[T_1, \ldots, T_n]}).

So let for a moment $R = \mathbb{Z}$ and $A = \mathbb{Z}[T_1, \ldots, T_n]$. We abbreviate $K,* := K,*(A, \Sigma)$ and $\Theta = \Theta(A, \Sigma)$. We get from Lemma 4.8 the following commutative diagram (where $D = D_A$)

$$
\begin{array}{cccccc}
C_X^{00} & \xrightarrow{\Xi^{00}} & D(C_X^{00})[r] & \xrightarrow{u} & K,*[-2q] & \xrightarrow{v} & C_X^{00}[-1] \\
\downarrow (-1)^{r(r+1)} & & \downarrow = & & \downarrow (-1)^{r(r+1)} & & \downarrow

\end{array}
$$

$$
\begin{array}{cccccc}
\mathbb{D}(C_X^{00}) \xrightarrow{D^{00} = \mathbb{D}(\Xi^{00})} D(C_X^{00})[r] & \xrightarrow{D(\Sigma)} D(K,*[-2q])[r + 1] & \xrightarrow{\mathbb{D}(\Theta)} D(C_X^{00})[1]

\end{array}
$$

whose rows are exact triangles for all $n \in \mathbb{N}$ (the bottom row is the dual of the upper one, shifted $r$ times). By the very basic properties of triangulated categories there exists an isomorphism

$$\varsigma : K,*[-2q] \longrightarrow D(K,*[-2q]) + 1 = (D(K,*))[n][-2q],$$

in $\mathbb{D}^b(VB_A)$ such that diagram (10) commutes. By Lemma 3.4 the isomorphism $\varsigma$ is equal to $\pm \Theta[-2q]$. Replacing if necessary $\Xi^{00}_\Sigma$ by $-\Xi^{00}_\Sigma$, i.e. replacing $\epsilon_n$ by $-\epsilon_n$ in the definition of $C_R^{00}$, we can assume that $\varsigma = \Theta[-2q]$ for all $n \in \mathbb{N}$, i.e. $(K,*[-2q], \varsigma) = K^{00}_\Sigma[-2q]$ for all $n \in \mathbb{N}$.

We fix $\epsilon_n$ as explained above.
We can now calculate $\text{cone}(C_X^{(n)})$ for any scheme $X$ and any $n \in \mathbb{N}$. The pull-back via the base-change morphism $\alpha_X : A_X^r \to A_Z^r$ of diagram (10) above is a cone diagram for the symmetric $r$-form $\alpha_X(C_X^{(n)})$. By Lemma 4.7 we have an isometry $C_X^{(n)} \simeq \alpha_X^*(C_Z^{(n)})$ and so we get $\text{cone}(C_X^{(n)}) \simeq \alpha_X^*(K_Z^{(n)}[-2q]) \simeq \alpha_X^*(K_X^{(n)})(-2q)$ (cf. Lemma B.1 for the later isometry). We have proven:

**Theorem 6.4.** With this choice of $\epsilon_n$, the cone of the symmetric pair $C_X^{(n)}$ is the Koszul symmetric space shifted as follows:

$$\text{cone}(C_X^{(n)}) = K_X^{(n)}[-2q].$$

In particular, $\Xi_X^{(n)}|_{U_X^n}$ is an isomorphism because the homology of $K_*|_{U_X^n}$ vanishes.

**Definition 6.5.** Let $X$ be a scheme. The symmetric $r$-space

$$B_X^{(n)} := C_X^{(n)}|_{U_X^n}$$

will be called the half-Koszul space over the scheme $X$. Its Witt class is denoted by

$$\varepsilon_X^{(n)} := [B_X^{(n)}] \in W^r(U_X^n).$$

The following is obvious (cf. Remark 6.2).

**Lemma 6.6.** Let $f : Y \to X$ be a morphism of schemes. Then there is a natural isometry

$$\nu_Y^*(\varepsilon_X^{(n)}) \simeq \varepsilon_Y^{(n)}.$$

By the main result of [3] we know that $B_X^{(n)}$ is Witt equivalent to a space living on a short complex (see 2.4). In fact, $B_X^{(n)}$ is not only Witt equivalent, but isometric to such a space on a “short complex”.

We use the notation of 4.3–4.6 with $R = \mathbb{Z}$, i.e. $A = \mathbb{Z}[T_1, \ldots, T_n]$ is the polynomial ring in $n$ variables over $\mathbb{Z}$, $T = (T_1, \ldots, T_n)$, and $K_* = K_*(A, T)$ is the Koszul complex of the sequence $T$ over $A$. As in 4.3–4.6 we denote the differential of this Koszul complex by $d_\bullet$ and set

$$E = E(A, T) = \text{Coker}(K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1}).$$

Note that $T$ is a regular sequence and so $K_\bullet$ is a finite free resolution of $\mathbb{Z} \simeq A/I$, where $I$ is the ideal generated by $T$. It follows that $K_*(A, T)|_{\text{Spec} A[T^{-1}]}$ is a split exact sequence and so

$$E_Z^{(n)} := \text{Coker}(K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1})|_{U_Z^n} = E|_{U_Z^n} \simeq \text{Ker} d_\ell|_{U_Z^n}$$

is a locally free $\mathcal{O}_{U_Z^n}$-module of rank $\ell \sum_{i=0}^{n} \binom{n}{i} = (n-1)^\ell$. Clearly the same is true for the pull-back

$$E_X^{(n)} := \nu_X^*(E_Z^{(n)}),$$

where $\nu_X : U_X^n \to U_Z^n$ is induced by base change, see (4). Note that we have

$$E_X^{(n)} = \text{Coker}(K_{X,\ell+2}^{(n)} \xrightarrow{K_{X,\ell+1}^{(n)}} K_{X,\ell+1}^{(n)})|_{U_X^n} \simeq \text{Ker}(K_{X,\ell+2}^{(n)} \xrightarrow{K_{X,\ell+1}^{(n)}} K_{X,\ell+1}^{(n)})|_{U_X^n},$$

where $K_{X}^{(n)} = K_{X,\bullet}^{(n)} = \alpha_X^*(K_Z^{(n)})$ (see Definition 5.3).

We consider now the case $n$ odd and $n$ even separately.
6.7. The space \( B^n_X \) if \( n = 2\ell + 1 \) is odd, i.e. \( r = 0 \) or \( r = 2 \).

