1 Introduction

In this paper we discuss results, obtained in the KAM for PDEs theory since it was originated 15 years ago in [Kuk87, Kuk88, Kuk89, Way90]. We hope that our work discusses all relevant topics, except the theory of time-periodic solutions of Hamiltonian PDEs (since, firstly, this theory is an extensive subject and, secondly, many results there have been recently proved and re-proved without the KAM-machinery, e.g. see [Bam00a]). We avoid completely the classical finite-dimensional KAM-theory which deals with time-quasiperiodic solutions.
of finite-dimensional Hamiltonian systems and instead refer the reader to the recent survey [Sev02]. Our bibliography is by no means complete.

**Notation** By $T^n$ we denote the torus $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ and write $T^1 = S^1$; by $\mathbb{R}^+_n$ – the open positive octant in $\mathbb{R}^n$; by $\mathbb{Z}_0$ – the set of nonzero integers. Abusing notation, we denote by $x$ both the space-variable and an element of an abstract Banach space $X$. For an invertible linear operator $J$ we set $J = -J^{-1}$. The Lipschitz norm of a map $f$ from a metric space $M$ to a Banach space is defined as $\sup_{m \in M} \|f(m)\| + \sup_{m_1 \neq m_2} \frac{\|f(m_1) - f(m_2)\|}{\text{dist}(m_1, m_2)}$.

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## 2 Hamiltonian partial differential equations

### 2.1 Hilbert scales and their morphisms

Let $X_0$ be a real Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$ and a Hilbert basis $\{\varphi_k, k \in \mathbb{Z}\}$, where $\mathbb{Z}$ is a countable set. Let us take a positive sequence $\{\theta_k, k \in \mathbb{Z}\}$ which goes to infinity with $k$. For any $s$ we define $X_s$ as a Hilbert space with the Hilbert basis $\{\varphi_k \theta_k^{-s}, k \in \mathbb{Z}\}$. By $\| \cdot \|_s$ and $\langle \cdot, \cdot \rangle_s$ we denote the norm and the scalar product in $X_s$ (in particular, $X_0 = X$). The totality $\{X_s\}$ is called a Hilbert scale, the basis $\{\varphi_k\}$ — the basis of the scale and the scalar product $\langle \cdot, \cdot \rangle$ — the basic scalar product. For any $s$, the basic scalar product extends to a paring $X_s \times X_{-s} \to \mathbb{R}$, still denoted as $\langle \cdot, \cdot \rangle_s$. This paring identifies the conjugated space $(X_s)^*$ with $X_{-s}$.

For a scale $\{X_s\}$ we denote by $X_{-\infty}$ and $X_\infty$ the linear spaces $X_{-\infty} = \bigcup X_s$ and $X_\infty = \bigcap X_s$.

A Hilbert scale may be continuous or discrete, depending on whether $s \in \mathbb{R}$ or $s \in \mathbb{Z}$. The objects we define below are meaningful and the theorems we discuss are valid in both cases.

**Example 2.1.** Basic for us is the Sobolev scale of functions on the $d$-dimensional torus $\{H^s(T^d; \mathbb{R}) = H^s(T^d)\}$. A space $H^s(T^d)$ is formed by functions $u : T^d \to \mathbb{R}$ such that

$$u = \sum_{l \in \mathbb{Z}^d} u_l e^{il \cdot x}, \quad C \ni u_l = \pi_{-l}, \quad \|u\|_s^2 = \sum_{l} (1 + |l|^2)^s |u_l|^2 < \infty.$$ 

The basis $\{\varphi_k\}$ is formed by properly normalised functions $\text{Re} e^{il \cdot x}$ and $\text{Im} e^{il \cdot x}$, $l \in \mathbb{Z}^d$.

We shall also use the sub-scale $\{H^s_0(\mathbb{T}^d)\}_0$, where a space $H^s_0(\mathbb{T}^d)$ consists of functions from $H^s(\mathbb{T}^d)$ with zero mean-value. 

**Example 2.2.** Consider the scale $\{H^s_0(0, \pi)\}$, where a space $H^s_0 = H^s_0(0, \pi)$ is formed by the odd $2\pi$-periodic functions $u = \sum_{k=1}^{\infty} u_k \sin kx$ such that $\|u\|_s^2 = \sum |k|^{2s} |u_k|^2 < \infty$. Since $\{\sin nx\}$ is a complete system of eigenfunctions of the
operator $-\Delta$ in $L^2(0, \pi)$ with the domain of definition $\{u \in H^2(0, \pi) \mid u(0) = u(\pi) = 0\}$, then an equivalent definition of these spaces is that $H_0^d = D((-\Delta)^{s/2})$ (see [RS75]). In particular,

$$
H_0^1 = \{u \in H^1(0, \pi) \mid u(0) = u(\pi) = 0\}, \quad H_0^2 = H^2(0, \pi) \cap H_0^1,
$$

$$
H_0^3 = \{u \in H^3(0, \pi) \mid u(0) = u_{xx}(0) = u(\pi) = u_{xx}(\pi) = 0\}. \square \ (2.1)
$$

Given a scale $\{X_s\}$ and a linear map $L: X_\infty \to X_{-\infty}$, we denote by $\|L\|_{s_1, s_2} \leq \infty$ its norm as a map $X_{s_1} \to X_{s_2}$. We say that $L$ defines a morphism of the scale of order $d$ and write $\text{ord} L = d$ if $\|L\|_{s, s - d} < \infty$ for every $s$. If in addition the inverse map $L^{-1}$ exists and defines a morphism of order $-d$, we say that $L$ defines an automorphism of order $d$. If $L$ is a morphism of order $d$, then the adjoint maps $L^*$ form a morphism of the same order, called the adjoint morphism. If $L = L^*$ ($L = -L^*$), then the morphism $L$ is called self-adjoint (anti-selfadjoint). For example, the operator $\Delta$ defines a selfadjoint morphism of order 2 of the Sobolev scale $\{H^s(\mathbb{T}^n)\}$ and the operators $\partial/\partial x_j$, $1 \leq j \leq n$, define anti-selfadjoint morphisms of order one.

Let $O_s \subset X_s$ be an open domain and $F$ be an analytic ($C^k$-smooth) map $F: O_s \to X_{s-d}$. Then $F$ is called an analytic ($C^k$-smooth) map of order $d$ on $O_s$ and we write $\text{ord} F = d$.

Example 2.3. Let $\{X_s\}$ be the Sobolev scale $\{H^s(\mathbb{T}^d)\}$ and $f(u, x)$ be a smooth function. Then the map $F: u(x) \mapsto f(u(x), x)$, $X_a \to X_b$, is smooth if $a > \frac{1}{2}$, so on these spaces $\text{ord} F = 0$. If $f$ is analytic, then so is $F$.

Now let us assume that $d = 1$, $f$ is analytic, $f(0, x) \equiv 0$ and consider $F$ as a map in the scale $\{H_a^s = H_0^s(0, \pi), s \in \mathbb{Z}\}$. For $s \geq 1$ the map $F: H_0^s \to H^0(0, \pi)$ is analytic. Since $F(0) = F(u(\pi)) = 0$, then due to (2.1) for $s = 1$ and $s = 2$ $F(H_0^s) \subset H_0^s$. So on the spaces $H_0^1$ and $H_0^2$ we have $\text{ord} F = 0$. Since in general for $u \in H_{0}^s$, $F(u) \in H_0^s$ but $\notin H_0^3$ (see (2.1)), then on the spaces $H_0^s$, $s \geq 3$ we have $\text{ord} F > 0$.

If $f(u, x)$ is odd in $u$ and even in $x$ (e.g., is $x$-independent), or vice versa, then $F(H_0^s) \subset H_0^s$ for $s \geq 1$, so $\text{ord} F = 0$ for any $s \geq 1$. \square

Given a $C^k$-smooth function $H: X_d \to O_d \to \mathbb{R}$, $k \geq 1$, we consider its gradient map with respect to the paring $\langle \cdot, \cdot \rangle$:

$$
\nabla H: O_d \to X_{-d}, \quad \langle \nabla H(u), v \rangle = dH(u)v \quad \forall v \in X_d.
$$

The map $\nabla H$ is $C^{k-1}$-smooth. If the gradient map $\nabla H$ defines a $C^{k-1}$-smooth map $O_d \to X_{d-d_H}$, we write that $\text{ord} \nabla H = d_H$.

### 2.2 Symplectic structures and Hamiltonian equations

For simplicity we restrict ourselves to constant-coefficient symplectic structures (for the general case see [Kuk00]). Let $\{X_s\}$ be a Hilbert scale and $J$ be its anti-selfadjoint automorphism of order $d_J$. Then the operator $\overline{J} = -J^{-1}$ defines an anti-selfadjoint automorphism of order $-d_J$. We define a two-form $\alpha_2$ as $\alpha_2 = \overline{J} dx \wedge dx$, where by definition $\overline{J} dx \wedge dx[\xi, \eta] = \langle \overline{J} \xi, \eta \rangle$. Clearly, $\overline{J} dx \wedge dx$ is
a continuous skew-symmetric bilinear form on $X_r \times X_r$ if $r \geq -d_f/2$. Therefore any space $X_r$, $r \geq -d_f/2$, becomes a symplectic (Hilbert) space and we shall write it as a pair $(X_r, \alpha_2)$. The pair $(\{X_s\}, \alpha_2)$ is called a symplectic (Hilbert) scale.

