Single Scale Analysis of Many Fermion Systems

Part 1: Insulators

Joel Feldman*
Department of Mathematics
University of British Columbia
Vancouver, B.C.
CANADA V6T 1Z2
feldman@math.ubc.ca
http://www.math.ubc.ca/~feldman/

Horst Knörrer, Eugene Trubowitz
Mathematik
ETH-Zentrum
CH-8092 Zürich
SWITZERLAND
knoerrer@math.ethz.ch, trub@math.ethz.ch
http://www.math.ethz.ch/~knoerrer/

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I. Introduction

The standard model for a gas of weakly interacting electrons in a $d$-dimensional crystal is given in terms of

a single particle dispersion relation (shifted by the chemical potential) $e(k)$ on $\mathbb{R}^d$,

an ultraviolet cutoff $U(k)$ on $\mathbb{R}^d$,

the interaction $v$.

Here $k$ is the momentum variable dual to the position variable $x \in \mathbb{R}^d$. The Fermi surface associated to the dispersion relation $e(k)$ is by definition

$$\text{F} = \{ k \in \mathbb{R}^d \mid e(k) = 0 \}$$

The ultraviolet cutoff is a smooth function with compact support that fulfils $0 \leq U(k) \leq 1$ for all $k \in \mathbb{R}^d$. We assume that it is identically one on a neighbourhood of the Fermi surface\(^{(1)}\). The function $v(x)$ is rapidly decaying; it gives a spin independent translation invariant interaction $v(x_1 - x_2)$ between the Fermions.

This situation can be described in field theoretic terms as follows: There are anti-commuting fields $\psi_\sigma(x_0, x), \bar{\psi}_\sigma(x_0, x)$, where $x_0 \in \mathbb{R}$ is the temperature (or Euclidean time) argument and $\sigma \in \{\uparrow, \downarrow\}$ is the spin argument. For $x = (x_0, x, \sigma)$ we write $\psi(x) = \psi_\sigma(x_0, x)$ and $\bar{\psi}(x) = \bar{\psi}_\sigma(x_0, x)$. The covariance of the Grassmann Gaussian measure, $d\mu_C$, for these fields has Fourier transform

$$C(k_0, k) = \frac{U(k)}{ik_0 - e(k)}$$

Precisely, for $x = (x_0, x, \sigma), x' = (x'_0, x', \sigma') \in \mathbb{R} \times \mathbb{R}^2 \times \{\uparrow, \downarrow\}$

$$C(x, x') = \int \psi(x)\bar{\psi}(x')d\mu_C(\psi, \bar{\psi}) = \delta_{\sigma, \sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i<k, x - x'>} - C(k)$$

where $<k, x - x'>_- = -k_0(x_0 - x'_0) + k \cdot (x - x')$ for $k = (k_0, k) \in \mathbb{R} \times \mathbb{R}^d$. The interaction between the Fermions is described by an effective potential

$$\mathcal{V}(\psi, \bar{\psi}) = \int_{(\mathbb{R} \times \mathbb{R}^2 \times \{\uparrow, \downarrow\})^4} V_0(x_1, x_2, x_3, x_4) \bar{\psi}(x_1)\psi(x_2)\bar{\psi}(x_3)\psi(x_4) \, dx_1 dx_2 dx_3 dx_4$$

with the interaction kernel

$$V_0((x_{1,0}, x_1, \sigma_1), \ldots, (x_{4,0}, x_4, \sigma_4)) = -\frac{1}{2} \delta(x_1, x_2)\delta(x_3, x_4)\delta(x_{1,0} - x_{3,0})v(x_1 - x_3)$$

\(^{(1)}\) In particular, we assume that $F$ is compact.
where \[ \delta((x_0, \mathbf{x}, \sigma), (x'_0, \mathbf{x}', \sigma')) = \delta(x_0 - x'_0)\delta(\mathbf{x} - \mathbf{x}')\delta_{\sigma, \sigma'}. \] More generally we will discuss translation invariant and spin independent interaction kernels \[ V_0(x_1, x_2, x_3, x_4). \]

Formally, the generating functional for the connected amputated Green’s functions is

\[ \mathcal{G}_{\text{amp}}(\phi, \bar{\phi}) = \log \frac{1}{Z} \int e^{\mathcal{V}(\psi+\phi, \bar{\psi}+\bar{\phi})} d\mu_C(\psi, \bar{\psi}) \]

where \[ Z = \int e^{\mathcal{V}(\psi, \bar{\psi})} d\mu_C(\psi, \bar{\psi}). \] The connected amputated Green’s functions themselves are determined by

\[ \mathcal{G}_{\text{amp}}(\phi, \bar{\phi}) = \sum_{n=1}^{\infty} \frac{1}{(n!)^2} \int \prod_{i=1}^{n} dx_i dy_i \ G_{2n}^{\text{amp}}(x_1, y_1, \cdots, x_n, y_n) \prod_{i=1}^{n} \bar{\phi}(x_i) \phi(y_i) \]

The fields \( \phi, \bar{\phi} \) are called source fields, the fields \( \psi, \bar{\psi} \) internal fields.

In a renormalization group analysis, the covariance \( C \) is written as a sum

\[ C = \sum_{j=0}^{\infty} C^{(j)} \]

where \( C^{(j)}(k) \) is supported on the set \( k = (k_0, k) \in \mathbb{R} \times \mathbb{R}^d \) for which \( |C(k)| \) is approximately \( M^j \). Here \( M > 1 \) is a scale parameter. To successively integrate out scales \( j = 0, 1, \cdots, \) we use the renormalization group map analysed in [FKTr1]. For any covariance \( S \) we set

\[ \Omega_S(\mathcal{W})(\phi, \bar{\phi}, \psi, \bar{\psi}) = \log \frac{1}{Z} \int e^{\mathcal{W}(\phi, \bar{\phi}, \psi, \bar{\psi})} d\mu_S(\zeta, \bar{\zeta}) \]

Here, \( \mathcal{W} \) is a Grassmann function and the partition function is \( Z = \int e^{\mathcal{W}(0, 0, \zeta, \bar{\zeta})} d\mu_S(\zeta, \bar{\zeta}). \) \( \Omega_S \) maps Grassmann functions in the variables \( \phi, \bar{\phi}, \psi, \bar{\psi} \) to Grassmann functions in the same variables. Clearly

\[ \mathcal{G}_{\text{amp}}(\phi, \bar{\phi}) = \Omega_C(\mathcal{V})(0, 0, \phi, \bar{\phi}) \] (I.1)

where we view \( \mathcal{V}(\psi, \bar{\psi}) \) as a function of the four variables \( \phi, \bar{\phi}, \psi, \bar{\psi} \) that happens to be independent of \( \phi, \bar{\phi} \). Also, \( \Omega \) obeys the semigroup property

\[ \Omega_{S_1+S_2} = \Omega_{S_1} \circ \Omega_{S_2} \]

Even for a cutoff covariance \( S \), it is not a priori clear that \( \Omega_S(\mathcal{W}) \) makes sense for a reasonable set of \( \mathcal{W} \)’s. On the other hand, it is easy to see, using graphs, that each term in the formal Taylor expansion of \( \Omega_S(\mathcal{W}) \) in powers of \( \mathcal{W} \) is well–defined for a large class of \( \mathcal{W} \)’s and cutoff

\(^{(2)} \) with respect to the bare propagator \( C(x, y) \)
$S$'s. The Taylor expansion of $\int e^{W(\phi, \bar{\phi}, \psi, \bar{\psi} + \zeta, \bar{\zeta})} d\mu_S(\zeta, \bar{\zeta})$ is $\sum_{n=1}^{\infty} G_n(W, \cdots, W)$ where the $n^{th}$ term is the multilinear form

$$G_n(W_1, \cdots, W_n) = \frac{1}{n!} \int W_1(\phi, \bar{\phi}, \psi + \zeta, \bar{\psi} + \bar{\zeta}) \cdots W_n(\phi, \bar{\phi}, \psi + \zeta, \bar{\psi} + \bar{\zeta}) d\mu_S(\zeta, \bar{\zeta})$$

restricted to the diagonal. Explicit evaluation of the Grassmann integral expresses $G_n$ as the sum of all graphs with vertices $W_1, \cdots, W_n$ and lines $S$. The (formal) Taylor coefficient $\frac{d}{dt_1} \cdots \frac{d}{dt_n} \Omega_S(t_1 W_1 + \cdots + t_n W_n) \bigg|_{t_1=\cdots=t_n=0}$ of $\Omega_S(W)$ is similar, but with only connected graphs. We prove in [FKTr1] that, under hypotheses that will be satisfied here, the formal Taylor series for $\Omega_S(W)$ converges to an analytic\(^{(3)}\) function of $W$.

In part 3 of this paper, we analyse the maps $\Omega_{C^{(j)}}$ in great detail. In this first part, we apply the general results of [FKTr1] to many Fermion systems. In particular, we introduce concrete norms that fulfill the conditions of §II.4 of [FKTr1] and develop contraction and integral bounds for them. Then, we apply Theorem II.32 of [FKTr1] and (I.1) to models for which the dispersion relation is both infrared and ultraviolet finite (insulators). For these models, no scale decomposition is necessary.

In the second part of this paper, we introduce scales and apply the results of part 1 to integrate out the first few scales. It turns out that for higher scales the norms introduced in parts 1 and 2 are inadequate and, in particular, power count poorly. Using sectors, we introduce finer norms that, in dimension two, power count appropriately. For these sectorized norms, passing from one scale to the next is not completely trivial. This question is dealt with in part 4.

\(^{(3)}\) For an elementary discussion of analytic maps between Banach spaces see, for example, Appendix A of [PT].
II. Norms

Let \( A \) be the Grassmann algebra freely generated by the fields \( \phi(y), \tilde{\phi}(y) \) with \( y \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \). The generating functional for the connected Greens functions is a Grassmann Gaussian integral in the Grassmann algebra with coefficients in \( A \) that is generated by the fields \( \psi(x), \tilde{\psi}(x) \) with \( x \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \). We want to apply the results of [FKTr1] to this situation.