Since the functor \((-)^\vee = \text{Hom}_A(-, A)\) is left exact we have \( E^\vee = \text{Ker}\ (K_{\ell+1} \xrightarrow{d_{\ell+1}} K_{\ell+2}) \) and hence a well defined homomorphism

\[
\varphi^0_Z \colon \epsilon_1, \theta_1 \cdot d_{\ell+1} |_{U_2} : E^0_Z \longrightarrow \text{Hom}_{O_{U_2}} (E^0_Z, O_{U_2}) = E^\vee |_{U_2}
\]

which is \((-1)^{\ell}\)-symmetric, where \( d_{\ell+1} \) is the unique morphism \( K_{\ell+1} \longrightarrow E \), such that \( d_{\ell+1} = d_{\ell+1} \cdot \text{pr}_E \) (cf. (8)). We set

\[
(F^0_X, \phi^0_X) := \left\{ \begin{array}{ll}
c_0(E^0_Z, \varphi^0_Z) & \text{if } \ell \text{ is even} \\
c_0(E^0_Z, \varphi^0_Z)[1] & \text{if } \ell \text{ is odd.}
\end{array} \right.
\]

This is a symmetric \( r \)-pair. Recall now from 4.3–4.6 that the projection \( \text{pr}_E : K_{\ell+1} \longrightarrow E \) induces a quasi-isomorphism (hence an isomorphism in \( D^b(V B_A) \))

\[
p|_{U_2} = p(Z, T) |_{U_2} : C^0_Z \xrightarrow{\cong} c_0(E^0_Z) \quad \text{ (respectively } C^0_Z \xrightarrow{\cong} c_0(E^0_Z)[1], \text{ if } c_0(E^0_Z) \text{ is odd).}
\]

which is easily seen to be an isomorphism \( B^0_Z \xrightarrow{\cong} (F^0_X, \phi^0_X) \). It follows that \( \phi^0_X \) is an isomorphism and so \((F^0_X, \phi^0_X)\) a symmetric \( r \)-space. In particular \((E^0_X, \phi^0_X)\) is a \((-1)^{\ell}\)-symmetric space over \( U_2 \). Applying the pull-back \( \nu_X^\ell \) we get:

(i) The pair

\[
(E^0_X, \varphi^0_X) := \nu_X^\ell(E^0_Z, \varphi^0_Z)
\]

is a \((-1)^{\ell}\)-symmetric space over \( U_X \).

(ii) The half Koszul space \( B^n_X \) is isometric to the short symmetric \( r \)-space:

\[
E^0_X := (F^0_X, \phi^0_X) : \nu_X^\ell(F^0_Z, \phi^0_Z) = \left\{ \begin{array}{ll}
c_0(E^0_X, \varphi^0_X) & \text{if } \ell \text{ is even} \\
c_0(E^0_X, \varphi^0_X)[1] & \text{if } \ell \text{ is odd.}
\end{array} \right.
\]

Example 6.8. If \( n = 1 \) then \((E^0_X, \varphi^0_X)\) is the symmetric space

\[
\mathcal{O}_{U_X} \xrightarrow{T} \mathcal{O}_{U_X} = \text{Hom}_{\mathcal{O}_{U_X}} (\mathcal{O}_{U_X}, \mathcal{O}_{U_X}).
\]

6.9. The space \( B^n_X \) if \( n = 2\ell \) is even, i.e. \( r = -1 \) or \( r = 1 \).

We fix \( L, M \subset K_{\ell} \) and \( \lambda : L \xrightarrow{\cong} M^\vee \) as in Lemma 4.1 (with \( R = Z \)), and let \( \text{pr}_L : K_{\ell} \longrightarrow L \) and \( \text{pr}_M : K_{\ell} \longrightarrow M \) be the respective projections. We denote \( L := L |_{U_2} \) and \( \text{pr}_L := \text{pr}_L |_{U_2} : K_{\ell+1} |_{U_2} \longrightarrow L. \)

On the complex \( F^0_Z := F_*(A, T) |_{U_2} \) we have the following symmetric \( r \)-form:

\[
\begin{array}{c}
\cdots \longrightarrow E^0_Z \xrightarrow{pr_L} \cdots \longrightarrow L' \xrightarrow{(pr_L \cdot d_{\ell+1}) |_{U_2}} (E^0_Z)^\vee \longrightarrow \\
\phi^0_Z |_{U_2} \biggm/ \Downarrow \epsilon_1, h |_{U_2} \biggm/ \Downarrow (1)^\ell \epsilon_1, h |_{U_2} \biggm/ \Downarrow (\text{can}_Z) \biggm/ \Downarrow (pr_L \cdot d_{\ell+1} |_{U_2}) \biggm/ \Downarrow (E^0_Z)^\vee
\end{array}
\]

if \( r = -1 \)

\[
\begin{array}{l}
\text{deg } 0 \\
\text{deg } -1
\end{array}
\]

if \( r = 1 \)

\[
\begin{array}{l}
\text{deg } 1 \\
\text{deg } 0
\end{array}
\]
where \( h = \lambda \cdot \text{can}_M \cdot \text{pr}_M \cdot d_{i+1} \). Since \( d_{\ell+1}|_{U^\ell} = (d_{\ell+1} \cdot \text{pr}_E)|_{U^\ell} \) we see that the quasi-isomorphism \( p|_{U^\ell} : \mathcal{F}_Z^{(\ell)} \cong \mathcal{C}_Z^{(\ell)} \) is an isomorphism \( B_Z^{(\ell)} \cong (\mathcal{F}_Z^{(\ell)}, \phi_Z^{(\ell)})_k \) and so \( \phi_Z^{(\ell)} \) is an isomorphism in \( \mathcal{D}(\mathbb{V}B_Z) \). Therefore \( (\mathcal{F}_Z^{(\ell)}, \phi_Z^{(\ell)})_k \) is a symmetric \( r \)-space over \( U^\ell \). Applying the pull-back \( v_X^\ell \) we get:

The half Koszul space \( B_X^{(\ell)} \) is isometric to the short symmetric \( r \)-space:

\[
E_X^{(\ell)} := (\mathcal{F}_X^{(\ell)}, \phi_X^{(\ell)}) := v_X^\ell(\mathcal{F}_Z^{(\ell)}, \phi_Z^{(\ell)}).
\]

Remark 6.10. We give in Appendix A a proof of the following fact. If \( n \geq 3 \) then the locally \( \mathcal{O}_{U^n} \)-module \( E_X^{(\ell)} \) can not be extended to a locally free \( \mathcal{O}_{\mathbb{A}^n_X} \)-module, and hence in particular is not extended from \( X \). Of course this is wrong for \( n = 1, 2 \), since then \( \ell = 0 \) and therefore \( E_Z^{(\ell)} \cong K_\bullet(\mathbb{Z}, \mathbb{T})|_{U^\ell} \) which is free.

It follows already from this that it is impossible to extend the symmetric \( r \)-space \( E_X^{(\ell)} \) to \( \mathbb{A}^n_X \) as long as \( n \geq 3 \). We will see in Theorem 8.2 that even more is true. It is impossible to extend \( E_X^{(\ell)} \) up to Witt equivalence to \( \mathbb{A}^n_X \), i.e. \( [B_X^{(\ell)}] \) is not in the image of \( W^\bullet(\mathbb{A}^n_X) \to W^\bullet(U^n_X) \), and this for any \( n \geq 1 \).

7. Witt groups of the punctured affine space

Recall the notations of Section 1, like formula \( (4) \), defining \( r \in \{-1, 0, 1, 2\} \) by \( n = 4q + r + 1 \). We begin with an easy application of triangular Witt theory:

**Theorem 7.1.** Let \( X \) be a regular scheme containing \( \frac{1}{2} \). There exists a split short exact sequence

\[
0 \longrightarrow W^i(X) \overset{\sigma_X^*}{\longrightarrow} W^i(U^n_X) \overset{\partial}{\longrightarrow} W^{i+1}(\mathbb{A}^n_X \setminus U^n_X) \longrightarrow 0,
\]

for all \( i \in \mathbb{Z} \), where \( \partial \) is the connecting homomorphism of the localization \( U^n_X \subset \mathbb{A}^n_X \).