**Example 2.4.** Let us take the index-set $\mathcal{Z}$ to be the union of non-intersecting subsets $\mathcal{Z}_+ \cup \mathcal{Z}_-$, provided with an involution $\mathcal{Z} \rightarrow \mathcal{Z}$ which will be denoted $j \mapsto -j$, such that $-\mathcal{Z}_\pm = \mathcal{Z}_\mp$. Let us consider a Hilbert scale $\{X_s\}$ with a basis $\{\phi_k, k \in \mathcal{Z}\}$ and a sequence $\theta_k, k \in \mathcal{Z}$, such that $\theta_{-j} \equiv \theta_j$. Take $J$ to be the linear operator, defined by the relations

$$J\phi_k = \phi_{-k} \quad \forall k \in \mathcal{Z}_+, \quad J\phi_k = -\phi_{-k} \quad \forall k \in \mathcal{Z}_-.$$  

It defines an anti-selfadjoint automorphism of the scale of zero order, and $\overline{J} = J$. The symplectic scale $(\{X_s\}, \alpha_2 = Jdx \wedge dx = Jdx \wedge dx)$ will be called a Darboux scale.  

To a $C^1$-smooth function $h$ on a domain $O_{d_0} \subset X_{d_0}$, the symplectic form $\alpha_2$ as above corresponds the Hamiltonian vector field $V_h$, defined by the usual relation $\alpha_2[V_h, \xi] = -dh(\xi) \quad \forall \xi$ (cf. [Arn89, HZ94]). That is, $\langle JV_h(x), \xi \rangle \equiv -\langle \nabla h(x), \xi \rangle$ and $V_h(x) = J\nabla h(x)$. So the corresponding Hamiltonian equation is

$$\dot{x} = J\nabla_x h(x) = V_h(x). \quad (2.2)$$

The vector field $V_h$ defines a continuous map $O_{d_0} \rightarrow X_{-d_0-d_f}$. Usually we shall assume that $V_h$ is smoother than that and ord $V_h = d_1 \leq 2d_0 + d_f$.

A partial differential equation is called a Hamiltonian partial differential equation (HPDE) if under a suitable choice of a symplectic Hilbert scale $(\{X_s\}, \alpha_2)$, a domain $O_{d_0} \subset X_{d_0}$ and a Hamiltonian $h$, it can be written in the form (2.2). In this case the vector field $V_h$ is unbounded, ord $V_h = d_1 > 0$.

A continuous curve $x: [t_0, t_1] \rightarrow O_{d_0}$ is called a solution of (2.2) (in the space $X_{d_0}$) if it defines a $C^1$-smooth map $x: [t_0, t_1] \rightarrow X_{d_0-d_f}$ and both parts of (2.2) coincide as curves in $X_{d_0-d_f}$. A solution $x$ is called smooth if it defines a smooth curve in each space $X_s$.

We shall call a Hamiltonian equation (2.2) quasilinear if $h(x) = \frac{1}{2}(Ax, x) + h_0(x)$, where $A$ is a linear operator which defines a selfadjoint morphism of the scale, and ord $\nabla h_0 < ord A$. Accordingly, (2.2) takes the form

$$\dot{x} = J(Ax + \nabla h_0(x)). \quad (2.3)$$

A quasilinear equation (2.3) will be called semilinear if ord $J\nabla h_0 \leq 0$.

**Example 2.5 (Korteweg–de Vries equation).** Let us take for $\{X_s\}$ the scale of zero mean-value Sobolev spaces $H^s(S^1)_0$ as in Example 2.1 and choose $J = \partial/\partial x$, so $d_f = 1$. For a Hamiltonian $h$ we take $h(u) = \int_0^{2\pi} (-\frac{1}{2}u'(x)^2 + \frac{1}{4}u^3) \, dx$. Then $\nabla h(u) = \frac{1}{4}(u'' + 3u^2)$ and (2.2) becomes the KdV equation:

$$\dot{u}(t, x) = \frac{1}{4} \frac{\partial}{\partial x}(u'' + 3u^2). \quad (2.4)$$

It has the form (2.3). Taking $O_{d_0} = X_{d_0}$ with any $d_0 \geq 1$, we have ord $JA = 3$ and ord $J\nabla h_0 = 1$. So it is quasilinear, but not semilinear.  

4
Example 2.6 (NLS – nonlinear Schrödinger equation). Let $X_s = H^s(\mathbb{T}^n; \mathbb{C})$, where this Sobolev space is treated as a real Hilbert space, and the basic scalar product is $(u, v) = \text{Re} \int u \overline{v} dx$. For $J$ we take the operator $J u(x) = i u(x)$, so ord $J = 0$ and $\{X_s\}, J du \wedge du$ is a Darboux scale. We choose

$$h(u) = \frac{1}{2} \int_{\mathbb{T}^n} \left( |\nabla u|^2 + V(x)|u|^2 + g(x, u, \bar{u}) \right) dx,$$

where $V$ is a smooth real function and $g(x, u, \bar{u})$ is a smooth function, real if $v = \bar{u}$. Then $\nabla h(u) = -\Delta u + V(x) u + \frac{\partial}{\partial \bar{u}} g(x, u, \bar{u})$, and (2.2) takes the form

$$\dot{u} = i (-\Delta u + V(x) u + \frac{\partial}{\partial \bar{u}} g(x, u, \bar{u})), \quad u = u(t, x), \quad x \in \mathbb{T}^n. \quad (2.5)$$

This is a semilinear Hamiltonian equation in any space $X_{d_0}, \ d_0 > d/2$, with ord $A = 2$ and ord $\nabla h_0 = 0$. Equation (2.5) with $V = 0$ and $g = C|u|^4$, $C \neq 0$, is called the Zakharov–Shabat equation. The equation with $C > 0$ is called defocusing and with $C < 0$ – focusing.

Example 2.7 (1D NLS with Dirichlet boundary conditions). Let us choose for $X_s$ the space $H^0_0(0, \pi; \mathbb{C})$ (see Example 2.2), $J u(x) = i u(x)$ and

$$h(u) = \frac{1}{2} \int_0^\pi \left( |u_x|^2 + V(x)|u|^2 + g(x, u) \right) dx,$$

where $g$ is smooth and $2\pi$-periodic in $x$. Now $\nabla h(u) = -u_{xx} + V(x) u + f(x, |u|^2) u$, where $f = \frac{\partial g}{\partial |u|^2}$, and (2.2) becomes

$$\dot{u} = i (-u_{xx} + V(x) u + f(x, |u|^2) u), \quad u(0) = u(\pi) = 0. \quad (2.6)$$

For $s = 1$ and $2$ the nonlinear term defines a smooth map $X_s \to X_s$ (see Example 2.3), so in these spaces this is a semilinear equation with ord $A = 2$ and ord $\nabla h_0 = 0$. If in addition $f$ is even in $x$, then the nonlinear term defines a smooth map for every $s \geq 1$. This map is analytic if $f$ is.

2.3 Integrable equations and their finite-gap solutions

Let us take a Hamiltonian PDE and write it as a Hamiltonian equation in a suitable symplectic Hilbert scale $\{X_s\}, \alpha_2 = J du \wedge du$:

$$\dot{u} = J \nabla H(u). \quad (2.7)$$

We assume that for any $u_0 \in X_\infty$ the equation has, locally in time, a unique smooth solution, equal $u_0$ at $t = 0$. Equation (2.7) is called Lax-integrable, or just integrable if there exists an additional real or complex Hilbert scale $\{Z_s\}$ and finite order linear morphisms of this scale $\mathcal{L}_u$ and $\mathcal{A}_u$ which depend on the parameter $u \in X_\infty$, such that a curve $u(t)$ is a smooth solution for (2.7) if and only if

$$\frac{d}{dt} \mathcal{L}_{u(t)} = [\mathcal{A}_{u(t)}, \mathcal{L}_{u(t)}]. \quad (2.8)$$
The operators $\mathcal{A}_u$ and $\mathcal{L}_u$, treated as morphisms of the scale $\{Z_s\}$, are assumed to depend smoothly on $u \in X_d$ with a sufficiently large $d$, so the left-hand side of (2.8) is well defined (for details see [Kuk00]). The pair of operators $\mathcal{L}, \mathcal{A}$ is called the Lax pair.

In most known examples of integrable equations relations between the scales $\{X_s\}$ and $\{Z_s\}$ are the following: spaces $X_s$ are formed by $T$-periodic Sobolev vector-functions, while $\mathcal{A}$ and $\mathcal{L}$ are differential or integro-differential operators with $u$-dependent coefficients, acting in a scale $\{Z_s\}$ of $TL$-periodic Sobolev vector-functions. Here $L$ is a fixed integer.

Let $u(t)$ be a smooth solution for (2.7). We set $L_t = L_{u(t)}$ and $A_t = A_{u(t)}$.