To simplify notation we define, for \( a = (x_0, x, \sigma, a) = (x, a) \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\} \), the internal fields

\[
\psi(\xi) = \begin{cases} 
\psi(x) & \text{if } a = 0 \\
\tilde{\psi}(x) & \text{if } a = 1
\end{cases}
\]

Similarly, we define for an external variable \( \eta = (y_0, y, \tau, b) = (y, b) \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\} \), the source fields

\[
\phi(\eta) = \begin{cases} 
\phi(y) & \text{if } b = 0 \\
\tilde{\phi}(y) & \text{if } b = 1
\end{cases}
\]

\( \mathcal{B} = \mathbb{R} \times \mathbb{R}^2 \times \{\uparrow, \downarrow\} \times \{0, 1\} \) is called the “base space” parameterizing the fields. The Grassmann algebra \( A \) is the direct sum of the vector spaces \( A_m \) generated by the products \( \phi(\eta_1) \cdots \phi(\eta_m) \). Let \( V \) be the vector space with basis \( \psi(\xi), \xi \in \mathcal{B} \). An antisymmetric function \( C(\xi, \xi') \) on \( \mathcal{B} \times \mathcal{B} \) defines a covariance on \( V \) by \( C(\psi(\xi), \psi(\xi')) = C(\xi, \xi') \). The Grassmann Gaussian integral with this covariance, \( \int d\mu_C(\psi) \), is a linear functional on the Grassmann algebra \( \wedge_A V \) with values in \( A \).

We shall define norms on \( \wedge_A V \) by specifying norms on the spaces of functions on \( \mathcal{B}^m \times \mathcal{B}^n \), \( m, n \geq 0 \). The rudiments of such norms and simple examples are discussed in this section. In the next section we recall the results of [FKTr1] in the current concrete situation.

The norms we construct are \((d+1)\)-dimensional seminorms in the sense of Definition II.15 of [FKTr1]. They measure the spatial decay of the functions, i.e. derivatives of their Fourier transforms.

**Definition II.1.** (Multiindices)

i) A multiindex is an element \( \delta = (\delta_0, \delta_1, \ldots, \delta_d) \in \mathbb{N}_0 \times \mathbb{N}_0^d \). The length of a multiindex \( \delta = (\delta_0, \delta_1, \ldots, \delta_d) \) is \( |\delta| = \delta_0 + \delta_1 + \cdots + \delta_d \) and its factorial is \( \delta! = \delta_0! \delta_1! \cdots \delta_d! \). For two multiindices \( \delta, \delta' \) we say that \( \delta \leq \delta' \) if \( \delta_i \leq \delta'_i \) for \( i = 0, 1, \ldots, d \). The spatial part of the multiindex \( \delta = (\delta_0, \delta_1, \ldots, \delta_d) \) is \( \delta = (\delta_1, \ldots, \delta_d) \in \mathbb{N}_0^d \). It has length \( |\delta| = \delta_1 + \cdots + \delta_d \).

ii) Let \( \delta, \delta^{(1)}, \ldots, \delta^{(r)} \) be multiindices such that \( \delta = \delta^{(1)} + \cdots + \delta^{(r)} \). Then by definition

\[
\left( \delta^{(1)}, \ldots, \delta^{(r)} \right) = \frac{\delta!}{\delta^{(1)}! \cdots \delta^{(r)}!}
\]
iii) For a multiindex $\delta$ and $x = (x_0, x, \sigma), x' = (x'_0, x', \sigma') \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}$ set

$$(x - x')^\delta = (x_0 - x'_0)^{\delta_0} (x_1 - x'_1)^{\delta_1} \cdots (x_d - x'_d)^{\delta_d}$$

If $\xi = (x, a), \xi' = (x', a') \in \mathcal{B}$ we define $(\xi - \xi')^\delta = (x - x')^\delta$.

iv) For a function $f(\xi_1, \ldots, \xi_n)$ on $\mathcal{B}^n$, a multiindex $\delta$, and $1 \leq i, j \leq n, i \neq j$ set

$$D_{i,j}^\delta f(\xi_1, \ldots, \xi_n) = (\xi_i - \xi_j)^\delta f(\xi_1, \ldots, \xi_n)$$

**Lemma II.2 (Leibniz’s rule)** Let $f(\xi_1, \ldots, \xi_n)$ be a function on $\mathcal{B}^n$ and $f'(\xi_1, \ldots, \xi_m)$ a function on $\mathcal{B}^m$. Set

$$g(\xi_1, \ldots, \xi_{n+m-2}) = \int_{\mathcal{B}} d\eta f(\xi_1, \ldots, \xi_{n-1}, \eta) f'(\eta, \xi_n, \ldots, \xi_{n+m-2})$$

Let $\delta$ be a multiindex and $1 \leq i \leq n-1, n \leq j \leq n+m-2$. Then

$$D_{i,j}^\delta g(\xi_1, \ldots, \xi_{n+m-2}) = \sum_{\delta' \leq \delta} \left(\begin{array}{c} \delta \\ \delta' \end{array}\right) \int_{\mathcal{B}} d\eta D_{i,n}^\delta f(\xi_1, \ldots, \xi_{n-1}, \eta) D_{1,j-n+2}^{\delta' - \delta} f'(\eta, \xi_n, \ldots, \xi_{n+m-2})$$

**Proof:** For each $\eta \in \mathcal{B}$

$$(\xi_i - \xi_j)^\delta = ((\xi_i - \eta) + (\eta - \xi_j))^\delta = \sum_{\delta' \leq \delta} \left(\begin{array}{c} \delta \\ \delta' \end{array}\right) (\xi_i - \eta)^{\delta'} (\eta - \xi_j)^{\delta - \delta'}$$

\[ \square \]

**Definition II.3 (Decay operators)** Let $n$ be a positive integer. A decay operator $\mathcal{D}$ on the set of functions on $\mathcal{B}^n$ is an operator of the form

$$\mathcal{D} = \mathcal{D}_{\delta^{(1)}}^{\delta_{u_1,v_1}} \cdots \mathcal{D}_{\delta^{(k)}}^{\delta_{u_k,v_k}}$$

with multiindices $\delta^{(1)}, \ldots, \delta^{(k)}$ and $1 \leq u_j, v_j \leq n, u_j \neq v_j$. The indices $u_j, v_j$ are called variable indices. The total order of derivatives in $\mathcal{D}$ is

$$\delta(\mathcal{D}) = \delta^{(1)} + \cdots + \delta^{(k)}$$

In a similar way, we define the action of a decay operator on the set of functions on $(\mathbb{R} \times \mathbb{R}^d)^n$ or on $(\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\})^n$. 

5
Definition II.4

i) On $\mathbb{R}_+ \cup \{\infty\} = \{ x \in \mathbb{R} \mid x \geq 0 \} \cup \{+\infty\}$, addition and the total ordering $\leq$ are defined in the standard way. With the convention that $0 \cdot \infty = \infty$, multiplication is also defined in the standard way.

ii) Let $d \geq -1$. For $d \geq 0$, the $(d+1)$-dimensional norm domain $\mathfrak{N}_{d+1}$ is the set of all formal power series

$$X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} X_\delta t_0^{\delta_0} t_1^{\delta_1} \cdots t_d^{\delta_d}$$

in the variables $t_0, t_1, \ldots, t_d$ with coefficients $X_\delta \in \mathbb{R}_+ \cup \{\infty\}$. To shorten notation, we set $t^\delta = t_0^{\delta_0} t_1^{\delta_1} \cdots t_d^{\delta_d}$. Addition and partial ordering on $\mathfrak{N}_{d+1}$ are defined componentwise. Multiplication is defined by

$$(X \cdot X')_\delta = \sum_{\beta + \gamma = \delta} X_\beta X'_\gamma$$

The max and min of two elements of $\mathfrak{N}_{d+1}$ are again defined componentwise.

The zero-dimensional norm domain $\mathfrak{N}_0$ is defined to be $\mathbb{R}_+ \cup \{\infty\}$. We also identify $\mathbb{R}_+ \cup \{\infty\}$ with the set of all $X \in \mathfrak{N}_{d+1}$ with $X_\delta = 0$ for all $\delta \neq 0 = (0, \cdots, 0)$.

If $a > 0$, $X_0 \neq \infty$ and $a - X_0 > 0$ then $(a - X)^{-1}$ is defined as

$$(a - X)^{-1} = \frac{1}{a - X_0} \sum_{n=0}^{\infty} \left( \frac{X - X_0}{a - X_0} \right)^n$$

For an element $X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} X_\delta t_0^{\delta_0} t_1^{\delta_1} \cdots t_d^{\delta_d}$ of $\mathfrak{N}_{d+1}$ and $0 \leq j \leq d$ the formal derivative $\frac{\partial}{\partial t_j} X$ is defined as

$$\frac{\partial}{\partial t_j} X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} (\delta_j + 1) X_{\delta + \epsilon_j} t^{\delta}$$

where $\epsilon_j$ is the $j$th unit vector.

Definition II.5 Let $E$ be a complex vector space. A $(d+1)$-dimensional seminorm on $E$ is a map $\| \cdot \| : E \to \mathfrak{N}_{d+1}$ such that

$$\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|, \quad \|\lambda e\| = |\lambda| \|e\|$$

for all $e, e_1, e_2 \in E$ and $\lambda \in \mathbb{C}$.

Example II.6 For a function $f$ on $\mathcal{B}^m \times \mathcal{B}^n$ we define the (scalar valued) $L_1-L_\infty$-norm as

$$\|f\|_{1, \infty} = \begin{cases} \max_{1 \leq j_0 \leq n} \sup_{\xi_{j_0} \in \mathcal{B}} \int \prod_{\substack{j=1 \neq j_0 \to n}} d\xi_j |f(\xi_1, \cdots, \xi_n)| & \text{if } m = 0 \\ \sup_{\eta_1, \cdots, \eta_m \in \mathcal{B}} \int \prod_{j=1, \cdots, n} d\xi_j |f(\eta_1, \cdots, \eta_m; \xi_1, \cdots, \xi_n)| & \text{if } m \neq 0 \end{cases}$$
and the \((d+1)\)-dimensional \(L_1-L_\infty\) seminorm

\[
\|f\|_{1,\infty} = \begin{cases} 
\sum_{\delta \in \mathbb{N}_0^n} \frac{1}{\delta!} \left( \max_{\text{decay operator with } \delta(D)=\delta} \|D\,f\|_{1,\infty} \right) e_{\delta_0}^1 \cdots e_{\delta_d}^d & \text{if } m = 0 \\
\|f\|_{1,\infty} & \text{if } m \neq 0
\end{cases}
\]

Here \(\|f\|_{1,\infty}\) stands for the formal power series with constant coefficient \(\|f\|_{1,\infty}\) and all other coefficients zero.