This sequence is natural in \( X \) in the obvious way. Moreover, a left inverse to \( \sigma_X^* \) is given by \( \gamma^* : W^i(U^n_X) \longrightarrow W^i(X) \) for any \( X \)-point \( \gamma : X \to U^n_X \), i.e. any morphism \( \gamma : X \to U^n_X \) such that \( \sigma_X \circ \gamma = \text{id}_X \).

**Proof.** Consider the commutative (plain) diagram:

\[
\begin{array}{ccccccccc}
\cdots & \overset{\partial}{\longrightarrow} & W^{i-1}(\mathbb{A}^n_X \setminus U^n_X) & \overset{\sigma_X^*}{\longrightarrow} & W^i(\mathbb{A}^n_X) & \overset{\gamma^*}{\longrightarrow} & W^i(U^n_X) & \overset{\partial}{\longrightarrow} & W^{i+1}(\mathbb{A}^n_X \setminus U^n_X) & \overset{\sigma_X^*}{\longrightarrow} & W^{i+1}(\mathbb{A}^n_X) & \cdots \\
\pi_X^* & & & & & & & & & & & \\
\end{array}
\]

The long sequence is exact by localization, see 2.9, and the homomorphism \( \pi_X^* \) is an isomorphism by homotopy invariance [4, Thm. 3.1]. Now, for any \( X \)-point \( \gamma : X \to U^n_X \), for instance \( (1, \ldots, 1) \), since \( \sigma_X \circ \gamma = \text{id}_X \), the homomorphism \( \gamma_X^* \) is split injective with the wanted left inverse. Hence the homomorphism \( \sigma_X^* \) is also split injective and the unlabeled morphism in the above diagram must be equal to zero, for all \( i \in \mathbb{Z} \), in particular for \( i + 1 \). This, in turn, gives the surjectivity of \( \partial \) and the result follows easily. \( \Box \)
We want to apply “dévissage” to the relative groups $W^{i+1}_{\mathcal{A}_X^n \setminus \mathcal{U}_X^n} (\mathcal{A}_X^n)$ and we will need

**Theorem 7.2.** Let $X$ be a regular $\mathbb{Z}[1/2]$-scheme of finite Krull dimension. Consider the structure morphism $\pi_X : \mathcal{A}_X^n \to \text{Spec } X$. Then, the homomorphism

$$\vartheta^{\mathcal{A}_X^n}_X : W^{i-n}(X) \longrightarrow W^i_{\mathcal{A}_X^n \setminus \mathcal{U}_X^n} (\mathcal{A}_X^n), \quad w \mapsto \pi_X^*(w) \star \kappa_{X}^{n_i}$$

is an isomorphism for all $i \in \mathbb{Z}$.

**Proof.** The affine case $X = \text{Spec}(R)$ is [9, Thm. 9.3].

Since the homomorphism $\vartheta^{\mathcal{A}_X^n}_X$ is given by the product with a “universal” Witt class $\kappa_{X}^{n_i} := [K_{X}^{n_i}]$, we can deduce the global statement from the affine case by Mayer-Vietoris as follows. If a regular scheme $V_0 = V_1 \cup V_2$ is covered by two open subschemes $V_1$ and $V_2$, we also have a covering of the corresponding affine spaces $\mathcal{A}_n$, in a compatible way with the closed subsets $\mathcal{U}_n \setminus \mathcal{U}_n$.

So, we get a Mayer-Vietoris long exact sequence [4] with supports, given in the first row of the diagram below, where we use the abbreviations $Y^n(X) := \mathcal{A}_n \setminus \mathcal{U}_n$ for all schemes $X$ and $V_3 := V_1 \cap V_2$:

$$\cdots \to W^i_{Y^n(V_1)}(\mathcal{A}_{V_1}^n) \oplus W^i_{Y^n(V_2)}(\mathcal{A}_{V_2}^n) \to W^i_{Y^n(V_3)}(\mathcal{A}_{V_3}^n) \xrightarrow{\partial} W^{i+1}_{Y^n(V_0)}(\mathcal{A}_{V_0}^n) \to \cdots$$

The second line is exact by the Mayer-Vietoris exact sequence for $V_0 = V_1 \cup V_2$. We claim that the diagram commutes. This is easy to check for the unmarked squares. For the square marked $(\dagger)$ this follows from the following calculation. Let $w \in W^{i-n}(V_3)$, then

$$\partial(\pi_{V_3}^*(w) \star [K_{V_3}^{n_i}]) = \partial(\pi_{V_3}^*(w) \star [K_{V_3}^{n_i}]) = (-1)^n \partial(\pi_{V_0}^*(w)) \star [K_{V_0}^{n_i}]$$

The first equality holds without $\partial$ and uses the following fact: before computing the product of the class $[K_{V_3}^{n_i}] \in W^{i-n}(V_3)(\mathcal{A}_{V_3}^n)$ with $\pi_{V_3}^*(w) \in W^{i-n}(\mathcal{A}_{V_3}^n)$, we can as well restrict $[K_{V_3}^{n_i}]$ to $\mathcal{A}_{V_3}^n$, and then multiply: this restriction of $[K_{V_3}^{n_i}]$ is precisely $[K_{V_3}^{n_i}]$ by Remark 5.4. The second equality is a consequence of (6). It then suffices to apply naturality of the localization sequence to replace $\partial(\pi_{V_3}^*(w))$ by $\pi_{V_0}^*(\partial(w))$ and we have the claimed commutativity of $(\dagger)$. The usual Five-Lemma gives the statement by induction on the number of open subschemes in an affine covering of the regular scheme $X$ (recall Convention 1.3).

**Remark 7.3.** We do not now whether $\vartheta^{\mathcal{A}_X^n}_X$ is an isomorphism for more general schemes $X$, like e.g. regular (affine) schemes of infinite Krull dimension. The proof of [9, Thm. 9.3] uses coherent Witt groups and therefore only applies to regular rings of finite Krull dimension.

**Theorem 7.4.** Let $X$ be a $\mathbb{Z}[1/2]$-scheme. Let $1 \leq i \leq n$ be an integer and consider the $X$-point $\gamma_i : X \to \mathcal{U}_X^n \subset \mathcal{A}_X^n$ corresponding to $T_i = 1$ and $T_j = 0$ for all $j \neq i$. 