**Lemma 2.8.** Let $\chi_0 \in Z_{\infty}$ be a smooth eigenvector of $L_0$, i.e., $L_0 \chi_0 = \lambda \chi_0$. Let us also assume that the initial-value problem

$$\dot{\chi} = \mathcal{A}\chi, \quad \chi(0) = \chi_0,$$  \quad (2.9)

has a unique smooth solution $\chi(t)$. Then

$$L_t \chi(t) = \lambda \chi(t) \quad \forall t.$$  \quad (2.10)

**Proof.** Let us denote the left-hand side of (2.10) by $\xi(t)$, the right-hand side — by $\eta(t)$ and calculate their derivatives. We have:

$$\frac{d}{dt} \xi = \frac{d}{dt} L \chi = [\mathcal{A}, L] \chi + L \mathcal{A} \chi = \mathcal{A} L \chi = \mathcal{A} \xi$$

and

$$\frac{d}{dt} \eta = \frac{d}{dt} \lambda \chi = \lambda \mathcal{A} \chi = \mathcal{A} \eta.$$  

Thus, both $\xi(t)$ and $\eta(t)$ solve the problem (2.9) with $\chi_0$ replaced by $\lambda \chi_0$ and coincide by the uniqueness assumption. \hfill \square

Due to this lemma the discrete spectrum of the operator $L_u$ is an integral of motion for equation (2.7). In particular, a set $\mathcal{T}$ formed by all smooth vectors $u \in X_{\infty}$ such that the eigenvalues of the operator $L_u$ belong to a fixed subset of $\mathbb{C} \times \mathbb{C} \times \ldots$, is invariant for the flow of equation (2.7). A remarkable discovery, made almost 30 years ago by Novikov [Nov74] and Lax [Lax75], is that for integrable Hamiltonian PDEs, considered on finite space-intervals with suitable boundary conditions, some sets $\mathcal{T}$ as above are finite dimensional symplectic submanifolds $T^{2n}$ of $X_{\infty}$, and restriction of equation (2.7) to any $T^{2n}$ is an integrable Hamiltonian system. Moreover, for some integrable equations it is known that the union of all these manifolds $T^{2n}$ is dense in every space $X_s$. Solutions that fill a manifold $T^{2n}$ are called finite-gap solutions, and the manifold itself — a finite-gap manifold. See e.g. [DMN76, ZMNP84, BBE+94, Kuk00].

In particular, the KdV equation (2.4) has the form (2.7) in the Hilbert scale $\{X_s = H^s(S^1)_0\}$, is integrable and the corresponding Lax pair is

$$L_u = -\frac{\partial^2}{\partial x^2} - u, \quad \mathcal{A}_u = \frac{\partial^3}{\partial x^3} + \frac{3}{2} u \frac{\partial}{\partial x} + \frac{3}{4} u_x.$$  

Taking for $\{Z_s\}$ the Sobolev scale of $4\pi$-periodic functions and applying the Lemma above, we get that smooth $4\pi$-periodic spectrum of the operator $L_u$ is
an integral of motion. It is well known that this spectrum has the form
\[ \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots < \infty. \]
Let us take any \( n \)-vector \( V = (V_1, \ldots, V_n) \in \mathbb{N}^n \) such that \( V_1 < \cdots < V_n \).
Denoting \( \Delta_j = \lambda_{2j} - \lambda_{2j-1} \geq 0, j = 1, 2, \ldots \), we define the set \( T^n_2/V \) as
\[ T^n_2/V = \{ u(x) \mid \Delta_j \neq 0 \text{ iff } j \in \{ V_1, \ldots, V_n \} \}. \]
Clearly \( T^n_2/V \) equals to the union \( T^n_2/V = \bigcup_{r \in \mathbb{R}_+^n} T^n_2/(r) \), where \( T^n_2/(r) = \{ u(x) \mid \lambda_{2j} - \lambda_{2j-1} = r_j \forall j \in \{ V_1, \ldots, V_n \} \} \).

The sets \( T^n_2/(r) \) are invariant for the KdV-flow. By the classical theory of the
Sturm–Liouville operator \( L_{u} \), the set \( T^n_2/V \) is a smooth submanifold of any space \( X_s \), foliated to the smooth \( n \)-tori \( T^n_2/(r) \) (see e.g. [KP03]).

Due to Novikov and Lax, there exist an analytic map \( \Phi = \Phi_V : \{(r, \xi)\} \to \mathbb{R}_+ \times \mathbb{T}^n \to X_s \) and an analytic function \( h = h^n(r) \) such that \( T^n_2/(r) = \Phi((r) \times \mathbb{T}^n) \), and for any \( \xi_0 \in \mathbb{T}^n \) the curve \( u(t) = \Phi(r, \xi_0 + t(\nabla h(r))) \) is a smooth solution for (2.4). As a function of \( t \), this solution is quasiperiodic. The celebrated Its–Matveev formula explicitly represents \( \Phi \) in terms of theta-functions, see in [DMN76, Dub81, BBE+94, Kuk00].

Situations with other Lax–integrable equations is similar. For example, see [BBE+94] concerning the Sine-Gordon (SG) and Zakharov-Shabat equations; concerning (SG) see also [Kuk00, EKMY02]. Integrable HPDEs with self-adjoint \( L \)-operator (e.g., the defocusing Zakharov–Shabat equation) are very similar to (KdV), while those with non-selfadjoint \( L \)-operator (e.g., (SG) and the focusing Zakharov–Shabat equation) differ from them in some respects and are more complicated. In particular, for those equations the manifolds \( T^{2n} \) are not smooth but have algebraic singularities.

3 Perturbations of parameter-depending linear problems

3.1 An abstract theorem

Let \( \{(X_s)\}, \alpha_2 = \text{ord} \ J du \wedge du \) be a symplectic Hilbert scale, \( -d_J = \text{ord} \ J \leq 0; A \) be an operator which defines a selfadjoint automorphism of the scale of order \( d_A \geq -d_J \) and \( H \) be a Frechet–analytic functional on \( X_d_0 \), \( d_0 \geq 0 \), such that \( \text{ord} \nabla H = d_H < d_A \):
\[ \nabla H : X_{d_0} \to X_{d_0 - d_H}. \]
We assume that \( d_A \leq 2d_0 \), so the quadratic form \( \frac{1}{2} \langle Au, u \rangle \) is well defined on the space \( X_{d_0} \).

In this section we consider the quasilinear Hamiltonian equations with the Hamiltonian \( H_\varepsilon(u) = \frac{1}{2} \langle Au, u \rangle + \varepsilon H(u) \):
\[ \dot{u}(t) = J(Au(t) + \varepsilon \nabla H(u(t))). \]
We assume that the scale $\{X_s\}$ admits a basis $\{\varphi_k, k \in \mathbb{Z}_0 = \mathbb{Z}\{0\}\}$ such that:

$$A\varphi_j^\pm = \lambda_j^A \varphi_j^\pm, \quad J\varphi_j^\pm = \mp \lambda_j^J \varphi_j^\pm \quad \forall j \geq 1,$$

(3.2)

with some real numbers $\lambda_j^A, \lambda_j^J$. In particular, the spectrum of the operator $JA$ is $\{\pm i \lambda_j | \lambda_j = \lambda_j^A \lambda_j^J\}$. The numbers $\lambda_j$ are called the frequencies of the linear system

$$\dot{u} = JAu. \quad \text{(3.3)}$$

Let us fix any $n \geq 1$. Then the $2n$-dimensional linear space

$$\text{span}\{\varphi_j^\pm | 1 \leq j \leq n\} \quad \text{(3.4)}$$
is invariant for the equation (3.3) and is foliated to the invariant tori

$$T^n = T^n(I) = \left\{ \sum_{j=1}^n u_j^+ \varphi_j^+ + u_j^- \varphi_j^- | u_j^+ + u_j^- = 2I_j \forall j \right\}. \quad \text{(3.5)}$$

If $I \in \mathbb{R}_+^n$, then $T^n(I)$ is an $n$-torus. Providing it with the coordinates $q = (q_1, \ldots, q_n)$, where $q_j = \arg(u_j^+ + iu_j^-)$, we see that equation (3.3) defines on $T^n(I)$ the motion

$$\dot{q} = (\lambda_1, \ldots, \lambda_n) =: \omega. \quad \text{(3.6)}$$

So all solutions for the linear equation in $T^n(I)$ are quasiperiodic curves with the frequency-vector $\omega$. Our goal in this section is to present and discuss a KAM-theorem which implies that under certain conditions 'most of' trajectories of the equation (3.6) on the torus $T^n(I)$ persist as time-quasiperiodic solutions of the perturbed equation (3.1), if $\varepsilon > 0$ is sufficiently small.

To state the result we assume that the operator $A$ and the function $H$ analytically depend on an additional $n$-dimensional parameter $a \in \mathcal{A}$, where $\mathcal{A}$ is a connected bounded open domain in $\mathbb{R}^n$. Then $\lambda_j = \lambda_j(a)$. We assume that the first $n$ frequencies $\lambda_1 = \omega_1$ depend on $a$ in the nondegenerate way:

\begin{itemize}
  \item[H1)] $\det[\partial \omega_l / \partial a_k | 1 \leq k, l \leq n] \neq 0$;
  \item[H2)] $|\lambda_j(a) - K_1 j^d_1 - K_1^1 j^d_1^1 - K_1^2 j^d_2 - \ldots | \leq K j^d$, $\Lip \lambda_j \leq j^d,$
\end{itemize}

where $d_1 := d_A + d_J \geq 1$, $K_1 > 0$, $d < d_1 - 1$ and the dots stand for a finite sum with exponents $d_1 > d_1^1 > d_2^1 > \ldots$.

Let us denote by $X_{\mathbb{C}}_a$ the complexification of a space $X_a$ and assume that the equation (3.1) is quasilinear 'uniformly in a complex neighbourhood':

\begin{itemize}
  \item[H3)] the set $X_{d_0} \times \mathcal{A}$ admits in $X_{d_0} \times \mathbb{C}^n$ a complex neighbourhood $Q$ such that the map $\nabla_a H : Q \to X_{d_0-d_H}$ is complex-analytic and bounded uniformly on bounded subsets of $Q$. Moreover, $d_H + d_J \leq d$.
\end{itemize}

Finally, we shall need the following non-resonance condition:
For all integer $n$-vectors $s$ and $(M_2 - n)$-vectors $l$ such that $|s| \leq M_1$, $1 \leq |l| \leq 2$ we have,

$$s \cdot \omega(a) + l_{n+1}\lambda_{n+1}(a) + \cdots + l_{M_2}\lambda_{M_2}(a) \neq 0,$$

where the integers $M_1 > 0$ and $M_2 > n$ are to be specified.