**Lemma II.7** Let \(f\) be a function on \(\mathcal{B}^m \times \mathcal{B}^n\) and \(f'\) a function on \(\mathcal{B}^{m'} \times \mathcal{B}^{n'}\). Let \(1 \leq \mu \leq n, \ 1 \leq \nu \leq n'\). Define the function \(g\) on \(\mathcal{B}^{m+m'} \times \mathcal{B}^{n+n'-2}\) by

\[
g(\eta_1, \ldots, \eta_{m+m'}; \xi_1, \ldots, \xi_{\mu-1}, \xi_{\mu+1}, \ldots, \xi_n, \ldots, \xi_{n+\nu-1}, \xi_{n+\nu+1}, \ldots, \xi_{n+n'})
= \int d\xi f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_{\mu-1}, \xi_{\mu+1}, \ldots, \xi_n) f'(\eta_{m+1}, \ldots, \eta_{m+m'}; \xi_{n+1}, \ldots, \xi_{n+\nu-1}, \xi_{n+\nu+1}, \ldots, \xi_{n+n'})
\]

If \(m = 0\) or \(m' = 0\)

\[
\|g\|_{1,\infty} \leq \|f\|_{1,\infty} \|f'\|_{1,\infty}
\]

\[
\|g\|_{1,\infty} \leq \|f\|_{1,\infty} \|f'\|_{1,\infty}
\]

**Proof:** We first consider the norm \(\|\cdot\|_{1,\infty}\). In the case \(m \neq 0, m' = 0\), for all \(\eta_1, \ldots, \eta_m \in \mathcal{B}\)

\[
\left| \int \prod_{j=1}^{n+n'} d\xi_j \; g(\eta_1, \ldots, \eta_{m+n}; \xi_1, \ldots, \xi_{\mu-1}, \xi_{\mu+1}, \ldots, \xi_n, \ldots, \xi_{n+\nu-1}, \xi_{n+\nu+1}, \ldots, \xi_{n+n'}) \right|
\]

\[
\leq \left| \int d\xi_1 \cdots d\xi_n \; f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n) \left( \sup_{\xi \in \mathcal{B}} \left| \int \prod_{j=1}^{n'} d\xi_j' \; f'(\xi_1, \ldots, \xi_{\nu-1}, \xi_{\nu+1}, \ldots, \xi_{n'}) \right| \right) \right|
\]

\[
\leq \|f\|_{1,\infty} \|f'\|_{1,\infty}
\]

The case \(m = 0, m' \neq 0\) is similar. In the case \(m = m' = 0\) first fix \(j_0 \in \{1, \ldots, n\} \setminus \{\mu\}\), and fix \(\xi_{j_0} \in \mathcal{B}\). As in the case \(m \neq 0, m' = 0\) one shows that

\[
\left| \int \prod_{j=1}^{n+n'} d\xi_j \; g(\xi_1, \ldots, \xi_{\mu-1}, \xi_{\mu+1}, \ldots, \xi_n, \ldots, \xi_{n+\nu-1}, \xi_{n+\nu+1}, \ldots, \xi_{n+n'}) \right|
\]

\[
\leq \left| \int \prod_{j=1}^{n} d\xi_j \; f(\xi_1, \ldots, \xi_n) \left( \sup_{\xi \in \mathcal{B}} \left| \int \prod_{j=1}^{n'} d\xi_j' \; f'(\xi_1, \ldots, \xi_{\nu-1}, \xi_{\nu+1}, \ldots, \xi_{n'}) \right| \right) \right|
\]

\[
\leq \|f\|_{1,\infty} \|f'\|_{1,\infty}
\]

If one fixes one of the variables \(\xi_{j_0}\) with \(j_0 \in \{n+1, \ldots n+n'\} \setminus \{n+\nu\}\), the argument is similar.
We now consider the norm $\| \cdot \|_{1,\infty}$. If $m \neq 0$ or $m' \neq 0$ this follows from the first part of this Lemma and

$$
\|g\|_{1,\infty} = \|g\|_{1,\infty} \leq \|f\|_{1,\infty} \|f'\|_{1,\infty} \leq \|f\|_{1,\infty} \|f'\|_{1,\infty}
$$

So assume that $m = m' = 0$. Set

$$
\|f\|_{1,\infty} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} \frac{1}{\delta!} X_\delta t^\delta \\
\|f'\|_{1,\infty} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} \frac{1}{\delta!} X'_\delta t^\delta
$$

with $X_\delta, X'_\delta, Y_\delta \in \mathbb{R}_+ \cup \{\infty\}$. Let $\mathcal{D}$ be a decay operator of degree $\delta$ acting on $g$. The variable indices for $g$ lie in the set $I \cup I'$, where

$$
I = \{1, \cdots , m - 1, m + 1, \cdots , n\} \\
I' = \{n + 1, \cdots , n + \nu - 1, n + \nu + 1, \cdots , n + n'\}
$$

We can factor the decay operator $\mathcal{D}$ in the form

$$
\mathcal{D} = \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3
$$

where all variable indices of $\mathcal{D}_1$ lie in $I$, all variable indices of $\mathcal{D}_2$ lie in $I'$, and

$$
\mathcal{D}_3 = \mathcal{D}^{\delta(1)}_{u_1,v_1} \cdots \mathcal{D}^{\delta(k)}_{u_k,v_k}
$$

with $u_1, \cdots , u_k \in I$, $v_1, \cdots , v_k \in I'$. Set $h = \mathcal{D}_1 f$ and $h' = \mathcal{D}_2 f'$. By Leibniz’s rule

$$
\mathcal{D} g = \pm \mathcal{D} \int_B d\zeta h(\xi_1, \cdots , \xi_{m-1}, \zeta, \xi_{m+1}, \cdots , \xi_n) h'(\xi_{n+1}, \cdots , \xi_{n+\nu-1}, \zeta, \xi_{n+\nu+1}, \cdots , \xi_{n+n'})
$$

$$
= \pm \sum_{\alpha^{(i)}_1 + \beta^{(i)}_1 = \delta^{(i)}} \left( \prod_{i=1}^k (\alpha^{(i)}_i, \beta^{(i)}_i) \right) \int d\zeta \left( \prod_{i=1}^k \mathcal{D}^{\alpha^{(i)}_i}_{u_i,v_i} h(\xi_1, \cdots , \xi_{m-1}, \zeta, \xi_{m+1}, \cdots , \xi_n) \right) \\
\left( \prod_{i=1}^k \mathcal{D}^{\beta^{(i)}_i}_{\nu_i,v_i} h'(\xi_{n+1}, \cdots , \xi_{n+\nu-1}, \zeta, \xi_{n+\nu+1}, \cdots , \xi_{n+n'}) \right)
$$

By the first part of this Lemma, the $L_1-L_\infty$-norm of each integral on the right hand side is bounded by

$$
\left\| \prod_{i=1}^k \mathcal{D}^{\alpha^{(i)}_i}_{u_i,v_i} h \right\|_{1,\infty} \left\| \prod_{i=1}^k \mathcal{D}^{\beta^{(i)}_i}_{\nu_i,v_i} h' \right\|_{1,\infty}
$$
Therefore, setting \( \delta = \delta(\bar{D}) = \delta^{(1)} + \cdots + \delta^{(k)} \),

\[
\|Dg\|_{1,\infty} t^\delta \leq \sum_{\alpha^{(i)} + \beta^{(i)} = \delta^{(i)}} \left( \prod_{i=1}^k \left( \delta^{(i)} \right) \right) t^{\delta(D_1)} X_{\delta(D_1)} + \alpha^\prime t^{\delta(D_2)} + \beta^\prime \|D_{\alpha^{(i)}, \beta^{(i)}}(D_1 f)\|_{1,\infty}
\]

\[
\leq \sum_{\alpha + \beta = \delta} \sum_{\alpha^{(i)} + \beta^{(i)} = \delta^{(i)}} \left( \prod_{i=1}^k \left( \delta^{(i)} \right) \right) X_{\delta(D_1)} + \alpha^\prime t^{\delta(D_2)} + \beta^\prime \|D_{\alpha^{(i)}, \beta^{(i)}}(D_2 f)\|_{1,\infty}
\]

\[
= \sum_{\alpha + \beta = \delta} \left( \delta \right) X_{\delta(D_1)} + \alpha^\prime t^{\delta(D_2)} + \beta^\prime \|D_{\alpha^{(i)}, \beta^{(i)}}(D_2 f)\|_{1,\infty}
\]

\[
\leq \sum_{\alpha + \beta = \delta} \left( \delta \right) X_{\delta(D_1)} + \alpha^\prime t^{\delta(D_2)} + \beta^\prime \|D_{\alpha^{(i)}, \beta^{(i)}}(D_2 f)\|_{1,\infty} \tag{II.1}
\]

In the equality, we used the fact that for each pair of multiindices \( \alpha, \beta \) with \( \alpha + \beta = \delta \) and each \( k \)-tuple of multiindices \( \delta^{(i)} \), \( 1 \leq i \leq k \), with \( \sum_i \delta^{(i)} = \delta \)

\[
\sum_{\alpha + \beta = \delta} \prod_{i=1}^k \left( \delta^{(i)} \right) = \left( \delta \right)
\]

This standard combinatorial identity follows from

\[
\sum_{\alpha + \beta = \delta} \left( \delta \right) x^\alpha y^\beta = (x + y)^\delta = \sum_{i=1}^k (x + y)^{\delta^{(i)}} = \sum_{i=1}^k \left( \delta^{(i)} \right) x^\alpha y^\beta
\]

\[
= \sum_{\alpha + \beta = \delta} \prod_{i=1}^k \left( \delta^{(i)} \right) x^{\alpha^{(1)} + \cdots + \alpha^{(k)}} y^{\beta^{(1)} + \cdots + \beta^{(k)}}
\]

by matching the coefficients of \( x^\alpha y^\beta \).