If $n \geq 2$, then the evaluation at this point of the Witt class $\varepsilon_X^{(n)} \in W^r(U^n_X)$ of the half-Koszul space is zero

$$
\gamma_i^{r}(\varepsilon_X^{(n)}) = 0 \text{ in } W^r(X).
$$

**Proof.** Also denote by $\gamma_i : \text{Spec}(\mathbb{Z}[1/2]) \rightarrow U^n_{\mathbb{Z}[1/2]}$ the “same” point. We have a commutative diagram

$$
\begin{array}{ccc}
W^r(U^n_{\mathbb{Z}[1/2]}) & \xrightarrow{\gamma_i} & W^r(\mathbb{Z}[1/2]) \\
\downarrow v_X & & \downarrow \\
W^r(U^n_X) & \xrightarrow{\gamma_i} & W^r(X)
\end{array}
$$

with the obvious morphisms and we know from 6.6 that $v_X^*(\varepsilon_X^{(n)}) = \varepsilon_X^{(n)}$. Therefore, it suffices to prove the result for $\mathbb{Z}$ since $\mathbb{Z}$ is a Dedekind domain, we have by [6, Thm. 10.1] that $W^r(\mathbb{Z}[1/2]) = W^2(\mathbb{Z}[1/2]) = 0$ and $W^1(\mathbb{Z}[1/2]) \cong \text{Coker} \left( W(\mathbb{Z}[1/2]) \xrightarrow{\sum \partial_p} \bigoplus_{p \neq 2} W(\mathbb{Z}/zp) \right)$, where $\partial_p$ is a second residue homomorphism associated with the prime number $p$. But this cokernel is also zero by the classical calculation of the Witt group of $\mathbb{Q}$, cf. e.g. [14, Thm. VI.6.11].

If $r = 0$, i.e. $n = 2\ell + 1$ with $\ell \neq 0$ even this follows from the following lemma. □

**Lemma 7.5.** Let $R$ be a ring and $n = 2\ell + 1$ with $\ell \geq 2$ even. Then $\gamma_i^r(B_R^{(n)}) \cong \gamma_i(B_R^{(n)})$ is trivial in $W(R) \cong W^0(R)$.

**Proof.** After renumbering we may assume $i = 1$. Let $K_s = K_s(A, T)$ be the Koszul complex of the regular sequence $T = (T_1, \ldots, T_n)$ over $A = R[T_1, \ldots, T_n]$, and $\Theta = \Theta(A, T) : K_s \xrightarrow{\cong} D_A(K_s)[n]$ the canonical symmetric $n$-form. Recall that the differential $d_s : K_s \rightarrow K_{s-1}$ is then given by

$$
e_i \land \ldots \land e_{i_s} \rightarrow \sum_{j=1}^s (-1)^{j-1}T_j \cdot e_i \land \ldots \land e_{i_{j-1}} \land e_{i_{j+1}} \land \ldots \land e_{i_s},$$

where $e_1, \ldots, e_n$ constitute a basis of $A^n$. We denote (cf. 1.4) $i_R$ the open immersion $U^n_R \hookrightarrow A^n_R$. Then we have

$$(C', \Xi') := \gamma_i(B_R^{(n)}) = \gamma_i^r(B_R^{(n)})$$

and $K'_s = \gamma_i^r(K_s)$ is the Koszul complex $K_s(R, (1,0,\ldots,0))$ for the sequence $(1,0,\ldots,0) \subseteq R$. We denote the differential of this Koszul complex by $d'$. Note that this complex is split exact. The isomorphism of complexes $\Theta' := \gamma_i^r(\Theta)$ is a symmetric $n$-form on $K'_s$ and $(C', \Xi')$ is the following symmetric 0-space:

$$
\begin{array}{cccccc}
0 & \rightarrow & K'_n & \xrightarrow{d'_n} & K'_{n-1} & \rightarrow & \cdots & \rightarrow & K'_{t+2} & \xrightarrow{d'_{t+2}} & K'_{t+1} & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow \epsilon_n \Theta' d'_{t+2} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 0 & \rightarrow & K''_{t+1} & \xrightarrow{d''_{t+1}} & K''_{t+2} & \rightarrow & \cdots & \rightarrow & K''_n & \rightarrow & 0 \\
\text{deg } \ell & & \text{deg } 0
\end{array}
$$
over $R$, where we have set $K'^{r}_{i} = \text{Hom}_{R}(K'_{i}, R)$. We give now a direct summand $E'$ of $K'_{\ell+1}$, such that the projection $K'_{\ell+1} \rightarrow E'$ induces a quasi-isomorphism $C_{*} \overset{\cong}{\rightarrow} c_{0}(E')$ ($c_{0} : VB_{R} \rightarrow \mathcal{D}^{b}(VB_{R})$ the natural embedding).

The elements $v_{i} = 1 \otimes e_{i}$ ($i = 1, \ldots, n$) are a basis of $R \otimes_{A} A^{n} = \gamma_{\ell}^{*}(A^{n})$, and so the exterior products $v_{i_{1}} \wedge \cdots \wedge v_{i_{s}}$ ($1 \leq i_{1} < \cdots < i_{s} \leq n$) are free generators of $K'_{s} \cong \wedge^{s} R^{n}$. The differential $d'_{s}$ acts on them as follows:

$$d'_{s}(v_{i_{1}} \wedge \cdots \wedge v_{i_{s}}) = \begin{cases} 0 & \text{if } 2 \leq i_{1} < \cdots < i_{s} \leq n; \\ v_{i_{1}} \wedge \cdots \wedge v_{i_{s}} & \text{if } 1 = i_{1} < i_{2} < \cdots < i_{s} \leq n, \end{cases}$$

therefore the $R$-module $E' := v_{1} \wedge (\bigwedge^{\ell} R^{n}) \subset K'_{\ell+1}$ is isomorphic to Coker $d'_{\ell+2}$. Hence $c_{0}(E') \cong C_{*}$ because $C_{*}$ has non vanishing homology only in degree 0, and we have an isometry $\varepsilon_{0}(E', \varphi') \cong (C_{*}, \Xi')$, where $\varphi' := \varepsilon_{0}(\Theta'_{\ell} : d'_{\ell+1} |_{E'})$. To see this we use the description of $\Theta$ given in Remark 3.3. Let $\omega : \wedge^{n} A^{n} \overset{\cong}{\rightarrow} A$ be as in this remark and $\omega' := \text{id}_{R} \otimes \omega$. Then $\omega'(v_{i_{1}} \wedge \cdots \wedge v_{i_{s}}) = 1$ and $\Theta'_{\ell}(x)(y) = \pm \omega'(x \wedge y)$ for all $x \in K'_{\ell}$ and $y \in K'_{\ell+1}$. If now $x, y \in M'$ then $y = v_{1} \wedge v_{2} \wedge y'$ and $d'_{\ell+1}(x) = v_{2} \wedge x'$ for some $x', y' \in \wedge^{\ell-1} R^{n}$, and so $\pm \varphi'(y) = \omega'(d'_{\ell+1}(x) \wedge y) = 0$ since $v_{2} \wedge v_{2} = 0$.