Let us fix any $I_0 \in \mathbb{R}^n_+$ and denote by $\Sigma_0$ the map $\mathbb{T}^n \times A \to X_{d_0}$ which sends $(q, a)$ to the point of the torus $T^n(I_0)$ with the coordinate $q$.

**Theorem 3.1.** Suppose the assumptions H1)-H3) hold. Then there exist integers $M_1 > 0$ and $M_2 > n$ such that if $H_4)$ is fulfilled, then for arbitrary $\gamma > 0$ and for sufficiently small $\varepsilon < \varepsilon(\gamma)$, a Borel subset $A_\varepsilon \subset A$ and a Lipschitz map $\Sigma_\varepsilon : \mathbb{T}^n \times A_\varepsilon \to X_{d_0}$, analytic in $q \in \mathbb{T}^n$, can be found with the following properties:

a) $\operatorname{mes}(A \setminus A_\varepsilon) \leq \gamma$;

b) the map $\Sigma_\varepsilon$ is $C\varepsilon$-close to $\Sigma_0$ on $\mathbb{T}^n \times A_\varepsilon$, in the Lipschitz norm;

c) each torus $\Sigma_\varepsilon(\mathbb{T}^n \times \{a\})$, $a \in A_\varepsilon$, is invariant for the flow of equation (3.1) and is filled with its time-quasiperiodic solutions of the form $u_\varepsilon(t; q) = \Sigma_\varepsilon(q + \omega t, a)$, $q \in \mathbb{T}^n$, where the frequency vector $\omega(a)$ is $C\varepsilon$-close to $\omega(a)$ in the Lipschitz norm;

d) the solutions $u_\varepsilon$ are linearly stable.

We note that, strictly speaking, if equation (3.1) is not semilinear (i.e., if $d_J + d_H > 0$), then the last assertion of the theorem is proved provided that the equation satisfies some mild regularity condition, see Theorem 8.4 in [Kuk00].

Let us assume that $\nabla H$ defines an analytic map of order $d_H$ on every space $X_{d_0}$. If $u_\varepsilon(t)$ is any solution of (3.1) as in the theorem, then due to the equation, $JAu(t)$ is a smooth curve in $X_{d_0 - d_H - d_J}$. Since $JA$ is an automorphism of the scale of order $d_1$, then $u_\varepsilon(t)$ is a smooth curve in $X_{d_0 - d_H - d_J + d_1} \subset X_{d_0 + 1}$. Iterating this argument we see that $u_\varepsilon$ is a smooth curve in each space $X_s$. I.e., it is a smooth solution of the equation.

In the semilinear case (i.e., when $d_H + d_J \leq \tilde{d} < d_1 - 1$ and $\tilde{d} \leq 0$) the theorem is proved in [Kuk87, Kuk88] (see also [Kuk93, Pos96a]). The semilinearity restriction $d \leq 0$ was removed in [Kuk98] (see also [Kuk00] and [KP03]).

We note that for some specific HPDEs (3.1) the assertions of Theorem 3.1 can be proven for any $n \geq 1$ even if the parameter $a$ is only one-dimensional. In particular, this can be done for the nonlinear wave equation as in Example 3.3 below, where $V(x) \equiv a$ and the constant $a$ is the one-dimensional parameter. See [Bou94] and [Ban00b].

The proof of Theorem 3.1 is rather technical. For its well-written outline in the semilinear case see [Cra00]. Below we present the proof’s scheme in the form which suits our further purposes.

### 3.2 On the proof of Theorem 3.1

We start with the semilinear case and assume for simplicity that $\lambda_J \equiv 1$. Then $I = (I_1, \ldots, I_n)$ and $q = (q_1, \ldots, q_n)$ form a symplectic coordinate system in the
We take these relations as equations on the following relations hold:

\( \nabla F \) of the same structure, so an additional hamiltonian \( F \) of variable is a transformation \( \varepsilon H \) change of variable to kill the part \( O \) of \( \varepsilon H \) in a sufficiently small neighbourhood of \( T^n \) transforms the hamiltonian \( \mathcal{H}_\varepsilon \) to a hamiltonian which differs from its integrable part by \( O(\varepsilon^2) \).

Next in the vicinity of the torus \( T^n = \{ p = 0, y = 0 \} \) we make a symplectic change of variable to kill the part \( \varepsilon H_1 \) of the perturbation \( \varepsilon H_1 \). This change of variable is a transformation \( S_1 \) which is the time-\( \varepsilon \) shift along trajectories of an additional hamiltonian \( F \). Here the recipe is that to kill \( H_1 \), \( F \) should be of the same structure, so \( F = f(q) + f^p(q) \cdot p + \langle f^y(q), y \rangle + \langle f^{yy}(q)y, y \rangle \). As in the finite-dimensional situation, the transformed hamiltonian \( \mathcal{H}_\varepsilon \circ S_1 \) can be written as the Lie series in \( \varepsilon \). Retaining only linear in \( \varepsilon \) terms, we have

\[
\mathcal{H}_\varepsilon \circ S_1 = H_0 + \varepsilon H_1 + \varepsilon \langle J \nabla_y F, \nabla_y H_0 \rangle + \varepsilon \nabla_y F \cdot \nabla_y H_0 - \varepsilon \nabla_y F \cdot \nabla_y H_0 + O(\varepsilon^2) + O.
\]

Since \( \nabla_y H_0 = \omega, \nabla_y H_0 = 0 \) and \( \nabla_y H_0 = Ay \), then the linear in \( \varepsilon \) term vanishes if the following relations hold:

\[
(\omega \cdot \nabla) f = h, \quad (\omega \cdot \nabla) f^p = h^p, \quad (\omega \cdot \nabla) f^{yy} + [f^{yy}, JA] = h^{yy}.
\]

We take these relations as equations on \( f, f^p, f^y \) and \( f^{yy} \) (called ‘the homological equations’) and try to solve them.

Since the equations are constant-coefficient, then decomposing \( f, f^p, \ldots \) in Fourier series in \( q \), we find for their components (and for matrix components of the operator \( f^{yy} \)) explicit formulae. Certain terms in these formulae contain small divisors, which vanish for some values of the vector \( \omega = \omega(a) \). Careful analysis of these divisors show that all of them are bounded away from zero if \( a \notin \mathcal{A}_1 \), where \( \mathcal{A}_1 \) is a Borel subset of \( \mathcal{A} \) of small measure. When the equations are solved, we get a transformation which in a sufficiently small neighbourhood of \( T^n \) transforms the hamiltonian \( \mathcal{H}_\varepsilon \) to a hamiltonian which differs from its integrable part by \( O(\varepsilon^2) \).

The explanation above has some flaws. The most important one is that the first and the second homological equations can be solved only if the mean values
of $h$ and $hp$ vanish. To fulfill the first condition we change the hamiltonian $\varepsilon H_1$ by a constant (this change is irrelevant since it does not affect the equations of motion), while to fulfill the second we subtract from $\varepsilon H_1$ the average $\varepsilon \langle hp \rangle \cdot p$ and add it to the integrable part $H_0$, thus changing the term $\omega \cdot p$ to $\omega^2 \cdot p$, where $\omega^2 = \omega + \varepsilon \langle hp \rangle$. Similar, to solve the last homological equation we subtract from the operator $h_{yy}$ the average of its diagonal part and add the corresponding quadratic form to $H_0$. Thus, the transformed hamiltonian becomes

$$H_2 := \mathcal{H}_c \circ S_1 = \omega_2 \cdot p + \frac{1}{2} \langle A_2 y, y \rangle + \varepsilon^2 H_2(p, q, y) + O(p, q, y).$$

This transformation is called the KAM-step.

Next we perform the second KAM-step. Under the condition that $a \notin A_2$ we find a transformation $S_2$ which sends the hamiltonian $H_2$ to $H_3 = H_2 \circ S_2 = \omega_3 \cdot p + \frac{1}{2} \langle A_3 y, y \rangle + (\varepsilon^2)^2 H_2 + O(p, q, y)$, etc. After $m$ steps we find transformations $S_1, \ldots, S_m$ such that

$$H_2 := \mathcal{H}_c \circ S_1 \circ \cdots \circ S_m = \omega_m \cdot p + \frac{1}{2} \langle A_m y, y \rangle + \varepsilon^{2m} H_m + O(p, q, y) =: H_m.$$ 

The torus $T^n = \{ p = 0, y = 0 \}$ is ‘almost invariant’ for the equation with the hamiltonian $H_m$. Hence, the torus $S_1 \circ \cdots \circ S_m(T^n)$ is ‘almost invariant’ for the original one. Since the sequence $\varepsilon^{2m}$ converges to zero superexponentially fast, we can choose the sets $A_1, A_2, \ldots$ in such a way that $\text{mes}(A_\infty := A_1 \cup A_2 \cup \ldots) < \gamma$, for any $a \notin A_\infty$ the vectors $\omega_m(a)$ converge to a limiting vector $\omega'(a)$, and the transformations $S_1 \circ \cdots \circ S_m$ converge to a limiting map $\Sigma_\varepsilon(\cdot, a)$, defined on $T^n$. Then the torus $\Sigma_\varepsilon(T^n, a)$ is invariant for the equation (3.1) and is filled with its quasiperiodic solutions $t \rightarrow \Sigma_\varepsilon(q + \omega't, a)$.