It follows from (II.1) that

\[
\frac{1}{\delta!} Y_\delta t^\delta \leq \sum_{\alpha' + \beta' = \delta} \frac{1}{\alpha'!} X_{\alpha'} t^{\alpha'} \frac{1}{\beta'!} X_{\beta'} t^{\beta'}
\]

and

\[
\|g\|_{1,\infty} \leq \sum_{\delta} \sum_{\alpha' + \beta' = \delta} \frac{1}{\alpha'!} X_{\alpha'} t^{\alpha'} \frac{1}{\beta'!} X_{\beta'} t^{\beta'} = \|f\|_{1,\infty} \|f'\|_{1,\infty}
\]
Corollary II.8 Let \( f \) be a function on \( B^n \), \( f' \) a function on \( B^{n'} \) and \( C_2, C_3 \) functions on \( B^2 \). Set
\[
h(\xi_1, \ldots, \xi_n, \xi'_1, \ldots, \xi'_{n'}) = \int d\zeta_2 d\eta_4 d\zeta_3 d\eta_4 f(\zeta, \xi_1, \xi_2, \xi_3, \ldots, \xi_n) C_2(\xi, \xi'_2) C_3(\eta, \eta'_3) f'(\xi, \xi'_2, \xi'_3, \ldots, \xi'_n)
\]
Then
\[
\|h\|_{1, \infty} \leq \sup_{\xi, \xi'} |C_2(\xi, \xi')| \sup_{\eta, \eta'} |C_3(\eta, \eta')| \|f\|_{1, \infty} \|f'\|_{1, \infty}
\]

Proof: Set
\[
g(\xi_1, \ldots, \xi_n, \xi'_1, \ldots, \xi'_{n'}) = \int d\zeta f(\zeta, \xi_1, \xi_2, \xi_3, \ldots, \xi_n) f'(\xi, \xi'_2, \xi'_3, \ldots, \xi'_n)
\]
Let \( D \) be a decay operator acting on \( h \). Then
\[
Dh = \int d\zeta_2 d\eta_4 d\zeta_3 d\eta_4 C_2(\xi, \xi'_2) C_3(\eta, \eta'_3) Dg(\xi, \ldots, \xi_n, \xi'_2, \ldots, \xi'_{n'})
\]
Consequently
\[
\|Dh\|_{1, \infty} \leq \sup |C_2| \sup |C_3| \|Dg\|_{1, \infty}
\]
and therefore
\[
\|h\|_{1, \infty} \leq \sup |C_2| \sup |C_3| \|g\|_{1, \infty}
\]
The Corollary now follows from Lemma II.7.

Definition II.9 Let \( \mathcal{F}_m(n) \) be the space of all functions \( f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n) \) on \( B^m \times B^n \) that are antisymmetric in the \( \eta \) variables. If \( f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n) \) is any function on \( B^m \times B^n \), its antisymmetrization in the external variables is
\[
\text{Ant}_{\text{ext}} f (\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n) = \frac{1}{m!} \sum_{\pi \in S_m} \text{sgn} (\pi) f(\eta_{\pi(1)}, \ldots, \eta_{\pi(m)}; \xi_1, \ldots, \xi_n)
\]
For \( m, n \geq 0 \), the symmetric group \( S_n \) acts on \( \mathcal{F}_m(n) \) from the right by
\[
f^\pi (\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n) = f(\eta_1, \ldots, \eta_m; \xi_{\pi(1)}, \ldots, \xi_{\pi(n)}) \quad \text{for } \pi \in S_n
\]

Definition II.10 A seminorm \( \| \cdot \| \) on \( \mathcal{F}_m(n) \) is called symmetric, if for every \( f \in \mathcal{F}_m(n) \) and \( \pi \in S_n \)
\[
\|f^\pi\| = \|f\|
\]
and \( \|f\| = 0 \) if \( m = n = 0 \).

For example, the seminorms \( \| \cdot \|_{1, \infty} \) of Example II.6 are symmetric.
III. Covariances and the Renormalization Group Map

Definition III.1 (Contraction) Let $C(\xi, \xi')$ be any skew symmetric function on $\mathcal{B} \times \mathcal{B}$. Let $m, n \geq 0$ and $1 \leq i < j \leq n$. For $f \in \mathcal{F}_m(n)$ the contraction $\operatorname{Con}_{i \rightarrow j} f \in \mathcal{F}_m(n-2)$ is defined as

$$\operatorname{Con}_{i \rightarrow j} f (\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_n) = (-1)^{j-i+1} \int d\zeta d\zeta' C(\zeta, \zeta') f (\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_{i-1}, \zeta, \xi_{i+1}, \ldots, \xi_{j-1}, \zeta', \xi_{j+1}, \ldots, \xi_n)$$

Definition III.2 (Contraction Bound) Let $\| \cdot \|$ be a family of symmetric seminorms on the spaces $\mathcal{F}_m(n)$. We say that $c \in \mathfrak{M}_{d+1}$ is a contraction bound for the covariance $C$ with respect to this family of seminorms, if for all $m, n, m', n' \geq 0$ there exist $i$ and $j$ with $1 \leq i \leq n$, $1 \leq j \leq n'$ such that

$$\| \operatorname{Con}_{i \rightarrow n+j} (\operatorname{Ant}_{\text{ext}} (f \otimes f')) \| \leq c \| f \| \| f' \|$$

for all $f \in \mathcal{F}_m(n)$, $f' \in \mathcal{F}_{m'}(n')$. Observe that $f \otimes f'$ is a function on $(\mathcal{B}^m \times \mathcal{B}^n) \times (\mathcal{B}^{m'} \times \mathcal{B}^{n'}) \cong \mathcal{B}^{m+m'} \times \mathcal{B}^{n+n'}$, so that $\operatorname{Ant}_{\text{ext}} (f \otimes f') \in \mathcal{F}_{m+m'}(n+n')$.

Remark III.3 If $c$ is a contraction bound for the covariance $C$ with respect to a family of symmetric seminorms, then, by symmetry,

$$\| \operatorname{Con}_{i \rightarrow n+j} (\operatorname{Ant}_{\text{ext}} (f \otimes f')) \| \leq c \| f \| \| f' \|$$

for all $1 \leq i \leq n$, $1 \leq j \leq n'$ and all $f \in \mathcal{F}_m(n)$, $f' \in \mathcal{F}_{m'}(n')$.

Example III.4 The $L_1-L_\infty$--norm introduced in Example II.6 has $\max\{\| C \|_{1, \infty}, \| C \|_\infty\}$ as a contraction bound for covariance $C$. Here, $\| C \|_\infty$ is the element of $\mathfrak{M}_{d+1}$ whose constant term is $\sup_{\xi, \xi'} \| C(\xi, \xi') \|$ and is the only nonzero term. This is easily proven by iterated application of Lemma II.7. See also Example II.25 in [FKTr1]. A more general statement will be formulated and proven in Lemma V.1.iii.

Definition III.5 (Integral Bound) Let $\| \cdot \|$ be a family of symmetric seminorms on the spaces $\mathcal{F}_m(n)$. We say that $b \in \mathbb{R}_+$ is an integral bound for the covariance $C$ with respect to this family of seminorms, if the following holds:
Let \( m \geq 0, 1 \leq n' \leq n \). For \( f \in \mathcal{F}_m(n) \) define \( f' \in \mathcal{F}_m(n-n') \) by

\[
f'(\eta_1, \ldots, \eta_{n'}; \xi_{n'+1}, \ldots, \xi_n) = \int_{\mathcal{B}^{n'}} \frac{d\xi_1 \cdots d\xi_n}{n!} f(\eta_1, \ldots, \eta_{n'}; \xi_1, \ldots, \xi_n) \psi(\xi_1) \cdots \psi(\xi_n) \, d\mu_C(\psi)
\]

Then

\[
\|f'\| \leq (b/2)^{n'} \|f\|
\]

**Remark III.6** Suppose that there is a constant \( S \) such that

\[
\left| \int \psi(\xi_1) \cdots \psi(\xi_n) \, d\mu_C(\psi) \right| \leq S^n
\]

for all \( \xi_1, \ldots, \xi_n \in \mathcal{B} \). Then \( 2S \) is an integral bound for \( C \) with respect to the \( L_1-L_\infty \)-norm introduced in Example II.6.

**Definition III.7**

i) We define \( A_m[n] \) as the subspace of the Grassmann algebra \( \bigwedge_A V \) that consists of all elements of the form

\[
\text{Gr}(f) = \int d\eta_1 \cdots d\eta_m \, d\xi_1 \cdots d\xi_n \, f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n) \phi(\eta_1) \cdots \phi(\eta_m) \psi(\xi_1) \cdots \psi(\xi_n)
\]

with a function \( f \) on \( \mathcal{B}^m \times \mathcal{B}^n \).

ii) Every element of \( A_m[n] \) has a unique representation of the form \( \text{Gr}(f) \) with a function \( f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n) \in \mathcal{F}_m(n) \) that is antisymmetric in its \( \xi \) variables. Therefore a seminorm \( \| \cdot \| \) on \( \mathcal{F}_m(n) \) defines a canonical seminorm on \( A_m[n] \), which we denote by the same symbol \( \| \cdot \| \).

**Remark III.8** For \( F \in A_m[n] \)

\[
\|F\| \leq \|f\| \quad \text{for all } f \in \mathcal{F}_m(n) \text{ with } \text{Gr}(f) = F
\]

**Proof:** Let \( f \in \mathcal{F}_m(n) \). Then \( f' = \frac{1}{n!} \sum_{\pi \in S_n} \text{sgn}(\pi) f^\pi \) is the unique element of \( \mathcal{F}_m(n) \) that is antisymmetric in its \( \xi \) variables such that \( \text{Gr}(f') = \text{Gr}(f) \). Therefore

\[
\|\text{Gr}(f)\| = \|f'\| \leq \frac{1}{n!} \sum_{\pi \in S_n} \|f^\pi\| = \frac{1}{n!} \sum_{\pi \in S_n} \|f\| = \|f\|
\]

since the seminorm is symmetric.
Definition III.9 Let $\| \cdot \|$ be a family of symmetric seminorms, and let $\mathcal{W}(\phi, \psi)$ be a Grassmann function. Write
\[ \mathcal{W} = \sum_{m,n \geq 0} \mathcal{W}_{m,n} \]
with $\mathcal{W}_{m,n} \in A_m[n]$. For any constants $c \in \mathfrak{m}_{d+1}$, $b > 0$ and $\alpha \geq 1$ set
\[ N(\mathcal{W}; c, b, \alpha) = \frac{1}{\alpha} c \sum_{m,n \geq 0} \alpha^n b^n \| \mathcal{W}_{m,n} \| \]

In practice, the quantities $b, c$ will reflect the “power counting” of $\mathcal{W}$ with respect to the covariance $C$ and the number $\alpha$ is proportional to an inverse power of the largest allowed modulus of the coupling constant.