**Theorem 7.6.** Let $X$ be a regular $\mathbb{Z}[1/2]$-scheme. The composition of the connecting homomorphism with the 4-periodicity isomorphism:

$$W^{r}(U_{X}^{n}) \xrightarrow{\partial} W^{r+1}_{\mathcal{K}_{X} \setminus U_{X}}(A^{n}_{X}) \xrightarrow{\tau_{n}^{X}} W^{r+1}_{\mathcal{K}_{X} \setminus U_{X}}(A^{n}_{X}),$$

maps the Witt class $\varepsilon^{[n]}_{X} \in W^{r}(U_{X}^{n})$ of the half-Koszul space to the Witt class $\kappa^{[n]}_{X} \in W^{r}_{\mathcal{K}_{X} \setminus U_{X}}(A^{n}_{X})$ of the Koszul space over $\mathcal{K}_{X}$.

**Proof.** Recall that we always have $n = 4q + r + 1$. The statement is a direct consequence of Theorem 6.4, using the definition of the connecting homomorphism $\partial$ via the symmetric cone 2.9 and the fact that $\varepsilon^{[n]}_{X} = [C^{[n]}_{X} |_{U_{X}}]$ by Definition 6.5.

**Theorem 7.7.** Let $X$ be a regular $\mathbb{Z}[1/2]$-scheme. For all $i \in \mathbb{Z}$, define the following homomorphism:

$$\rho_{X}^{[n]} : W^{i-r}(X) \rightarrow W^{i}(U_{X}^{n}), \quad w \mapsto \sigma_{X}^{*}(w) \ast \varepsilon^{[n]}_{X}.$$

Then the following diagram commutes:

$$W^{i-r}(X) \xrightarrow{\tau_{n}^{X}} W^{i+1-n}(X) \quad \rho_{X}^{[n]} \bigg|_{W^{i}(U_{X}^{n})} \xrightarrow{\partial_{X}^{[n]}} W^{i+1}_{\mathcal{K}_{X} \setminus U_{X}}(A^{n}_{X})$$

for all $i \in \mathbb{Z}$, where the isomorphism $\partial_{X}^{[n]}$ is the one of Theorem 7.2 and where $\tau$ is the 4-periodicity isomorphism.
Proof. Recall of course that $1 - n + 4q = r$ by (2). We have to show that
\[
\partial \rho_X^o([x]) = \partial \rho_X^o([x]|2q])\]
for all $[x] \in \mathcal{W}^{i-r}(X)$. Using the fact that $\sigma_X : \mathbb{U}_X^n \longrightarrow \text{Spec } X$ composes $U_X^n \xrightarrow{\Delta} A_X^n \xrightarrow{\pi_X} \text{Spec } X$,
we get:
\[
\partial \rho_X^o([x]) = \pi_X^o([x]) \ast \partial \epsilon_X^o \quad \text{by equation (5)}
\]
\[
= \pi_X^o([x]) \ast \tau^{-q}(\kappa_X^o) \quad \text{by Theorem 6.4}.
\]
But this is equal to the right hand side of (13) because:
\[
\partial \rho_X^o([x]|2q]) = \pi_X^o([x]|2q]) \ast \kappa_X^o \quad \text{by Lemma B.1}
\]
\[
= \pi_X^o([x]) \ast \tau^{-q}(\kappa_X^o) \quad \text{by Lemma B.3}.
\]
\[\square\]

**Corollary 7.8.** Let $X$ be a regular $\mathbb{Z}[1/2]$-scheme. We have an isomorphism
\[
(\sigma_X^*, \rho_X^o) : \mathcal{W}^i(X) \oplus \mathcal{W}^{i-r}(X) \xrightarrow{\Xi} \mathcal{W}^i(U_X^n).
\]
for all $i \in \mathbb{Z}$.

**Proof.** From Theorem 7.1, it suffices to show that the homomorphism
\[
\rho_X^o : \mathcal{W}^{i-r}(X) \longrightarrow \mathcal{W}^i(U_X^n)
\]
has the two following properties: first $\partial \cdot \rho_X^o$ is an isomorphism; secondly that $\gamma^* \cdot \rho_X^o$ is zero. The first one follows from Theorem 7.7 and the second one from the definition of $\rho_X^o$ and Theorem 7.4, since $\gamma^*$ is a morphism of graded rings by [10, Thm. 3.2]. \[\square\]

**Remark 7.9.** Note that this result generalizes the well-known calculation of the Witt group of a Laurent ring (cf. e.g. [13]).

As for the Laurent ring it is likely that the result is true for a bigger class of rings, e.g. all regular rings. But already in the Laurent ring case it fails to be true for all (commutative) rings as loc. cit. Examples 8.1 and 8.2 show.

To understand the ring structure on $\mathcal{W}^{tot}(U_X^n)$, we need some properties of the symmetric spaces $K_X^n$ and $B_X^n$, which can be proven for non necessarily regular schemes as well. The case $n = 1$, i.e. the “Laurent scheme” case, is well-known, so we have to deal with $n \geq 2$.

**Theorem 7.10.** Let $X$ be a $\mathbb{Z}[1/2]$-scheme. If $n \geq 2$ then the symmetric $r$-space $B_X^n$ is locally trivial, i.e. for any $x \in U_X^n$ we have $[B_X^n]_x = 0$ in $\mathcal{W}^r(\mathcal{O}_{U_X^n,x})$.

**Proof.** Define for all $i \in \{1, \ldots, n\}$ the principal open $V_X^n(i)$ of $A_X^n$ given by the equation $T_i \neq 0$. Let $J \subseteq \{1, \ldots, n\} \subset \mathbb{N}$. We define
\[
V_X^n(J) := \bigcup_{j \in J} V_X^n(j) \subseteq U_X^n = V_X^n(\{1, \ldots, n\})
\]
and denote $\iota_J : V_X^n(J) \longrightarrow U_X^n$ the corresponding open immersion. Since $n \geq 2$, we can cover $U_X^n$ with the open subschemes $V_X^n(J)$ with $|J| \leq n - 1$. So it suffices to prove the following stronger result. \[\square\]
Theorem 7.11. With the above notations, if $|J| \leq (n - 1)$ then $[B^\circ_X|_{V^n_X(J)}] = 0$ in $W^r(V^n_X(J))$.

Proof. We easily reduce to the case $X = \text{Spec} \mathbb{Z}[1/2]$. In this case we argue as follows. For brevity we set $R := \mathbb{Z}[1/2]$.