If the equation is not semilinear, then the situation complicates since to solve the forth homological equation we have to remove from the operator $h_{yy}$ the whole of its diagonal part (not only its average). Because of that the operator $A$ in the integrable part of the hamiltonian gets terms which form a small $q$-dependent diagonal operator of a positive order. Accordingly, the forth homological equation becomes more difficult and cannot be solved by the direct Fourier method. Its resolution follows from a non-trivial lemma, based on properties of fast-oscillating Fourier integrals, proved in [Kuk98] (see also [Kuk00, KP03]).

### 3.3 Applications to 1D HPDEs

Theorem 1 well applies to parameter-depending quasilinear HPDEs with one-dimensional space variable in a finite interval, supplemented by boundary conditions such that spectrum of the linear operator $JA$ is not multiple. Indeed, for such equations assumption H2) follows from usual spectral asymptotics, H3) is obvious if the nonlinearity is analytic, while H1) and H4) hold if the equation depends on the additional parameter in a non-degenerate way. More explicitly it means the following. In the examples which we consider below, the equations depend on a potential $V(x; a)$, which is analytic in $a$ and smooth in $x$. The
non-degeneracy means that in a functional space, formed by functions of \(x\) and \(a\) of the required smoothness, the potential \(V\) should not belong to some analytic subset of infinite codimension.

Below we just list the examples. In each case application of Theorem 3.1 is straightforward. The theorem applies if dimension of the parameter \(a\) is \(\geq n\) and dependence of the potential \(V\) on \(a\) is nondegenerate as it was explained above. In the first three examples the potential \(V(x; a)\) is real, smooth in \(x\) and analytic in \(a\). The function \(f(x, v; a)\) is real, smooth in \(x\) and analytic in \(v\) and \(a\). Details can be found in [Kuk93, Kuk00, Kuk94, Kuk98].

**Example 3.2.** Nonlinear Schr"odinger equation (NLS), cf. Example 2.6:

\[
\dot{u} = i(-u_{xx} + V(x; a)u + \varepsilon f(x, |u|^2; a)u), \quad (3.9) \\
u = u(t, x), \quad x \in [0, \pi]; \quad u(t, 0) \equiv u(t, \pi) \equiv 0. \quad (3.10)
\]

Now \(d_J = 0, d_A = 2, \tilde{d} = d_H = 0\) and we view the Dirichlet boundary conditions as odd periodic ones (cf. Example 2.7). The theorem applies in the scale of odd periodic functions with \(d_0 = 1\) or \(2\). If \(f\) is even and \(2\pi\)-periodic in \(x\), then the constructed quasiperiodic solutions are smooth.

**Example 3.3.** Nonlinear string equation: \(w(t, x)\) satisfies (3.10) and

\[
\ddot{w} = w_{xx} - V(x; a)w + \varepsilon f(x, w; a),
\]

where now \(V > 0\) and \(f = 0\) if \(w = 0\). Let us denote \(U = (u, (-\Delta)^{-1/2} \dot{u})\).

It is a matter of direct verification that \(U\) satisfies a semilinear Hamiltonian equation (3.1) in a suitable symplectic Hilbert scale, formed by odd periodic Sobolev functions. Now \(d_A = 1, d_J = 0, \tilde{d} = d_H = -1\). Cf. [Way90]

**Example 3.4.** KdV-type equations:

\[
\dot{u} = \frac{\partial}{\partial x} (-u_{xx} + V(x; a)u + \varepsilon f(x, u; a)); \quad x \in S^1, \int_{S^1} u\, dx \equiv 0, \quad (3.11)
\]

cf. Example 2.5. Now \(d_J = 1, d_A = 2, \tilde{d} = d_H = 0\).

Theorem 3.1 also applies if \(x \in \mathbb{R}^1\) and the potential \(V(x; a)\) grows sufficiently fast when \(x \to \infty\).

**Example 3.5.** Nonlinear Schrödinger equation on the line:

\[
\dot{u} = i(-u_{xx} + (x^2 + \mu x^4 + V(x; a))u + \varepsilon f(|u|^2; a)u), \quad \mu > 0, \\
u = u(t, x), \quad x \in \mathbb{R}, \quad u \to 0 \text{ as } |x| \to \infty.
\]

Here the potential \(V\) is smooth, analytic in \(a\) and vanishes as \(|x| \to \infty\). The real-valued function \(f\) is analytic. Now \(d_J = 0, d_A = 4/3, d_H = 0\).

The time-quasiperiodic solutions, constructed in Examples 3.2 - 3.5, are linearly stable. Therefore they should be observable in numerical models for the corresponding equations. Indeed, quasiperiodic behaviour of solutions for 1D HPDEs with small nonlinearity was observed in many experiments, starting from the famous numerics of Fermi–Pasta–Ulam [FPU65]; e.g., see [ZIS79].
3.4 Multiple spectrum

In Examples 3.2, 3.3 the equations are considered under the Dirichlet boundary conditions. If we replace them by the periodic ones

\[ u(t, x) \equiv u(t, x + 2\pi), \]

then Theorem 3.1 would not apply since now the frequencies of the corresponding linear equations are asymptotically double: they have the form \( \{\lambda^+_j, j \geq 1\} \), where \( |\lambda^+_j - \lambda^-_j| \to 0 \) as \( j \to \infty \). It is clear that the numbers \( \{\lambda^\pm_j\} \) cannot be re-ordered to meet the spectral asymptotic condition H2). Still, for some semi-linear equations (3.1) assertions of the theorem remain true if the frequencies \( \lambda_j \) are not single, but asymptotically they have the same multiplicity \( m \geq 2 \) and behave regularly. Corresponding result is proved by Chierchia–You in [CY00], using the scheme, explained in section 3.2. We do not give precise statement of their theorem, but note that it applies to the nonlinear string equation in Examples 3.3 under the periodic boundary conditions. The result is the same: if the non-degeneracy condition holds, then for \( \epsilon \) small enough and for most (in the sense of measure) values of the \( n \)-dimensional parameter \( a \), solutions of the linear equation (3.3) which fill in a torus \( T^n(I), I \in \mathbb{R}^n_+ \), persist as linearly stable time-quasiperiodic solutions of the corresponding non-linear equation (3.1).

We note that earlier this persistence result was proven by Bourgain [Bou94], who used another KAM-scheme, discussed in the next section.

3.5 Space-multidimensional problem

The abstract Theorem 3.1 is a flexible tool to study 1D HPDEs, but it never applies to space-multidimensional equations since the spectral assumption H2) never holds in high dimensions. At the time of this writing, the only published KAM result, which applies to higher-dimensional HPDEs, is due to J.Bourgain [Bou98]. In that work the 2D NLS equation as in the Example 2.6 is considered. For technical reasons the potential term \( V \) is replaced there by the convolution \( V * u \):

\[ \dot{u} = i(-\Delta u + V(x; a) * u + \epsilon \frac{\partial}{\partial \bar{u}} g(u, \bar{u})), \quad u = u(t, x), \quad x \in \mathbb{T}^2. \quad (3.12) \]

The potential \( V(x; a) \) is real analytic and \( g(u, \bar{u}) \) is a real-valued polynomial of \( u \) and \( \bar{u} \). This equation has the form (3.1), where \( Au = -\Delta u + V * u \) and \( JU = iu \). The basis \( \{\varphi_k\} \) as in (3.2) is formed by the exponents \( \{e^{i\alpha s}, s \in \mathbb{Z}^2\} \), renumerated properly, and

\[ \lambda^f_s \equiv 1, \quad \lambda^A_s = |s|^2 + \hat{V}(s; a), \]

where \( \{\hat{V}(s; a)\} \) are the Fourier coefficients of \( V \). For any \( n \), the linear equation (3.12) \( \epsilon = 0 \) has quasiperiodic solutions

\[ u = \sum_{j=1}^{n} z_{\lambda^f_j} e^{i\lambda^f_j t} \varphi_{\lambda^f_j}(x) \quad (3.13) \]
(these are trajectories of the equation (3.6) on the torus $T^n(I)$, where $I_j = \frac{1}{2}|z_j|^2$). For simplicity let us assume that $a_j = \hat{V}(s_j; a)$, $j = 1, \ldots, n$. Then the result of [Bou98] is that for most values of the parameter $a$ (in the same sense as in Theorem 3.1), the solution (3.13) persists as a time-quasiperiodic solution of the equation (3.12). In difference with the 1D case, it is unknown if the persisted solutions are linearly stable.

Presumably, in the nearest future similar results will be obtained for a number of other semilinear equations. We do not expect that an abstract KAM-theorem, which applies for different classes of space-multidimensional HPDEs, will be proven. Certainly it will be much more difficult to handle quasi-linear equations that are not semi-linear. It is very plausible that KAM-solutions for some space-multidimensional HPDEs are not linearly stable. If so, then it will be hard to observe them in numerical experiments. Unfortunately, we are not aware of any related reliable numerics.

The proof in [Bou98] is based on a KAM-scheme, different from that described in section 3.2. Originally this scheme is due to Craig and Wayne [CW93] who used it to construct periodic solutions of nonlinear wave equations. Also see [Bou94].

Now we briefly describe the scheme, using the notations from section 3.2. When the perturbation $\varepsilon H_1$ is decomposed as in (3.8), we extract the term $\varepsilon \langle h^{yy}(q)y, y \rangle$ from $\varepsilon H_1^1$ and add it to the integrable part $H_0$. After this the hamiltonian to be killed is the sum of the three terms $h(q) + h^p(q) + \langle h^y(q), y \rangle$; accordingly the hamiltonian $F$ is a sum of three terms as well. We have to find them from the first three homological equations. The first two are not difficult, but the third one is a real problem since the operator $A$ is not any more constant-coefficient but equals $A_0 + \hat{A}(q)$, where $\hat{A}$ is a bounded operator of order $\varepsilon$. The resolution of this equation for high KAM steps is the most difficult part of implementation of the Craig–Wayne–Bourgain KAM-scheme.