In this paper, we will derive bounds on the renormalization group map for several kinds of seminorms. The main ingredients from [FKTr1] are

Theorem III.10 Let $\| \cdot \|$ be a family of symmetric seminorms and let $C$ be a covariance on $V$ with contraction bound $c$ and integral bound $b$. Then the formal Taylor series $\Omega_C(\mathcal{W})$ converges to an analytic map on $\{ \mathcal{W} \mid \mathcal{W}$ even, $N(\mathcal{W}; c, b, 8\alpha)_0 < \frac{\alpha^2}{4} \}$. Furthermore, if $\mathcal{W}(\phi, \psi)$ is an even Grassmann function such that
\[ N(\mathcal{W}; c, b, 8\alpha)_0 < \frac{\alpha^2}{4} \]
then
\[ N(\Omega_C(\mathcal{W}) - \mathcal{W}; c, b, \alpha) \leq \frac{2}{\alpha^2} \frac{N(\mathcal{W}; c, b, 8\alpha)^2}{1 - \frac{\alpha}{\alpha} N(\mathcal{W}; c, b, 8\alpha)} \]
Here, $::$ denotes Wick ordering with respect to the covariance $C$.

In §V we will use this Theorem to discuss the situation of an insulator. More generally we have

Theorem III.11 Let, for $\kappa$ in a neighbourhood of 0, $\mathcal{W}_\kappa(\phi, \psi)$ be an even Grassmann function and $C_\kappa, D_\kappa$ be antisymmetric functions on $\mathcal{B} \times \mathcal{B}$. Assume that $\alpha \geq 1$ and
\[ N(\mathcal{W}_0; c, b, 32\alpha)_0 < \alpha^2 \]
and that
\begin{align*}
C_0 & \text{ has contraction bound } c \\
\frac{d}{d\kappa} C_\kappa |_{\kappa=0} & \text{ has contraction bound } c' \\
\frac{1}{2} b & \text{ is an integral bound for } C_0, D_0 \\
\frac{1}{2} b' & \text{ is an integral bound for } \frac{d}{d\kappa} D_\kappa |_{\kappa=0}
\end{align*}
and that $\epsilon \leq \frac{1}{\mu} \epsilon^2$. Set

$$\tilde{W}_\kappa(\phi, \psi); \psi, D_\kappa = \Omega_{C_\kappa}(; W_\kappa; \psi, C_\kappa + D_\kappa)$$

Then

$$N\left(\frac{d}{dx} [\tilde{W}_\kappa - W_\kappa]; \kappa = 0 ; \epsilon, b, \alpha\right) \leq \frac{1}{2\alpha^2} \frac{N(W_0; \epsilon, b, 32\alpha)}{N(W_0; \epsilon, b, 32\alpha)} N\left(\frac{d}{dx} W_\kappa; \kappa = 0 ; \epsilon, b, 8\alpha\right)$$

$$+ \frac{1}{2\alpha^2} \frac{N(W_0; \epsilon, b, 32\alpha)^2}{N(W_0; \epsilon, b, 32\alpha)} \left\{ \frac{1}{4\alpha} \epsilon' + \left(\frac{\epsilon'}{b}\right)^2 \right\}$$

Proof of Theorems III.10 and III.11:

If $f(\eta_1, \cdots, \eta_m; \xi_1, \cdots, \xi_n)$ is a function on $\mathcal{B}^m \times \mathcal{B}^n$ we define the corresponding element of $A_m \otimes V^\otimes n$ as

$$\text{Tens}(f) = \int \prod_{i=1}^m d\eta_i \prod_{j=1}^n d\xi_j f(\eta_1, \cdots, \eta_m; \xi_1, \cdots, \xi_n) \phi(\eta_1) \cdots \phi(\eta_m) \psi(\xi_1) \otimes \cdots \otimes \psi(\xi_n)$$

Each element of $A_m \otimes V^\otimes n$ can be uniquely written in the form $\text{Tens}(f)$ with a function $f \in \mathcal{F}_m(n)$. Therefore a seminorm on $\mathcal{F}_m(n)$ defines a seminorm on $A_m \otimes V^\otimes n$ and conversely. Under this correspondence, symmetric seminorms on $\mathcal{F}_m(n)$ in the sense of Definition II.10 correspond to symmetric seminorms on $A_m \otimes V^\otimes n$ in the sense of Definition II.18 of [FKTr1], contraction bounds as in Definition III.2 correspond, by Remark III.3, to contraction bounds as in Definition II.24.i of [FKTr1] and integral bounds as in Definition III.5 correspond to integral bounds as in Definition II.24.ii of [FKTr1]. Furthermore the norms on the spaces $A_m[n]$ defined in Definition III.7.ii agrees with those of Lemma II.21 of [FKTr1]. Therefore Theorem III.10 follows directly from Theorem II.32 of [FKTr1] and Theorem III.11 follows from Theorem IV.4 of [FKTr1].
IV. Bounds for Covariances

Integral Bounds

Definition IV.1  For any covariance $C = C(\xi, \xi')$ we define

$$S(C) = \sup_{m} \sup_{\xi_1, \ldots, \xi_m \in B} \left( \left| \int \psi(\xi_1) \cdots \psi(\xi_m) \, d\mu_C(\psi) \right| \right)^{1/m}$$

Remark IV.2

i) By Remark III.6, $2S(C)$ is an integral bound for $C$ with respect to the $L_1-L_\infty$-norms introduced in Example II.6.

ii) For any two covariances $C, C'$

$$S(C + C') \leq S(C) + S(C')$$

Proof of (ii): For $\xi_1, \ldots, \xi_m \in B$

$$\int \psi(\xi_1) \cdots \psi(\xi_m) \, d\mu_{(C+C')}(\psi) = \int (\psi(\xi_1) + \psi'(\xi_1)) \cdots (\psi(\xi_m) + \psi'(\xi_m)) \, d\mu_C(\psi) \, d\mu_{C'}(\psi')$$

Multiplying out one sees that

$$\left( \psi(\xi_1) + \psi'(\xi_1) \right) \cdots \left( \psi(\xi_m) + \psi'(\xi_m) \right) = \sum_{p=0}^{m} \sum_{I \subseteq \{1, \ldots, m\}} \mathcal{M}(p, I)$$

with

$$\mathcal{M}(p, I) = \pm \prod_{i \in I} \psi(\xi_i) \prod_{j \notin I} \psi'(\xi_j)$$

Therefore

$$\left| \int \psi(\xi_1) \cdots \psi(\xi_m) \, d\mu_{(C+C')}(\psi) \right| \leq \sum_{p=0}^{m} \sum_{I \subseteq \{1, \ldots, m\}} \left| \int \mathcal{M}(p, I) \, d\mu_C(\psi) \, d\mu_{C'}(\psi') \right|$$

$$\leq \sum_{p=0}^{m} \sum_{I \subseteq \{1, \ldots, m\}} S(C)^p \, S(C')^{m-p}$$

$$= (S(C) + S(C'))^m$$

\[\blacksquare\]
In this Section, we assume that there is a function \( C(k) \) such that for \( \xi = (x, a) = (x_0, x', \sigma, a') \in \mathcal{B} \)

\[
C(\xi, \xi') = \begin{cases} 
\delta_{\sigma,a'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i<k, x-x'>} C(k) & \text{if } a = 0, a' = 1 \\
-\delta_{\sigma,a'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i<k, x-x'>} C(k) & \text{if } a = 1, a' = 0 \\
0 & \text{if } a = a'
\end{cases} 
\tag{IV.1}
\]

(as usual, the case \( x_0 = x'_0 = 0 \) is defined through the limit \( x_0 - x'_0 \to 0^- \)) and derive bounds for \( S(C) \) in terms of norms of \( C(k) \).

**Proposition IV.3 (Gram’s estimate)**

\[ i) \quad S(C) \leq \sqrt{\int \frac{d^{d+1}k}{(2\pi)^{d+1}} |C(k)|} \]

\[ ii) \text{ Let, for each } s \text{ in a finite set } \Sigma, \chi_s(k) \text{ be a function on } \mathbb{R} \times \mathbb{R}^d. \text{ Set, for } a \in \{0,1\}, \]

\[ \hat{\chi}_s(x-x',a) = \int e^{(-1)^a i<k, x-x'>} \chi_s(k) \frac{d^{d+1}k}{(2\pi)^{d+1}} \]

and

\[ \psi_s(x,a) = \int d^{d+1}x' \hat{\chi}_s(x-x',a) \psi(x',a) \]

Then for all \( \xi_1, \ldots, \xi_m \in \mathcal{B} \) and all \( s_1, \ldots, s_m \in \Sigma \)

\[ \left| \int \psi_{s_1}(\xi_1) \cdots \psi_{s_m}(\xi_m) \text{d}\mu_C(\psi) \right| \leq \left[ \max_{s \in \Sigma} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |C(k)\chi_s(k)|^2 \right]^{m/2} \]

**Proof:** Let \( \mathcal{H} \) be the Hilbert space \( \mathcal{H} = L^2(\mathbb{R} \times \mathbb{R}^d) \otimes \mathcal{F}^2 \). For \( \sigma \in \{\uparrow, \downarrow\} \) define the element \( \omega(\sigma) \in \mathcal{F}^2 \) by

\[ \omega(\sigma) = \begin{cases} 
(1,0) & \text{if } \sigma = \uparrow \\
(0,1) & \text{if } \sigma = \downarrow
\end{cases} \]

For each \( \xi = (x, a) = (x_0, x', \sigma, a) \in \mathcal{B} \) define \( w(\xi) \in \mathcal{H} \) by

\[ w(\xi) = \begin{cases} 
e^{-i<k, x'>} \sqrt{|C(k)|} \otimes \omega(\sigma) & \text{if } a = 0 \\
e^{-i<k, x'>} \frac{C(k)}{|C(k)|} \otimes \omega(\sigma) & \text{if } a = 1
\end{cases} \]

Then

\[ \|w(\xi)\|_\mathcal{H}^2 = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |C(k)| \quad \text{for all } \xi \in \mathcal{B} \]
and

\[ C(\xi, \xi') = \langle w(\xi), w(\xi') \rangle_\mathcal{H} \]

if \( \xi = (x, \sigma, 0), \xi' = (x', \sigma', 1) \in \mathcal{B} \). Part (i) of the Proposition now follows from part (i) of Proposition B.1 in [FKTr1].