For $J$ empty, the result is trivial since $V^n_R(J) = \emptyset$ and its Witt group is zero. So we assume that $J \neq \emptyset$. Consider the closed complement $V^n_R(J) := \mathbb{A}^n_R \setminus V^n_R(J)$ of $V^n_R(J) \subset \mathbb{A}^n_R$. Note that $\mathbb{A}^n_R - U^n_R \subset V^n_R(J)$. By Theorem 7.1, we have the following commutative diagram with exact rows:

\[
\begin{array}{c}
0 \rightarrow W^r(R) \rightarrow W^r(U^n_R) \rightarrow W^r+1(\mathbb{A}^n_R) \rightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow W^r(R) \rightarrow W^r(V^n_R(J)) \rightarrow W^r+1(Z^n_R(J)) \rightarrow 0.
\end{array}
\]

We get from the right-hand commutative square, from Theorem 7.6, and from Proposition 3.6 that $\partial(\iota^*_J(e^n_R)) = 0$. Therefore, by exactness of the above second row, there exists a unique class $w \in W^r(R)$ such that $\iota^*_J(B^n_R) = \iota^*_J(\sigma^n_R(v))$. In fact, $w = \gamma^*(\iota^*_J(B^n_R))$ for any $R$-point $\gamma : \text{Spec}(R) \to V^n_R(J)$, which exists by assumption $J \neq \emptyset$. Choose $j \in J$ and define the $R$-point $\gamma_j : \text{Spec}(R) \to V^n_R(J)$ to be given by $T_j = 1$ and $T_i = 0$ for $i \neq j$. Since $w = \gamma^*(\iota^*_J(B^n_R)) = (\iota^*_j \cdot \gamma)^*(\mathcal{B}^n_R)$ and since $\iota^*_j \cdot \gamma : \text{Spec}(R) \to U^n_R$ is simply the $R$-point $\gamma_j$ of Theorem 7.4, we conclude from it that $w = 0$. Hence $\iota^*_J(B^n_R) = 0$ as wanted.$\square$

Remark 7.12. The statement of Theorem 7.10 is obviously not true for $n = 1$. The proof fails for $n = 1$ because then $V^n_X(J) = \emptyset$ for any $J$ such that $|J| \leq n - 1 = 0$ and hence we can not cover $U^n_X$ with these.

It follows from this theorem above and [5, Thm. 4.2] that if $n \geq 2$ the space $\varepsilon^n_X$ is nilpotent in $W^\text{tot}(U^n_X)$. We prove a more precise result.

Theorem 7.13. Let $X$ be a $\mathbb{Z}[1/2]$-scheme. Assume that $n \geq 2$ then

\[ (\varepsilon^n_X)^2 = \varepsilon^n_X \ast \varepsilon^n_X = 0 \]

in $W^\text{tot}(U^n_X)$. If $n = 1$ then $(\varepsilon^n_X)^2 = 1$ in $W^\text{tot}(U^n_X)$.

Proof. If $n = 1$ this is an obvious consequence of Example 6.8.

So let $n \geq 2$. Since $\alpha^n_X : W^\text{tot}(\mathbb{Z}[1/2]) \rightarrow W^\text{tot}(X)$ is a morphism of graded rings (c.f. [10, Thm. 3.2]) it is enough to prove this for the affine scheme $X = \text{Spec} \mathbb{Z}[1/2]$.

Because we assume $n \geq 2$ there exists non-empty subsets $J_1, J_2 \subset \{1, \ldots, n\}$ with $J_1 \neq J_2$ and $J_1 \cup J_2 = \{1, \ldots, n\}$. We define $V^n_{\mathbb{Z}[1/2]}(J_i) \subseteq U^n_{\mathbb{Z}[1/2]}$ as in the proof of Theorem 7.10 above and let $V^n_{\mathbb{Z}[1/2]}(J_i) : = U^n_{\mathbb{Z}[1/2]} \setminus V^n_{\mathbb{Z}[1/2]}(J_i)$ be the complement ($i = 1, 2$). Note that $J_1 \cup J_2 = \{1, \ldots, n\}$ implies $V^n_{\mathbb{Z}[1/2]}(J_1) \cap V^n_{\mathbb{Z}[1/2]}(J_2) = \emptyset$.

By Theorem 7.11 we know that $[B^\circ_{\mathbb{Z}[1/2]}|_{V^n_{\mathbb{Z}[1/2]}(J_i)}] = 0$ for $i = 1, 2$. Therefore by the localization sequence there exists $x_1 \in W^r_{V^n_{\mathbb{Z}[1/2]}(J_1)}(U^n_{\mathbb{Z}[1/2]})$ with $x_1 = \varepsilon^n_{\mathbb{Z}[1/2]}$ in $W^r(U^n_{\mathbb{Z}[1/2]})$ for $i = 1, 2$, and so $x_1 \ast x_2 = (\varepsilon^n_{\mathbb{Z}[1/2]})^2$ in $W^2r(U^n_{\mathbb{Z}[1/2]})$. But the space $x_1 \ast x_2$ lives on a complex with support in $V^n_{\mathbb{Z}[1/2]}(J_1) \cap V^n_{\mathbb{Z}[1/2]}(J_2) = \emptyset$ and so $(\varepsilon^n_{\mathbb{Z}[1/2]})^2 = 0$. $\square$
Denote $W^\text{tot}(X)[\varepsilon]$ the graded skew polynomial ring in one variable $\varepsilon$ of degree $r$ over the graded ring $W^\text{tot}(X)$. Recall that this means that $c \cdot \varepsilon = (-1)^r \deg(c) \cdot \varepsilon$ for a homogeneous element $c \in W^\text{tot}(X)$. We have a homogeneous homomorphism of graded rings given by

$$W^\text{tot}(X)[\varepsilon] \longrightarrow W^\text{tot}(U^n_X) \quad \sum_{i=0}^m c_i \varepsilon^i \longmapsto \sum_{i=0}^m \sigma_X^*(c_i) \star (\varepsilon^{(n)} i).$$

Using this morphism we can restate Corollary 7.8 and Theorem 7.13 as follows

**Theorem 7.14.** Let $X$ be a regular scheme of finite Krull dimension over $\mathbb{Z}[1/2]$. Then we have an isomorphism of graded rings

$$\begin{aligned}
\text{if } n &\geq 2 : & W^\text{tot}(X)[\varepsilon] / (\varepsilon^2) &\cong W^\text{tot}(U^n_X), \\
\text{if } n = 1 : & W^\text{tot}(X)[\varepsilon] / (\varepsilon^2 - 1) & \cong W^\text{tot}(U^n_X).
\end{aligned}$$

8. Witt non-triviality of the (half) Koszul spaces

**Theorem 8.1.** Let $X$ be a scheme which is not of equicharacteristic 2. Then the Witt class of the symmetric $n$-space $K^n_X$ is non-trivial in the Witt group with support $W^\text{tot}_n(A^n_X \setminus U^n_X)$. 

*Proof.* By assumption, there is a point $x \in X$ whose residue field $k(x)$ has characteristic different from 2. By specialization at $x$ (see Remark 5.4 for naturality), it suffices to prove the result for the regular $\mathbb{Z}[1/2]$-scheme $X := \text{Spec}(k(x))$. Here, we apply Theorem 7.2 with $i := n$ and $w := 1 \in W^0(X)$, the unit of the Witt ring. \hfill \Box

**Theorem 8.2.** Let $X$ be any scheme which is not of equicharacteristic 2. Then the Witt class $\varepsilon^n_X$ of the symmetric $r$-space $B^n_X$ is not in the image of the natural homomorphism $W^r(A^n_X) \longrightarrow W^r(U^n_X)$. In particular, $B^n_X$ cannot be extended to the whole affine space $A^n_X$.