4 Perturbations of integrable equations

Let us consider a quasilinear HPDE on a finite space-interval, which is an integrable Hamiltonian equation (2.7) in some symplectic Hilbert scale $\{X_s\}$, $\alpha_2 = Jdx \wedge dx$. As we explained in section 3.1, this equation has invariant finite-gap symplectic manifolds $T^{2n}$ such that restriction of (2.7) to any of them is integrable. In this section we discuss the results on persistence of quasiperiodic solutions in these manifolds, provided by the KAM for PDEs theory. We shall see that they are very similar to the celebrated Kolmogorov theorem, which states that most of quasiperiodic solutions of a nondegenerate analytic integrable (finite-dimensional) Hamiltonian system persist under small perturbations of the hamiltonian; see [Arn63, MS71] and Addendum in [Kuk00]. We state the main result as a

**Theorem 4.1 (Metatheorem).** Most of quasiperiodic solutions that fill in any finite-gap manifold $T^{2n}$ as above persist under small Hamiltonian quasilinear
analytic perturbations of the integrable equation. If the finite-gap solutions in $T^{2n}$ are linearly stable, then the persisted solutions are linearly stable as well.

In the assertion above the statement ‘most of quasiperiodic solutions persist’ means the following. Due to the Liouville–Arnold theorem [Arn89, HZ94], the manifold $T^{2n}$ can be covered by charts, diffeomorphic to $B_1 \times T^n = \{ p, q \} \ (B_1$ is a one-ball in $\mathbb{R}^n$), with chart-maps $\Phi_0 : B_1 \times T^n \to T^{2n}$ such that $\Phi_0^{\alpha_2} = dp \wedge dq$, and the curves $\Phi_0(p, q + t \nabla h(p))$ are solutions of the integrable equation, where $h(p) = H \circ \Phi_0(p, q)$. Let us denote by $\varepsilon$ the small coefficient in front of the perturbation. Then for every chart there exists a Borel subset $B_\varepsilon \subset B_1$ and a map $\Phi_\varepsilon : B_\varepsilon \times T^n \to X_d$ ($d$ is fixed), with the following properties:

i) $\text{mes}(B \setminus B_\varepsilon) \to 0$ as $\varepsilon \to 0$;

ii) the map $\Phi_\varepsilon : B_\varepsilon \times T^n \to X_d$ is $C_\sqrt{\varepsilon}$-close to $\Phi_0$ in the Lipschitz norm and is analytic in $q \in T^n$;

iii) there exists a map $\omega_\varepsilon : B_\varepsilon \to \mathbb{R}^n$, $C_\varepsilon$-close to the gradient map $\nabla h$ in the Lipschitz norm, such that the curves $t \mapsto \Phi_\varepsilon(p, q + t \omega_\varepsilon(p)), \ p \in B_\varepsilon, \ q \in T^n$, are solutions for the perturbed equation.

The statement of Theorem 4.1 is proven under a number of assumptions (see [Kuk00], [EKMY02]). These assumptions are checked for such basic integrable HPDEs as KdV, Sine– and Sinh–Gordon equations. There are no doubts that they also hold for the Zakharov–Shabat equations $^1$ (but the theorem in [Kuk00, EKMY02] does not apply to the Kadomtsev–Petviashvili equation). Below we present a scheme of the proof and discuss the restrictions on the integrable HPDE which allow to implement it.

We view (2.7) as an equation in the Hilbert space $X_d$, and denote the quasi-linear hamiltonian of the perturbed equation as

$$H_\varepsilon = \frac{1}{2} \langle Ax, x \rangle + h_0(x) + \varepsilon h_1(x).$$

Accordingly, $H_0 = \frac{1}{2} \langle Ax, x \rangle + h_0$ is the hamiltonian $H$ of the unperturbed equation (2.7).

Step 1. Let us consider any finite-gap solution $u_0(t) = \Phi_0(p_0, q_0 + t \nabla h(p_0))$ and linearize (2.7) about it:

$$\dot{v} = J(\nabla H(u_0(t)))_* v.$$  \hfill (4.1)

The theory of integrable equations provides tools to reduce this equation to constant coefficients by means of time-quasiperiodic substitution $v(t) = G(p_0, q_0 + t \nabla h(p_0)) \tilde{v}(t)$, where $G(p, q), \ (p, q) \in B_1 \times T^n$, is a symplectic linear map $G(p, q) : Y_d \to Z_d$. Here $Y_d$ is a fixed symplectic subspace of $Z_d$ of codimension $2n$. The restriction, which we impose at this step, is that the operator $G(p, q)$ is a compact perturbation of the embedding $Y_d \to Z_d$, which analytically depends on $(p, q)$.

Step 2. The map $G$ from the Step 1 defines an analytic map

$$B_1 \times T^n \times Y_d \to X_d,$$

$^1$See [BK03] for an ad hoc KAM-theorem for the defocusing equation.
linear and symplectic in $y \in Y_d$. This map defines a symplectomorphism

$$B_1 \times T^n \times B_\delta(Y_d) \to X_d, \quad B_\delta(Y_d) = \{ \|y\|_d < \delta \},$$

(4.2)
such that linearisation in $y$ at $y = 0$ of the latter equals the former.

Step 3. We use the map (4.2) to pass in the Hamiltonian $H_\varepsilon$ to the variables $(p, q, y)$. We get

$$H_\varepsilon(p, q, y) = h(p) + \frac{1}{2} \langle A(p)y, y \rangle + h_3(p, q, y) + \varepsilon h_1(p, q, y),$$

(4.3)

where $h_3 = O(\|y\|_d^3)$. Calculations show that $h_3(p, q, y)$ contains terms such that their gradient maps have the same order as the operator $A(p)$. If this really was the case, then the Hamiltonian equation would not be quasilinear, which would complicate its study a lot. Fortunately, this does not happen due to a cancellation of a very general nature (see Lemma 7.5 in [Kuk00] and Lemma 12 in [EKMY02]), and we have

$$\text{ord} \nabla h_3 < \text{ord} A(p) - 1.$$  

(4.4)

Step 4. Invariant tori of the unperturbed system with the Hamiltonian $H_0(p, q, y)$ have the form $\{ p = \text{const} \}$. Let us scale the variables near a torus $\{ p = a \}$: $p = a + \varepsilon^{2/3} \tilde{p}$, $q = \tilde{q}$, $y = \varepsilon^{1/3} \tilde{y}$. In the scaled variables the perturbed equation has the Hamiltonian

$$\text{const} + \omega(a) \cdot \tilde{p} + \frac{1}{2} \langle A(a) \tilde{y}, \tilde{y} \rangle + O(\varepsilon^{1/3}), \quad \omega(a) = \nabla h(a).$$

(4.5)

So we have got the system (3.1), written in the form (3.8), with $\varepsilon$ replaced by $\varepsilon^{1/3}$. If Theorem 3.1 applies, then most of the finite-gap tori $\{ p = \text{const} \}$ persist in the perturbed equation, as states the Metatheorem. To be able to use the theorem we have to check the assumptions H1) - H4).

The condition H2) holds if the integrable equation is 1D (if the spectrum is asymptotically double, e.g. if the unperturbed equation is the Sine–Gordon equation under the periodic boundary conditions, then one should use a version of the Metatheorem, based on the Chierchia–You result). The quasilinearity condition H3) holds due to (4.4). The assumption H1) now takes the form

$$\text{Hess} h(p) \not\equiv 0.$$  

(4.6)

This is exactly Kolmogorov’s nondegeneracy condition for the integrable system on $T^{2n}$. The assumption H4) with $\omega = \nabla h(a)$ is the second nondegeneracy condition, which needs verification.

Summing up what was said above, we see that Theorem 3.1 implies the Metatheorem if the unperturbed integrable equation is 1D quasilinear, the linear operator $G(p, q)$ from Step 1 possesses the required regularity properly and the nondegeneracy assumptions (4.6) and (3.7) hold true.

The scheme we have just explained was suggested in [Kuk89], where it was used to prove an abstract KAM-theorem, which next was applied to Birkhoff-integrable infinite dimensional systems and to perturbed KdV equations. See
[Kuk00], [EKMY02] for a more general abstract theorem, based on the same scheme.

Steps 1–2 are not the only way to reduce an integrable equation to the normal form (4.3). Another approach to get it has been initiated by Kappeler [Kap91]. In [KM01] it is proved that the KdV equation is Birkhoff-integrable. It means the following. Let us take the Darboux scale \( \{X_s, \alpha_2\} \) with the index-set \( Z = \mathbb{Z}_0 \), and \( \theta_k = |k| \) (see Example 2.4). Then there exists a map \( \Phi : X_\infty \to H_0^\infty(S^1) \) which extends to analytic maps \( X_s \to H^s(S^1) \), \( s \geq 0 \), such that

\[
h \circ \Phi(u) = \sum_{j=1}^{\infty} j^3(u_j^2 + u_{-j}^2) + \langle \text{a function of } u_l^2 + u_{-l}^2, l = 1, 2, \ldots \rangle.
\]

(4.7)

Here \( \{u_k, k \in \mathbb{Z}_0\} \) are coefficients of decomposition of \( u \in X_s \) in the basis \( \{\varphi_k\} \) and \( h \) is the KdV-hamiltonian (see Example 2.5). Moreover, the hamiltonian (4.7) defines an analytic Hamiltonian vector field of order three in each space \( X_d, d \geq 1 \). In the transformed variables the \( N \)-gap tori of the KdV equation take the form (3.5), where \( n \geq N \) and exactly \( N \) numbers \( I_j \) are non-zero. Now let us take a torus (3.5), where \( I \in \mathbb{R}_n^+ \). Making a change of variables as in section 3.2, we arrive at the hamiltonian (4.5). Detailed and readable derivation of the normal form (4.7) see in [KP03].