ii) For each \( \xi = (x, a) = (x_0, x, \sigma, a) \in \mathcal{B} \) and \( s \in \Sigma \) define \( w'(\xi, s) \in \mathcal{H} \) by

\[ w'(\xi, s) = \begin{cases} \frac{e^{-i \langle k, x \rangle}}{(2\pi)^{d+1/2}} \sqrt{|C(k)|} \chi_s(k) \otimes \omega(\sigma) & \text{if } a = 0 \\ \frac{e^{-i \langle k, x' \rangle}}{(2\pi)^{d+1/2}} \sqrt{|C(k)|} \chi_s(k) \otimes \omega(\sigma) & \text{if } a = 1 \end{cases} \]

Then

\[ \| w'(\xi, s) \|_\mathcal{H}^2 = \int \frac{d^{d+1}k}{(2\pi)^d} |C(k)| |\chi_s(k)|^2 \]

and

\[ \int \psi_s(\xi) \psi_{s'}(\xi') d\mu_C(\xi) = \langle w(\xi, s), w(\xi', s') \rangle_\mathcal{H} \]

if \( \xi = (x_0, x, \sigma, 0), \xi' = (x'_0, x', \sigma', 1) \in \mathcal{B} \). Part (ii) of the Proposition now follows from part (i) of Proposition B.1 in [FKTr1], applied to the generating system of fields \( \psi_s(\xi) \).

\[ \blacksquare \]

**Lemma IV.4** Let \( \Lambda > 0 \) and \( U(k) \) a function on \( \mathbb{R}^d \). Assume that

\[ C(k) = \frac{U(k)}{i k_0 - \Lambda} \]

Then

\[ S(C) \leq \sqrt{\int \frac{d^d k}{(2\pi)^d} |U(k)|} \]

**Proof:** For \( a = 0, a' = 1 \)

\[ C((x_0, x, \sigma, a), (x'_0, x', \sigma', a')) = \delta_{\sigma, \sigma'} \int \frac{d k_0}{2\pi} e^{-i k_0 (x_0 - x'_0)} \int \frac{d^d k}{(2\pi)^d} e^{i \langle k, x - x' \rangle} U(k) \]

\[ = -\delta_{\sigma, \sigma'} \int \frac{d^d k}{(2\pi)^d} e^{i \langle k, x - x' \rangle} U(k) \begin{cases} e^{-\Lambda (x_0 - x'_0)} & \text{if } x_0 > x'_0 \\ 0 & \text{if } x_0 \leq x'_0 \end{cases} \]

Let \( \mathcal{H} \) be the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{Q}^2 \). For \( \sigma \in \{\uparrow, \downarrow\} \) define the element \( \omega(\sigma) \in \mathcal{Q}^2 \) as in the proof of Proposition IV.3, and for each \( \xi = (x_0, x, \sigma, a) \in \mathcal{B} \) define \( w(\xi) \in \mathcal{H} \) by

\[ w(\xi) = \begin{cases} \frac{e^{-i \langle k, x \rangle}}{(2\pi)^{d/2}} \sqrt{|U(k)|} \otimes \omega(\sigma) & \text{if } a = 0 \\ -\frac{e^{-i \langle k, x \rangle}}{(2\pi)^{d/2}} \sqrt{|U(k)|} \otimes \omega(\sigma) & \text{if } a = 1 \end{cases} \]
Again
\[ \|w(\xi)\|_H^2 = \frac{1}{(2\pi)^d} \int d^d k \, |U(k)| \] for all \( \xi \in \mathcal{B} \)

Furthermore set \( \tau(x_0, x, \sigma, a) = \Lambda x_0 \). Then for \( \xi = (x_0, x, \sigma, 0), \xi' = (x_0', x', \sigma', 1) \in \mathcal{B} \)

\[
C(\xi, \xi') = \begin{cases} 
\exp(-\tau(\xi) - \tau(\xi')) \langle w(\xi), w(\xi') \rangle_H & \text{if } \tau(\xi) > \tau(\xi') \\
0 & \text{if } \tau(\xi) \leq \tau(\xi')
\end{cases}
\]

The Lemma now follows from part(ii) of Proposition B.1 in [FKTr1].

Proposition IV.5 Assume that \( C \) is of the form

\[
C(k) = \frac{U(k) - \chi(k)}{ik_0 - e(k)}
\]

with real valued measurable functions \( U(k), e(k) \) on \( \mathbb{R}^d \) and \( \chi(k) \) on \( \mathbb{R} \times \mathbb{R}^d \) such that

\[ 0 \leq \chi(k) \leq U(k) \leq 1 \quad \text{for all } k = (k_0, k) \in \mathbb{R} \times \mathbb{R}^d. \]

Then

\[
S(C)^2 \leq 9 \int \frac{d^d k}{(2\pi)^d} \, |U(k)| + \frac{3}{2} \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \chi(k) + 6 \int_{|k_0| \leq E} \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \frac{U(k) - \chi(k)}{|ik_0 - e(k)|}
\]

where \( E = \sup_{k \in \text{supp} U} |e(k)| \).

Proof: Write

\[
C(k) = \frac{U(k) - \chi(k)}{ik_0 - E} + \frac{e(k) - E}{(ik_0 - e(k))(ik_0 - E)}(U(k) - \chi(k))
\]

By Remark IV.2, Lemma IV.4 and Proposition IV.3.i

\[
\frac{1}{3} S(C)^2 \leq \int \frac{d^d k}{(2\pi)^d} \, |U(k)| + \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \frac{\chi(k)}{|ik_0 - E|} + \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \frac{e(k) - E}{(ik_0 - e(k))(ik_0 - E)}(U(k) - \chi(k))
\]

The first two terms are bounded by

\[
\int \frac{d^d k}{(2\pi)^d} \, U(k) + \frac{1}{E} \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \chi(k)
\]

The contribution to the third term having \( |k_0| \leq E \) is bounded by

\[
\int_{|k_0| \leq E} \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \frac{e(k) - E}{(ik_0 - e(k))(ik_0 - E)}(U(k) - \chi(k)) \leq 2 \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \frac{U(k) - \chi(k)}{|ik_0 - e(k)|}
\]

The contribution to the third term having \( |k_0| > E \) is bounded by

\[
\int_{|k_0| > E} \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \frac{e(k) - E}{(ik_0 - e(k))(ik_0 - E)}(U(k) - \chi(k)) \leq 4 \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \frac{E}{|ik_0 - E|^2} U(k)
\]

\[ = 2 \int \frac{d^d k}{(2\pi)^d} U(k) \]

Hence

\[
\frac{1}{3} S(C)^2 \leq 3 \int \frac{d^d k}{(2\pi)^d} U(k) + \frac{1}{E} \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \chi(k) + 2 \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \frac{U(k) - \chi(k)}{|ik_0 - e(k)|}
\]

\[
= 3 \int \frac{d^d k}{(2\pi)^d} U(k) + \frac{1}{E} \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \chi(k) + 2 \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \, \frac{U(k) - \chi(k)}{|ik_0 - e(k)|}
\]
Contraction Bounds

We have observed in Example III.4 that the $L_1-L_\infty$-norm introduced in Example II.6 has $\max\{\|C\|_{1,\infty},\|C\|_{\infty}\}$ as a contraction bound for covariance $C$. For the propagators of the form (IV.1), we estimate these position space quantities by norms of derivatives of $C(k)$ in momentum space.

Definition IV.6

i) For a function $f(k)$ on $\mathbb{R} \times \mathbb{R}^d$ and a multiindex $\delta$ we set

$$D^\delta f(k) = \frac{\partial^{\delta_0}}{\partial k_0^{\delta_0}} \frac{\partial^{\delta_1}}{\partial k_1^{\delta_1}} \cdots \frac{\partial^{\delta_d}}{\partial k_d^{\delta_d}} f(k)$$

and

$$\|f(k)\|_{\infty} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} \frac{1}{\delta!} \left( \sup_{k} \left| D^\delta f(k) \right| \right) t^\delta \in \mathfrak{R}_{d+1}$$

$$\|f(k)\|_1 = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} \frac{1}{\delta!} \left( \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left| D^\delta f(k) \right| \right) t^\delta \in \mathfrak{R}_{d+1}$$

If $B$ is a measurable subset of $\mathbb{R} \times \mathbb{R}^d$,

$$\|f(k)\|_{\infty,B} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} \frac{1}{\delta!} \left( \sup_{k \in B} \left| D^\delta f(k) \right| \right) t^\delta \in \mathfrak{R}_{d+1}$$

$$\|f(k)\|_{1,B} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} \frac{1}{\delta!} \left( \int_B \frac{d^{d+1}k}{(2\pi)^{d+1}} \left| D^\delta f(k) \right| \right) t^\delta \in \mathfrak{R}_{d+1}$$

ii) For $\mu > 0$ and $X \in \mathfrak{R}_{d+1}$

$$T_\mu X = \frac{1}{\mu^d+1} X + \frac{\mu}{d+1} \sum_{j=0}^d \left( \frac{\partial}{\partial t_0} \cdots \frac{\partial}{\partial t_d} \right) \frac{\partial}{\partial t_j} X$$

Remark IV.7  For functions $f(k)$ and $g(k)$ on $B \subset \mathbb{R} \times \mathbb{R}^d$

$$\|f(k) g(k)\|_{1,B} \leq \|f(k)\|_{1,B} \|g(k)\|_{\infty,B}$$

by Leibniz’s rule for derivatives. The proof is similar to that of Lemma II.7.
Proposition IV.8  Let $d \geq 1$. Assume that there is a function $C(k)$ such that for $\xi = (x, a) = (x_0, x, \sigma, a)$, $\xi' = (x', a') = (x_0', x', \sigma', a') \in B$

$$C(\xi, \xi') = \begin{cases} \delta_{\sigma, \sigma'} \int \frac{d^{d+1} k}{(2\pi)^{d+1}} e^{\langle k, x-x' \rangle} - C(k) & \text{if } a = 0, a' = 1 \\ 0 & \text{if } a = a' \\ -C(\xi', \xi) & \text{if } a = 1, a' = 0 \end{cases}$$

Let $\delta$ be a multiindex and $0 < \mu \leq 1$.

i)$$\|D_{1,2}^\delta C\|_\infty \leq \int \frac{d^{d+1} k}{(2\pi)^{d+1}} |D^\delta C(k)| \leq \frac{\text{vol}}{(2\pi)^{d+1}} \sup_{k \in \mathbb{R} \times \mathbb{R}^d} |D^\delta C(k)|$$
and

$$\|C\|_{1,\infty} \leq \text{const} \|T_\mu\|_1 \|T_\mu\|_{\infty} \leq \frac{\text{vol}}{(2\pi)^{d+1}} \text{const} |D^\delta C(k)|$$

where $\text{vol}$ is the volume of the support of $C(k)$ in $\mathbb{R} \times \mathbb{R}^d$ and the constant $\text{const}$ depends only on the dimension $d$.