*Proof.* Again, by specialization at a point $x$ with char($k(x)$) $\neq 2$, we are reduced to prove the result for the $\mathbb{Z}[1/2]$-regular scheme $X := \text{Spec}(k(x))$. In this case, the following composition vanishes:

$$W^r(A^n_X) \xrightarrow{\varepsilon_X} W^r(U^n_X) \xrightarrow{\partial} W^r_{A^n_X \setminus U^n_X}(A^n_X).$$

Here, the connecting homomorphism $\partial$ is, for instance, as in Theorem 7.6, where we proved that $\partial(\varepsilon^n_X)$ coincides with $[K^n_X]$, up to 4-periodicity. So, $\varepsilon^n_X$ cannot be extended to $A^n_X$ since $[K^n_X] \neq 0 \in W^r_{A^n_X \setminus U^n_X}(A^n_X)$ by Theorem 8.1. Note that we have to pass via the regular case otherwise the connecting homomorphism $\partial$ is not defined. \hfill \Box
Appendix A. The locally free module $E_X^n$

We use the notation of the main part of the text. We want to prove:

**Theorem A.1.** Let $X$ be a noetherian scheme and $n \geq 3$. Then there does not exist a locally free $\mathcal{O}_{X^n}$-module $F$, such that

$$
F \mid_{U_X^n} \simeq E_X^n.
$$

In particular $E_X^n$ is not a free $\mathcal{O}_{U_X^n}$-module.

Let $x \in X$ and $\text{Spec} \ k(x) \to X$ be the corresponding point. If there exists a locally free $\mathcal{O}_{X^n}$-module $F$, such that $F \mid_{U_X^n} \simeq E_X^n$, then

$$
E_X^n \mid_{\text{Spec} \ k(x)} \simeq \upsilon^* f(E_X^n) \simeq \upsilon^* f(F) \mid_{U_k^n},
$$

and so it is enough to show the theorem for $X = \text{Spec} \ R$ with $R$ a field. Similarly, localizing $R[T_1, \ldots, T_n]$ at the origin, we are reduced to the local case which follows from the following result of commutative algebra.

**Theorem A.2.** Let $(A, \mathfrak{m})$ be a regular local ring, $T = (T_1, \ldots, T_n)$ a regular system of parameters (see [7, Def. 2.2.1]), and $U = \bigcup_{i=1}^n \text{Spec} \ A_{T_i} = \text{Spec} \ A \setminus \{\mathfrak{m}\}$ the punctured spectrum of $A$. Assume that $\dim A = n \geq 3$. Then

$$
S_j := \text{Ker} \left( \dfrac{K_j(A, T)}{\mathfrak{d}_j(A, T)} \to K_{j-1}(A, T) \right) \mid_{U}
$$

can not be extended to a free $A$-module if $n > j \geq 2$.

Let in the following $I_j = \text{Ker} \ d_j(A, T)$, i.e. $S_j = I_j \mid_{U}$. Recall also that $(-)^\vee = \text{Hom}_A(-, A)$. For the proof we need:

**Proposition A.3.**

(i) Let $j \geq 2$. Then the $A$-module $I_j$ is reflexive, i.e. the natural morphism $I_j \longrightarrow I_j^{\vee \vee}$ is an isomorphism.

(ii) If $M$ and $N$ are finitely generated $A$-modules, such that $M \mid_{U} \simeq N \mid_{U}$, and both $M \mid_{U}$ and $N \mid_{U}$ are locally free, then $M^{\vee} \simeq N^{\vee}$.

**Proof.** By assumption $I_j$ is a second Syzygy and so (i) is a consequence of [8, Thm. 3.6]. For (ii), by [11, Thm. 6.9.17] there exists $c \geq 0$, such that the given isomorphism $M \mid_{U} \overset{\sim}{\longrightarrow} N \mid_{U}$ is the restriction of a morphism $\mathfrak{m}^c M \longrightarrow N$. Therefore we can assume that there exists $g : M \longrightarrow N$, such that $g \mid_{U}$ is an isomorphism, i.e. $\text{Ker} \ g$ and $\text{Coker} \ g$ have finite length.

Now we use the following fact (see [7, Thm. 1.2.8]). Since $\dim A \geq 2$ and $A$ is regular (hence in particular Cohen-Macaulay) we have $\text{Ext}^1_A(G, A) = 0$ for any finite length module $G$ and $i = 0, 1$.

This and the exact sequences

$$
0 \longrightarrow \text{Ker} \ g \longrightarrow M \longrightarrow \text{Im} \ g \longrightarrow 0
$$

and

$$
0 \longrightarrow \text{Im} \ g \longrightarrow N \longrightarrow \text{Coker} \ g \longrightarrow 0
$$

give $M^{\vee} \simeq (\text{Im} \ g)^{\vee} \simeq N^{\vee}$. \qed
Proof. (of Theorem A.2) Assume that $P$ is a free $A$-module, such that $P_{[\omega]} \cong S_j$. We have $\mathcal{T}_j_{[\omega]} \cong S_j$, too, and so $\mathcal{T}_j \cong P^\vee$ by Proposition A.3 (ii). Part (i) of this proposition tells us that $\mathcal{T}_j$ is reflexive and hence $\mathcal{T}_j \cong \mathcal{T}_j^{\vee\vee} \cong P^{\vee\vee}$ is free. But this is impossible, because

\[ \text{Tor}_n^A(\mathcal{T}_j, A/m) \cong \text{Tor}_n^A(A/m, A/m) \cong A/m \neq 0 \]

(note that we need here $j < n = \dim A$). We are done. \qed

Appendix B. The product and 4-periodicity

We have used in this work the fact that the product commutes with the translation. This has not been established in [10]. For the sake of completeness we give here a proof but refer to loc. cit. for unexplained notations and definitions.

To start with let $A^{(0)} = (\mathcal{A}, \mathcal{D}_A, \delta_A, \omega_A)$ and $B^{(0)} = (\mathcal{B}, \mathcal{D}_B, \delta_B, \omega_B)$ be triangulated categories with $\delta_A$, respectively $\delta_B$-exact duality (like e.g. $\mathbb{D}^b(\text{VB}_X)$ with the usual 1-exact duality as in the main part of this work). We denote the shift functor in these triangulated categories by $\Sigma_A$ respectively $\Sigma_B$ (to distinguish we do not use $X \mapsto X[1]$). A symmetric $i$-space in $A^{(0)}$ is a pair $(X, \psi)$ consisting of an object $X \in \mathcal{A}$ and a symmetric $i$-form $X, \psi : \Sigma_A^i \mathcal{D}_A X$ which is an isomorphism, the symmetry of an $i$-form reads $\Sigma_A^i \mathcal{D}_A (\psi) \cdot \Sigma_A^i = (-1)^{\frac{m+1}{2}} \delta_A \cdot \psi$. As in the case of derived categories if $(X, \psi)$ is a symmetric $i$-form then $(\Sigma_A^i X, \Sigma_A^i (\psi))$ is a symmetric $(i + 4)$-form.