Reduction to the Birkhoff normal form (4.7) uses essentially specifics of the KdV’s \( L \)-operator. Still, similar arguments apply to some other integrable HPDEs. In particular, to the defocusing Zakharov–Shabat equation.

Presumably, Birkhoff normal forms exist for many integrable equations with self-adjoint \( L \)-operators (see [BK03] for the defocusing Zakharov–Shabat), but not for equations with non-selfadjoint operators. In particular, the focusing Zakharov–Shabat equation cannot be reduced to the form (4.7) since for this equation some finite-gap tori are linearly unstable [CMM02], while all invariant tori of the form (3.5) for the hamiltonian (4.7) are linearly stable.

Example 4.2 (perturbed KdV equation). Consider the equation

\[
\dot{u}(t, x) = \frac{1}{4} \frac{\partial}{\partial x}(u'' + 3u^2 + \varepsilon f(x, u)), \quad x \in S^1; \quad \int_{S^1} u \, dx \equiv 0,
\]

(4.8)

where \( f \) is smooth in \( x, u \) and analytic in \( u \). The Metatheorem applies and implies that most of finite-gap KdV-solutions persist as time-quasiperiodic solutions of (4.8). Moreover, these solutions are smooth and linearly stable. \( \square \)

This result was first proved in [Kuk89] with a number of omissions. Two the most serious are that Theorem 3.1, proved then only for semilinear equations, was used in a quasilinear case, and that the non-degeneracy assumptions (4.6) and (3.7) were taken for granted. These omissions were filled in later. The quasilinear version of Theorem 3.1 was proved in [Kuk98] and the non-degeneracy conditions were verified in [BK91]. The verification follows the following scheme: the finite-gap manifold \( \mathcal{T}_V^{2n} \) can be uniformised near its singular set \( \{r_j = 0 \text{ for some } j\} \) by means of coordinates \((R, q), R \in \mathbb{R}_+^N, q \in \mathbb{T}^n, R \geq 0\),
such that $R_j = 0$ iff $r_j = 0$, $j = 1, \ldots, n$. The functions $\omega_j$ and $\lambda_j$ all are $q$-independent and analytic in $R$, and their 1-jets at $R = 0$ can be calculated explicitly. Assuming that the l.h.s. of (3.7) vanishes identically, taking its 1-jet at $R = 0$ and using elementary Diophantine arguments, we arrive at a contradiction. This way to prove the non-degeneracy is very general and applies to other equations.

For complete proofs see [Kuk00, EKMY02] and [KP03].

The Metatheorem (in its rigorous form as in [Kuk00, EKMY02] and [KP03]), applies to quasilinear Hamiltonian perturbations of higher equations from the KdV-hierarchy, provided that the non-degeneracy relations are checked for these equations. This can be done in the same way as in Example 4.2. See [KP03], where the nondegeneracy of the second KdV equation is verified.

**Example 4.3 (perturbed SG equation).** Consider the equation

$$\ddot{u} = u_{xx} - \sin u + \varepsilon f(u, x), \quad u(t, 0) = u(t, \pi) = 0,$$

where $f(0, x) \equiv 0$ (and $f \in C^\infty$ is analytic in $u$). The Metatheorem applies to prove persistence most of finite-gap solutions of the SG-equation, see [BK93, Kuk00, EKMY02]. In general, due to the phenomenon explained in Example 2.7, the persisted solutions are only $H^2$-smooth in $x$. But if $f$ is $x$-independent and odd in $u$, then these solutions are smooth.

In difference with the KdV-case, large amplitude finite-gap SG-solutions, as well as the corresponding persisted solutions of (4.9), in general are not linearly stable.

To end this section we note that since the persisted solutions $u_\varepsilon(t)$ have the form

$$u_\varepsilon(t) = \Phi_\varepsilon(p, q + t\omega_\varepsilon(p)) = \Phi_0(p, q + t\omega_\varepsilon(p)) + O(\sqrt{\varepsilon}),$$

then to calculate them with the accuracy $\sqrt{\varepsilon}$ for all values of time $t$, we can use the “finite gap map” $\Phi_0$ with the corrected frequency vector. Moreover, $\omega_\varepsilon(p) = \nabla h(p) + \varepsilon W_1(p) + O(\varepsilon^2)$, where the vector $W_1(p)$ can be obtained by averaging over the corresponding finite-gap torus of some explicit quantity, see [Kuk00], p.147.

### 5 Small amplitude solutions of HPDEs

Let us consider the nonlinear string equation

$$u_t = u_{xx} - mu + f(u), \quad u = u(t, x), \quad 0 \leq x \leq \pi; \quad u(t, 0) = u(t, \pi) = 0.$$

Here $m > 0$ and $f$ is an odd analytic function of the form

$$f(u) = \kappa u^3 + O(|u|^5), \quad \kappa > 0.$$

Since $m, \kappa > 0$, then constants $a, b > 0$ can be found such that $-mu + f(u) = -a \sin bu$. Hence, the equation (5.1) can be written as

$$u_{tt} = u_{xx} - a \sin bu + O(|u|^5).$$
After the scaling $u = \varepsilon w$, $\varepsilon \ll 1$, the higher-order perturbation transforms to a small one, and we can apply the Metatheorem (cf. Example 4.3) to prove that small-amplitude parts of the finite-gap manifolds $T^{2n}$, $n = 1, 2, \ldots$, for the SG equation $u_{tt} = u_{xx} - a \sin bu$ with the Dirichlet boundary conditions mostly persist in (5.1). To put this scheme through, small-amplitude parts of the manifolds $T^{2n}$,

$$T^{2n}_\delta = \{(u, \dot{u}) \in T^{2n} | \|u\| + \|\dot{u}\| < \delta\}, \quad 0 < \delta \ll 1,$$

have to be studied in details. This task was done in [BK95b], where the following results were proved:

i) the sets $T^{2n}_\delta$ are smooth manifolds,

ii) they are in one-to-one correspondence with their tangent spaces at zero,

iii) these tangent spaces are the invariant spaces for the Klein–Gordon equation $u_{tt} = u_{xx} - (ab)u$.

Another proof of i)-iii) was suggested in [Kuk00]. It is based on some ideas from [Kap91] and applies to other integrable equations. After i)-iii) are obtained, a version of the Metatheorem (or a version of Theorem 3.1) applies to prove that most of finite-gap solutions from a manifold $T^{2n}_\delta$ persist in (5.1) in the following sense: the $2n$-dimensional Hausdorff measure of the persisted part of the manifold, divided by a similar measure of $T^{2n}_\delta$, converges to one as $\delta \to 0$.

See [BK95a] for a proof and [Kuk94] for discussion.

Similar results hold for the NLS equation

$$i\dot{u} = u_{xx} + mu + f(|u|^2)u,$$  \hspace{1cm} (5.2)

where $f(0) = 0$, $f'(0) = \gamma \neq 0$, since it is a higher-order perturbation of the Zakharov–Shabat equation $i\dot{u} = u_{xx} + mu + \gamma|u|^2u$. But it turns out that it is easier to approximate (5.2) near the origin by its partial Birkhoff normal form. The latter is an integrable infinite-dimensional Hamiltonian system (which is not an HPDE), and a sibling of the Metatheorem applies to prove that most of its time-quasiperiodic solutions persist in (5.2), see [KP96]. More on the techniques of Birkhoff normal forms in HPDE see in [Pos96b] and [KP03]. The classical reference for finite-dimensional Birkhoff normal forms is the book [MS71].

6 Around the Nekhoroshev theorem

The classical Nekhoroshev theorem [Nek77] deals with nearly-integrable Hamiltonian systems with analytic hamiltonians $H_\varepsilon(p, q) = h(p) + \varepsilon H(p, q)$ on the phase-space $P \times \mathbb{T}^n$, $P \subset \mathbb{R}^n$, given the usual symplectic structure $dp \wedge dq$. Under the assumption that the hamiltonian $h(p)$ satisfies a mild non-degeneracy assumption called the steepness, the theorem states that the action variables change exponentially slow along trajectories of the system. Namely, there exist constants $a, b \in (0, 1)$ such that for any trajectory $(p(t)), q(t))$ of the system we have

$$|p(t) - p(0)| \leq C\varepsilon^a \text{ if } |t| \leq \exp(\varepsilon^{-b}).$$  \hspace{1cm} (6.1)
Strictly convex functions \( h(p) \) form an important class of the steep hamiltonians. An alternative proof of the theorem which applies in the convex case was suggested by Lochak [Loc92]. It is based on clever approximation of a trajectory \((p(t), q(t))\) by a time-periodic solution of the equation which is a high-order normal form for \( H_\epsilon \). So rational frequency-vectors play for the Lochak approach very important role.