ii) Assume that there is an $r$-times differentiable real valued function $e(k)$ on $\mathbb{R}^d$ such that $|e(k)| \geq \mu$ for all $k \in \mathbb{R}^d$ and a real valued, compactly supported, smooth, non negative function $U(k)$ on $\mathbb{R}^d$ such that

$$C(k) = \frac{U(k)}{|k_0 - e(k)|}$$

Set

$$g_1 = \int_{\text{supp} U} d^d k \frac{1}{|e(k)|}, \quad g_2 = \int_{\text{supp} U} d^d k \frac{\mu}{|e(k)|^2}$$

Then there is a constant $\text{const}$ such that, for all multiindices $\delta$ whose spatial part $|\delta| \leq r - d - 1$,

$$\|C\|_\infty \leq \text{const} \|D_{1,2}^\delta C\|_{1,\infty} \leq \begin{cases} \frac{1}{\alpha} & \text{if } |\delta| = 0 \\ \frac{\text{const}}{\mu^{d+1}} & \text{if } |\delta| \geq 1 \end{cases}$$

The constant $\text{const}$ depends only on the dimension $d$, the degree of differentiability $r$, the ultraviolet cutoff $U(k)$ and the quantities $\sup_k |D^\gamma e(k)|$, $\gamma \in \mathbb{N}_0^d$, $|\gamma| \leq r$.

iii) Assume that $C$ is of the form

$$C(k) = \frac{U(k) - \chi(k)}{|k_0 - e(k)|}$$

with real valued functions $U(k)$, $e(k)$ on $\mathbb{R}^d$ and $\chi(k)$ on $\mathbb{R} \times \mathbb{R}^d$ that fulfil the following conditions:

The function $e(k)$ is $r$ times differentiable. $|ik_0 - e(k)| \geq \mu$ for all $k = (k_0, k)$ in the support of $U(k) - \chi(k)$. The function $U(k)$ is smooth and has compact support. The
function $\chi(k)$ is smooth and has compact support and $0 \leq \chi(k) \leq U(k) \leq 1$ for all $k = (k_0, k) \in \mathbb{R} \times \mathbb{R}^d$.

There is a constant const such that

$$\|C\|_{\infty} \leq \text{const}$$  \hspace{1cm} (IV.2)

The constant const depends on $d$, $\mu$ and the supports of $U(k)$ and $\chi$. Let $r_0 \in \mathbb{N}$. There is a constant const such that, for all multiindices $\delta$ whose spatial part $|\delta| \leq r - d - 1$ and whose temporal part $|\delta_0| \leq r_0 - 2$,

$$\|D_{1,2}^{\delta}C\|_{1,\infty} \leq \text{const}$$  \hspace{1cm} (IV.3)

The constant const depends on $d$, $r$, $r_0$, $\mu$, $U(k)$ and the quantities $\sup_k |D^{\gamma}e(k)|$ with $\gamma \in \mathbb{N}_0^d$, $|\gamma| \leq r$ and $\sup_k |D^{\beta}\chi(k)|$ with $\beta \in \mathbb{N}_0 \times \mathbb{N}_0^d$, $\beta_0 \leq r_0$, $|\beta| \leq r$.

Proof:

i) As the Fourier transform of the operator $D^{\delta'}$ is, up to a sign, multiplication by $[-i(x-x')]^{\delta'}$, we have for $\xi = (x, \sigma, a)$ and $\xi' = (x', \sigma', a')$

$$|x-x'|^{\delta'}|D_{1,2}^{\delta}(\xi, \xi')| \leq \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |D^{\delta+\delta'}C(k)|$$

In particular

$$|D_{1,2}^{\delta}C(\xi, \xi')| \leq \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |D^{\delta}C(k)|$$  \hspace{1cm} (IV.4)

and, for $j = 0, 1, \ldots, d$,

$$\mu^{d+2} |x_j - x'_j| \prod_{i=0}^{d} |x_i - x'_i| |D_{1,2}^{\delta}C(\xi, \xi')| \leq \mu^{d+2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |D^{\delta+\epsilon_j}C(k)|$$  \hspace{1cm} (IV.5)

where $\epsilon = (1, 1, \ldots, 1)$ and $\epsilon_j$ is the $j$th unit vector. Taking the geometric mean of (IV.5), \ldots, (IV.5.d) on the left hand side and the arithmetic mean on the right hand side gives

$$\mu^{d+2} \prod_{i=0}^{d} |x_i - x'_i|^{1 + \frac{1}{d+1}} |D_{1,2}^{\delta}C(\xi, \xi')| \leq \mu^{d+2} \frac{1}{d+1} \sum_{j=0}^{d} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |D^{\delta+\epsilon_j}C(k)|$$  \hspace{1cm} (IV.6)

Adding (IV.4) and (IV.6) gives

$$\left(1 + \mu^{d+2} \prod_{i=0}^{d} |x_i - x'_i|^{1 + \frac{1}{d+1}} \right) |D_{1,2}^{\delta}C(\xi, \xi')|$$

$$\leq \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |D^{\delta}C(k)| + \mu^{d+2} \frac{1}{d+1} \sum_{j=0}^{d} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |D^{\delta+\epsilon_j}C(k)|$$

$$\leq \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |D^{\delta}C(k)| + C \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |D^{\delta+\epsilon_j}C(k)|$$  \hspace{1cm} (IV.7)
Dividing across and using \( \int \frac{d^{d+1}x}{1 + \mu^{d+1} \prod_{i=0}^{d} |x_i|^{1+\frac{1}{\mu}}} \leq const \frac{1}{\mu^{d+1}} \) we get

\[
\| D_{1,2}^\delta C(\xi, \xi') \|_{1,\infty} \leq const \left( \frac{1}{\mu^{d+1}} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |D^\delta C(k)| + \mu \sum_{j=0}^d \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |D^{\delta + \epsilon_j} C(k)| \right)
\]

The contents of the bracket on the right hand side are, up to a factor of \( \frac{1}{\delta} \), the coefficient of \( t^\delta \) in \( T_\mu \| C(k) \|_1 \).

ii) Denote by

\[
C(t, k) = \int \frac{dk_0}{2\pi} e^{-ik_0 t} \frac{U(k)}{sk_0 - e(k)} = U(k) e^{-e(k)t} \left\{ \begin{array}{ll} -\chi(e(k) > 0) & \text{if } t > 0 \\ \chi(e(k) < 0) & \text{if } t \leq 0 \end{array} \right.
\]

the partial Fourier transform of \( C(k) \) in the \( k_0 \) direction. (As usual, the case \( t = 0 \) is defined through the limit \( t \to 0^- \).) Then, for \( |\delta| + |\delta'| \leq r \),

\[
| (x - x')^{\delta'} | | D_{1,2}^\delta C(\xi, \xi') | \leq \int \frac{d^d k}{(2\pi)^d} |D_{1,2}^{\delta_0} D^{\delta + \delta'} C(t - t', k)|
\]

\[
\leq const \int_{supp U} d^d k \left( |t - t'| / e(k) \right)^{\delta_0 + |\delta'|} e^{-|e(k)(t-t')|/3} \leq const 2^{\delta_0} \int_{supp U} \frac{1}{|e(k)|^{\delta + |\delta'|}} e^{-|e(k)(t-t')|/3}
\]

In particular, \( \| C \|_\infty \leq const \) and

\[
| (x - x')^{\delta'} | \int dt' | D_{1,2}^\delta C(\xi, \xi') | \leq const 2^{\delta_0} \int_{supp U} \frac{1}{|e(k)|^{\delta + |\delta'| + 1}}
\]

\[
\leq const 2^{\delta_0} \left\{ \begin{array}{ll} g_1 & \text{if } |\delta| + |\delta'| = 0 \\ \frac{g_2}{\mu^{\delta + |\delta'|}} & \text{if } |\delta| + |\delta'| > 0 \end{array} \right.
\]

\[
\leq const \frac{2^{\delta_0} \delta_1}{\mu^{\delta + |\delta'|}} \left\{ \begin{array}{ll} g_1 & \text{if } |\delta| = 0 \\ \frac{g_2}{\mu^{\delta + |\delta'|}} & \text{if } |\delta| > 0 \end{array} \right.
\]

since \( g_1 \geq g_2 \). As in equations (IV.4) – (IV.7), choosing various \( \delta' \)'s with \( |\delta'| = d + 1 \),

\[
\int dt' | D_{1,2}^\delta C(\xi, \xi') | \leq const 2^{\delta_0} \delta_1 \frac{1}{1 + \mu^{d+1} \prod_{i=1}^d |x_i - x'_i|^{1+\frac{1}{\mu}}} \left\{ \begin{array}{ll} g_1 & \text{if } |\delta| = 0 \\ \frac{g_2}{\mu_0^{\delta + |\delta'|}} & \text{if } |\delta| > 0 \end{array} \right.
\]