Let $(F, \rho) : A^{(0)} \rightarrow B^{(0)}$ be a duality preserving functor, i.e. $\rho : \mathcal{D}_A \cong \mathcal{D}_B F$ is an isomorphism of functors satisfying some compatibility axioms. We will only use the following. Since $F$ is a covariant exact functor between triangulated categories there exists a family of isomorphisms of functors $\theta^{(i)} : F \Sigma_A^i \cong \Sigma_B^i F$ $(i \in \mathbb{Z})$ which are related by the following formulas:

\[ \theta^{(i+j)} = \Sigma_B^i (\theta^{(j)}) \cdot \theta^{(i)}_{\Sigma_A^i} \]

(14)

$(i, j \in \mathbb{Z})$. Then we have

\[ \mathcal{D}_B \Sigma_B^{-1}(\theta^{(1)}_{\Sigma_A^1}) \cdot \rho_{\Sigma_A^{-1}} = (\delta_A \delta_B) \cdot \Sigma_B(\rho) \cdot \theta^{(1)}_{\mathcal{D}_A} \]

(15)

(cf. loc. cit. Definition 1.8). This axioms are made such that if $(X, \psi)$ is a symmetric $i$-space in $A^{(0)}$ then

\[ (F, \rho)_*(X, \psi) := (FX, (\delta_A \rho) \cdot \Sigma_B(\rho) \cdot \theta^{(i)}_{\mathcal{D}_A} \cdot F(\psi)) \]

is a symmetric $i$-space in $B^{(0)}$.

Lemma B.1. Let $(X, \psi)$ be a symmetric $i$-space in $\mathcal{A}$. Then there is an isometry

\[ (F, \rho)_*(\Sigma_A^2 X, \Sigma_A^2 (\psi)) \cong \Sigma_B^2 ((F, \rho)_*(X, \psi)) . \]

Proof. We claim that $\theta_{X}^{(2)} : F \Sigma_A^2 X \cong \Sigma_B^2 FX$ is an isometry, i.e. we have to show

\[ (\delta_A \delta_B)^{-1} \Sigma_A^{i+4} \mathcal{D}_A (\theta_{X}^{(2)}) \cdot \Sigma_B^{i+4} (\rho X) \cdot \Sigma_B^2 (\theta^{(i)}_{\mathcal{D}_A}) \cdot \Sigma_B^2 F(\psi) \cdot \theta_{X}^{(2)} \]

(16)
We observe first that
\[ \Sigma^2_B F(\psi) \cdot \theta_X^{(2)} = \theta_X^{(2)} \Sigma_A^2 X \cdot F \Sigma^2_A (\psi) \]
since \( \theta^{(2)} \) is a natural transformation. By using (14) three times we get (recall that by definition \( \Sigma_A D_A = D_A \Sigma_A^{-1} \))
\[ \theta_D^{(i+4)} A = \Sigma^2_B \left( \theta_D^{(i)} A \Sigma_A^2 X \right) \cdot \Sigma^2_B \left( \theta_D^{(i)} A \Sigma_A X \right) \cdot \Sigma^2_B \left( \theta_D^{(i)} A \Sigma_A X \right) \cdot \theta_D^{(2)} A \Sigma_A X \]
and so (16) is equivalent to
\[ \Sigma^2_B (\rho_{\Sigma_A X}) \cdot \Sigma^2_B (\rho_{\Sigma_A X}) \cdot \Sigma^2_B (\rho_{\Sigma_A X}) \cdot \Sigma^2_B (\rho_{\Sigma_A X}) \]
Hence the result follows from the following calculation:
\[ \Sigma^2_B \left( \Sigma B (\rho_{\Sigma_A X}) \cdot \theta_X^{(1)} A \Sigma_A X \right) \cdot \Sigma^2_B \left( \rho_{\Sigma_A X} \right) \]
\[ = (\delta A \delta A) \Sigma B \left( \Sigma B (\rho_{\Sigma_A X}) \cdot \theta_X^{(1)} A \Sigma_A X \right) \cdot \Sigma^2_B \left( \rho_{\Sigma_A X} \right) \]
\[ = (\delta A \delta A) \Sigma B \left( \Sigma B (\rho_{\Sigma_A X}) \cdot \theta_X^{(1)} A \Sigma_A X \right) \cdot \Sigma^2_B \left( \rho_{\Sigma_A X} \right) \]
\[ = \Sigma^2_B \left( \Sigma B (\rho_{\Sigma_A X}) \cdot \theta_X^{(1)} A \Sigma_A X \right) \cdot \Sigma^2_B \left( \rho_{\Sigma_A X} \right) \]
since \( \theta_X^{(1)} = \Sigma B (\rho_{\Sigma_A X}) \cdot \theta_X^{(1)} A \Sigma_A X \) by Eq. (14). \( \square \)

Assume now we have a third triangulated category with duality, say \( C^{(0)} = (\mathcal{C}, D_C, \delta_C, \psi C) \), and a dualizing pairing \( \text{(loc. cit.)} \) Definition 1.11)
\[ \otimes : A^{(0)} \times B^{(0)} \rightarrow C^{(0)} \].

**Example B.2.** Let \( X \) be a scheme and \( Z \subseteq X \) a closed subset. Then the (derived) tensor product
\[ \otimes_{\sigma_X} : \mathbb{D}^b(VB_X) \times \mathbb{D}^b(VB_X) \rightarrow \mathbb{D}^b(Z(VB_X)) \]
is a dualizing pairing. Note that in this case \( \delta_A = \delta_B = \delta_C = 1 \).

Let \( (X, \psi) \) be a symmetric \( i \)-space in \( A^{(0)} \) and \( (Y, \phi) \) a symmetric \( j \)-space in \( B^{(0)} \). The left product \( (X, \psi) \star_l (Y, \phi) \) is then defined by considering \( X \boxtimes \mathbb{C} \) as duality preserving functor with the aid of a duality transformation \( L(\psi) \) which depends on \( \psi \), i.e. the left product \( (X, \psi) \star_l (Y, \phi) \) of these spaces is by definition \( (X \boxtimes \mathbb{C} \mathbb{C} \mathbb{C}, L(\psi))(Y, \phi) \). The right product \( \star_r \) is defined analogously by making the functor \( - \boxtimes \mathbb{C} Y \) duality preserving using the symmetric \( j \)-form \( \psi \). Both products are related by the following isometry:
\[ (X, \psi) \star_l (Y, \phi) \simeq (\delta_A \delta_C)^j \cdot (\delta_B \delta_C)^j \cdot (-1)^{ij} \cdot (X, \psi) \star_r (Y, \phi) \] (loc. cit. Theorem 2.9). From this we easily deduce

**Lemma B.3.** There is an isometry
\[ (\Sigma^2_A X, \Sigma^2_A (\psi)) \star_l (Y, \phi) \simeq (X, \psi) \star_r (\Sigma^2_B Y, \Sigma^2_B (\phi)) \], and the same is true for the right product.
Proof. From Lemma B.1 we get isometries \((X, \psi) \star_1 (\Sigma^2_B Y, \Sigma^2_B \phi) \cong \Sigma^2_C ((X, \psi) \star_1 (Y, \phi))\) and \((\Sigma^2_A X, \Sigma^2_A \psi) \star_r (Y, \phi) \cong \Sigma^2_C ((X, \psi) \star_r (Y, \phi))\). Hence the theorem follows by applying (17) twice. \(\square\)

References