Original Nekhoroshev’s proof contains two parts, analytical and geometrical. The techniques, developed in the analytical part of the proof, allow to get the following result, which we call below the quasi-Nekhoroshev theorem: Let us consider the hamiltonian \( H_\epsilon \), depending on an additional vector-parameter \( \omega \in \Omega \subset \mathbb{R}^n \), \( H_\epsilon = p \cdot \omega + \varepsilon H(p, q) \). Then for any \( \gamma > 0 \) there exists a Borel subset \( \Omega_\gamma \subset \Omega \) (the Diophantine subset) such that \( \text{mes}(\Omega \setminus \Omega_\gamma) < \gamma \), and (6.1) with \( C = C_\gamma \) holds if \( \omega \in \Omega_\gamma \). Note that in the Cartesian coordinates \((x, y)\), corresponding to the action-angle variables \((p, q)\) (i.e., \( x_j = \sqrt{2p_j} \cos q_j \), \( y_j = \sqrt{2p_j} \sin q_j \)), the hamiltonian \( H_\epsilon \) reeds as

\[
H_\epsilon = \frac{1}{2} \sum \omega_j (x_j^2 + y_j^2) + \varepsilon H(x, y).
\]

That is, \( H_\epsilon \) is a perturbation of the quadratic hamiltonian \( H_0 \). Therefore the quasi-Nekhoroshev theorem should apply to study the dynamics in the vicinity of the origin of an analytical hamiltonian

\[
H(x, y) = H_0 + h, \quad h = O((|x, y|)^3).
\]

To get a corresponding theorem which applies to all small initial data is a non-trivial task, resolved by Niederman [Nie98] by means of the Lochak approach.

No analogy of the Nekhoroshev theorem for PDEs is known yet, but a number of ad hoc quasi-Nekhoroshev theorems for PDEs were proved, mostly by Bourgain and Bambusi, see [Bam99, Bam00b, Bou00] and references therein. These works discuss stability of the equilibrium for HPDEs (mostly 1D) with hamiltonians of the form (6.2). Under some restrictions on the quadratic part \( H_0 \) and on the higher-order part \( h \), it is proved that if the initial data \( u_0 \) is an \( \varepsilon \)-small and ‘very’ smooth function, then a solution stays very close to the corresponding invariant torus of the linear system with the hamiltonian \( H_0 \), during the time which is polinomially large in \( \varepsilon^{-1} \), or even exponentially large. This result is obtained either under the ‘quasi-Nekhoroshev’ condition that the spectrum of the operator \( A \) is ‘highly non-resonant’, or under the opposite assumption (needed to apply the Lochak–Niederman technique) that the spectrum is ‘very resonant’. In particular, the following result is proved in [Bam99] (also see [Pos99, Bou00]): Let us consider the NLS equation

\[
\dot{u} = i(-u_{xx} + \varphi(|u|^2)u), \quad \varphi'(0) \neq 0,
\]

where \( \varphi \) is an analytic function and the equation is being studied in the scale \( \{H_0(0, \pi)\} \) of odd 2\( \pi \)-periodic functions. Assume that \( u_0(x) = \sum_{k=1}^{N} u_{k0} \sin kx \), \( k \geq 1 \),
denote $\epsilon = |u_0(x)|_{L^2} \ll 1$ and write the solution $u(t, x)$ of (6.3) as $u = \sum u_k(t) \sin kx$. Then there exist $\epsilon_* > 0$ and constants $C_1, C_2 > 0$ such that for $\epsilon < \epsilon_*$ and $|t| \leq C_1 \exp(\epsilon_* \epsilon)^{1/N} =: T_\epsilon$ we have

$$\sum_{k=1}^{\infty} \left( |u_k(t)|^2 - |u_{k0}|^2 \right)^2 \leq C_2 \epsilon^{4+1/N}.$$  \hspace{1cm} (6.4)

Let us set $T^n = \{ u(x) = \sum_{k=1}^{N} u_k \sin kx \mid |u_k| = |u_{k0}| \}$. This is an $n$-torus of diameter $\sim \epsilon$ and (6.4) implies that

$$\text{dist}_{H^s}(u(t), T^n) \leq C_s \epsilon^{1+1/N} \forall |t| \leq T_\epsilon,$$

if $s < -1/4$. Thus, during the time $T_\epsilon$ the trajectory $u(t)$ remains very close to its projection to $T^N$. The latter is a trajectory of an $N$-dimensional dynamical system, so the time of its return to a $\rho\epsilon$-neighbourhood ($\rho \ll 1$) of the initial point 'should' be of order $\rho^{-N}$. Same is true for the trajectory $u(t)$, if $\epsilon$ is small in terms of $\rho$. The phenomenon of the pathologically good recurrence properties of small-amplitude trajectories of some 1D HPDEs is well known from numerics.\footnote{E.g. I observed it in late 80's in numerics which I run in the Institute for Control Sciences (Moscow) to get numerical evidence for the persistence-result, described in Section 3.3. Instead of verifying that a solution for a non-linear HPDE is quasiperiodic, I checked that it has good recurrence property. To my surprise, not only some solutions (those close to the persisted tori) were recurrent, but practically all of them were. With the 15-years delay I thank Rauf Izmailov for helping me with the numerics.}

We have seen that quasi-Nekhoroshev theorems as above explain it up to some extend.

## 7 Open problems

In this section we state problems, related to qualitative properties of nearly-integrable and nearly-linear HPDEs which we consider as important.\footnote{E.g. I observed it in late 80's in numerics which I run in the Institute for Control Sciences (Moscow) to get numerical evidence for the persistence-result, described in Section 3.3. Instead of verifying that a solution for a non-linear HPDE is quasiperiodic, I checked that it has good recurrence property. To my surprise, not only some solutions (those close to the persisted tori) were recurrent, but practically all of them were. With the 15-years delay I thank Rauf Izmailov for helping me with the numerics.}

The KAM-theory for 1D HPDEs is now reasonably complete (see sections 3.1 - 3.4 and 4). The main open problem here is the following:

**Problem 7.1 (infinite-dimensional tori).** What happens under perturbations to infinite-dimensional tori (3.5) with $n = \infty$, where the actions $I_j > 0$ decays as $j^{-M}$ for some $M > 0$?

Perturbation theory for infinite-dimensional tori with very fast decaying actions is similar to that for finite-dimensional ones, see [Bou96, Pos02].

For nD HPDEs with $n \geq 2$ the only published KAM-result is [Bou98]. The main problems in the nD-case seem to be the following:

**Problem 7.2 (nD equations).** i) Find an abstract theorem which applies to various nD equations (cf. Theorem 3.1 and its applications in section 3.3).

ii) Prove a KAM-theorem for some nD HPDE ($n \geq 2$) which is quasilinear but not semilinear.

iii) Analyse linearized stability of the persisted solutions.
iv) (a counter-example). Find a parameter-depending HPDE (3.1) such that H1) holds, but the $n$-torus (3.5) does not persist in (3.1) for arbitrarily small $\varepsilon$, for all parameters from a set of positive $\varepsilon$-independent measure.

KAM-theory for perturbations of integrable HPDEs is rather complete since ‘almost all’ of these equations are 1D. The Kadomtsev–Petviashvili (KP) equation (see e.g. in [Dub81, BBE*94]) is a notable example of an integrable HPDE which is not 1D:

Problem 7.3 (perturbed KP equation). Analyse KAM-stability of finite-gap solutions for the KP equation under Hamiltonian perturbations of the equation.

Nothing is known about small-amplitude time-quasiperiodic solutions for a nonlinear HPDE, which is not a perturbation of an integrable equation by a higher-order term (cf. section 5). In particular we have:

Problem 7.4 ($\varphi^4$-equation). Construct small-amplitude time-quasiperiodic solutions for the zero mass $\varphi^4$-equation $u_{tt} = u_{xx} - u^3$ under, say, odd periodic boundary conditions (see [ZIS79] for related numerics).

In the Nekhoroshev theory it is desirable to get an abstract quasi-Nekhoroshev theorem which would apply to various HPDEs. Still the main challenge here is to prove or disprove the Nekhoroshev theorem:

Problem 7.5 (Nekhoroshev for PDEs). Prove or disprove the following statement: For any $m \geq 1$ and $K > 0$ there exist $a, b \in (0, 1)$ and constants $C_1, \ldots, C_m$ such that if $u(t)$ is a solution for the perturbed KdV (4.8) with $u(0) = u_0$, and

$$H_1(u_0) \leq K, \ldots, H_m(u_0) \leq K,$$

then for $0 \leq t \leq \exp \varepsilon^{-b}$ and $j = 1, \ldots, \tilde{m}$ we have $|H_j(u(t)) - H_j(u_0)| \leq \varepsilon^a$. Here $\tilde{m} = \tilde{m}(m) \leq m$ goes to infinity with $m$ and $H_m$ is the $m$-th hamiltonian from the KdV-hierarchy.

If this statement is true, then it might be easier to get it under the assumption that the function $u_0(x)$ is analytic (and $|u_0(x)| \leq K$ in a fixed complex strip).

As usual, closely related is the problem of diffusion. It can be understood differently. For example:

Problem 7.6 (diffusion). For any $\varepsilon > 0$ find a smooth solution $u(t)$ of (4.8) such that $|H_j(u(T_\varepsilon)) - H_j(u(0))| \geq \delta > 0$ for some $T_\varepsilon > 0$, where $j$ and $\delta$ are $\varepsilon$-independent.

Problem 7.7 (diffusion). Find a quasilinear nD HPDE ($n \geq 1$) in a suitable Sobolev scale $\{X_s\}$ such that for all sufficiently large $s$, for any $\varepsilon > 0$ and for a ‘typical’ initial condition $u(0) \in \{\|u\|_s = \varepsilon\}$, for the corresponding solution $u(t)$ we have $\|u(t)\|_s \to \infty$ (or $\limsup \|u(t)\|_s = \infty$) as $t \to \infty$. Does this property hold for a typical quasilinear HPDE?

Concerning the last problem see [Bou00] and p. 188 in [EKMY02].
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