Integrating \( x' \) gives the desired bound on \( \| D_{1,2}^\delta C \|_{1,\infty} \).
iii) Write
\[ C(k) = C_1(k) - C_2(k) + C_3(k) \]
with
\[ C_1(k) = \frac{U(k)}{i k_0 - E} \]
\[ C_2(k) = \frac{\chi(k)}{i k_0 - E} \]
\[ C_3(k) = \frac{e(k) - E}{(i k_0 - e(k))(i k_0 - E)} (U(k) - \chi(k)) \]
and define the covariances \( C_j \) by
\[
C_j(\xi, \xi') = \begin{cases} 
  \delta_{\sigma, \sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i<k, x - x'>} - C_j(k) & \text{if } a = 0, a' = 1 \\
  0 & \text{if } a = a' \\
  -C_j(\xi', \xi) & \text{if } a = 1, a' = 0
\end{cases}
\]
for \( j = 1, 2, 3 \). For \( a = 0, a' = 1 \)
\[ C_1((x_0, x, \sigma, a), (x'_0, x', \sigma', a')) = -\delta_{\sigma, \sigma'} \int \frac{d^{d}k}{(2\pi)^d} e^{i<k, x - x'>} U(k) \left\{ \begin{array}{ll} 
  e^{-E(x_0 - x'_0)} & \text{if } x_0 > x'_0 \\
  0 & \text{if } x_0 \leq x'_0
\end{array} \right. \]
and, for \( |\delta| \leq r, |\delta_0| \leq r_0 \),
\[
\| D^\delta_{1,2} C_1 \|_\infty \leq \frac{\text{const}}{|E|^{\delta_0}} \| \delta_0! \| \leq \text{const} \quad \| D^\delta_{1,2} C_1 \|_1, \infty \leq \frac{\text{const}}{|E|^{\delta_0 + r}} \| \delta_0! \| \leq \text{const}
\]
By Remark IV.7
\[
\| C_2(k) \|_1 \leq \| \chi(k) \|_1 \int \frac{1}{i k_0 - E} \| \chi(k) \|_\infty \leq \| \chi(k) \|_1 \left( \sum_{n=0}^{\infty} \frac{1}{E^{\mu+1} t^0_0} \right)
\]
so that, for \( |\delta_0| \leq r_0 - 2 \) and \( |\delta| \leq r - d - 1 \),
\[
\| D^\delta_{1,2} C_2 \|_\infty \leq \text{const} \quad \| D^\delta_{1,2} C_2 \|_1, \infty \leq \text{const}
\]
by part (i).
We now bound \( C_3 \). Let \( B \) be the support of \( U(k) - \chi(k) \). On \( B \), \( |i k_0 - e(k)| \geq \mu > 0 \) and \( |e(k)| \leq E \), so we have, for \( \delta = (\delta_0, \delta) \neq 0 \) with \( |\delta| \leq r \) and \( \delta_0 \leq r_0 \),
\[
\left| D^\delta \frac{e(k) - E}{(i k_0 - e(k))(i k_0 - E)} \right| \leq \text{const} \frac{E}{|i k_0 - E|} \left( \frac{1}{|i k_0 - e(k)|^{\mu+1}} + \frac{1}{|i k_0 - e(k)|} \right) \leq \text{const} \frac{1}{|\delta|} \frac{E}{|i k_0 - E| |i k_0 - \mu|}
\]
Integrating
\[
\frac{1}{\delta} \int_B \frac{d^{d+1}k}{(2\pi)^{d+1}} \left| D^\delta \frac{e(k) - E}{(i k_0 - e(k))(i k_0 - E)} \right| \leq \text{const}
\]
It follows that
\[
\left\| \frac{e(k)-E}{(ik_0-e(k))(ik_0-E)} \right\|_{1,B}^r \leq \text{const} \sum_{|\delta| \leq r} t^\delta + \sum_{|\delta| > r} \infty t^\delta
\]
and, by Remark IV.7, that
\[
\left\| C^0(k) \right\|_1^r \leq \text{const} \left( \sum_{|\delta| \leq r} t^\delta + \sum_{|\delta| > r} \infty t^\delta \right) \left( \| U(k) \|_\infty + \| \chi(k) \|_\infty \right)
\]
By part (i) of this Proposition and the previous bounds on $C_1$ and $C_2$, this concludes the proof of part (iii).

**Corollary IV.9** Under the hypotheses of Proposition IV.8.ii, the $(d+1)$-dimensional norm
\[
\| C \|_{1,\infty} \leq \frac{\text{const} g_1 + g_2}{\mu^d} \sum_{|\delta| \geq 1} \left( \sum_{|\delta| \leq r-d-1} (2^\delta |\delta| t^\delta + \sum_{|\delta| \geq r-d} \infty t^\delta) \right)
\]
Under the hypotheses of Proposition IV.8.iii
\[
\| C \|_{1,\infty} \leq \text{const} \left( \sum_{|\delta| \leq r-d-1} t^\delta + \sum_{|\delta| > r-d-1} \infty t^\delta \right)
\]
In the renormalization group analysis we shall add a counterterm $\delta e(k)$ to the dispersion relation $e(k)$. For such a counterterm, we define the Fourier transform\(^{(1)}\)
\[
\delta \hat{e}(\xi, \xi') = \delta_{\sigma,\sigma'} \delta_{a,a'} \delta(x_0 - x'_0) \int e^{(-1)^a \xi \cdot k} e^{(x_0 - x'_0) \cdot \xi} \delta e(k) \frac{d^d k}{(2\pi)^d}
\]
for $\xi = (x, a) = (x_0, x, \sigma, a)$, $\xi' = (x', a') = (x'_0, x', \sigma', a') \in B$.

**Definition IV.10** Fix $r_0$ and $r$. Let
\[
c^0 = \sum_{|\delta| \leq r} t^\delta + \sum_{|\delta| > r} \infty t^\delta \in \mathcal{M}_{d+1}
\]
The map $c_0(X) = \frac{c_0}{1-X}$ from $X \in \mathcal{M}_{d+1}$ with $X_0 < 1$ to $\mathcal{M}_{d+1}$ is used to implement the differentiability properties of various kernels depending on a counterterm whose norm is bounded by $X$.

\(^{(1)}\) A comprehensive set of Fourier transform conventions are formulated in §IX.
Proposition IV.11 Let

\[ C(k) = \frac{U(k) - \chi(k)}{i k_0 - e(k) + \delta e(k)} \quad C_0(k) = \frac{U(k) - \chi(k)}{i k_0 - e(k)} \]

with real valued functions \( U(k), e(k), \delta e(k) \) on \( \mathbb{R}^d \) and \( \chi(k) \) on \( \mathbb{R} \times \mathbb{R}^d \) that fulfils the following conditions:

The function \( e(k) \) is \( r + d + 1 \) times differentiable. \( |i k_0 - e(k)| \geq \mu_0 > 0 \) for all \( k = (k_0, k) \) in the support of \( U(k) - \chi(k) \). The function \( U(k) \) is smooth and has compact support. The function \( \chi(k) \) is smooth and has compact support and \( 0 \leq \chi(k) \leq U(k) \leq 1 \) for all \( k = (k_0, k) \in \mathbb{R} \times \mathbb{R}^d \). The function \( \delta e(k) \) obeys

\[
\| \delta \hat{e} \|_{1, \infty} < \mu + \sum_{\delta \neq 0} \infty \ t^\delta
\]

Then, there is a constant \( \mu_1 > 0 \) such that if \( \mu < \mu_1 \), the following hold

i) \( C \) is an analytic function of \( \delta e \) and

\[
\| C \|_{\infty} \leq \text{const} \quad \| C - C_0 \|_{\infty} \leq \text{const} \| \delta \hat{e} \|_{1, \infty}
\]

and

\[
\| C \|_{1, \infty} \leq \text{const} e_0(\| \delta \hat{e} \|_{1, \infty}) \quad \| C - C_0 \|_{1, \infty} \leq \text{const} e_0(\| \delta \hat{e} \|_{1, \infty}) \| \delta \hat{e} \|_{1, \infty}
\]

ii) Let

\[
C_s(k) = \frac{U(k) - \chi(k)}{i k_0 - e(k) + s \delta e(k)}
\]

Then

\[
\left\| \frac{d}{ds} C_s \bigg|_{s=0} \right\|_{\infty} \leq \text{const} \| \delta \hat{e}' \|_{1, \infty} \quad \left\| \frac{d}{ds} C_s \bigg|_{s=0} \right\|_{1, \infty} \leq \text{const} e_0(\| \delta \hat{e} \|_{1, \infty}) \| \delta \hat{e}' \|_{1, \infty}
\]

Proof: i) The first bound follows from (IV.2), by replacing \( e \) by \( e - \delta e \).

Select a smooth, compactly support function \( \tilde{U}(k) \) and a smooth compactly supported function \( \tilde{\chi}(k) \) such that \( 0 \leq \tilde{\chi}(k) \leq \tilde{U}(k) \leq 1 \) for all \( k = (k_0, k) \in \mathbb{R} \times \mathbb{R}^d \). \( \tilde{U}(k) - \tilde{\chi}(k) \) is identically 1 on the support of \( U(k) - \chi(k) \) and \( |i k_0 - e(k)| \geq \frac{1}{2} \mu_e \) for all \( k = (k_0, k) \) in the support of \( \tilde{U}(k) - \tilde{\chi}(k) \). Let

\[
\tilde{C}_0(k) = \frac{\tilde{U}(k) - \tilde{\chi}(k)}{i k_0 - e(k)}
\]

Then

\[
C(k) = \frac{C_0(k)}{1 + \frac{\delta e(k)}{i k_0 - e(k)}} = \frac{C_0(k)}{1 + \frac{\delta e(k)(U(k) - \chi(k))}{i k_0 - e(k)}} = \frac{C_0(k)}{1 + \delta e(k) \tilde{C}_0(k)}
\]

\[
= C_0(k) \sum_{n=0}^{\infty} \left( -\delta e(k) \tilde{C}_0(k) \right)^n
\]

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Then, by iterated application of Lemma II.7 and the second part of Corollary IV.9, with \( r \) replaced by \( r + d + 1 \) and \( r_0 \) replaced by \( r_0 + 2 \),

\[
\| C \|_{1, \infty} \leq \| C_0 \|_{1, \infty} \sum_{n=0}^{\infty} \left( \| \delta \hat{e} \|_{1, \infty} \| \hat{C}_0 \|_{1, \infty} \right)^n
\]

\[
\leq \text{const} \; c_0 \sum_{n=0}^{\infty} \left( \text{const} ' \; c_0 \| \delta \hat{e} \|_{1, \infty} \right)^n
\]

\[
= \text{const} \frac{c_0}{1 - \| \delta \hat{e} \|_{1, \infty}}
\]

If \( \mu_1 < \min \{ \frac{1}{2 \text{const} '}, 1 \} \), then, by Corollary A.5.i, with \( \Delta = \{ \delta \in \mathcal{R}_{d+1} \mid |\delta| \leq r, |\delta_0| \leq r_0 \} \), \( \mu = \text{const} ' \), \( \Lambda = 1 \) and \( X = \| \delta \hat{e} \|_{1, \infty} \),

\[
\| C \|_{1, \infty} \leq \text{const} \frac{c_0}{1 - \| \delta \hat{e} \|_{1, \infty}}
\]

Similarly

\[
\| C - C_0 \|_{1, \infty} \leq \| C_0 \|_{1, \infty} \sum_{n=1}^{\infty} \left( \| \delta \hat{e} \|_{1, \infty} \| \hat{C}_0 \|_{1, \infty} \right)^n
\]

\[
\leq \text{const} \; c_0 \sum_{n=1}^{\infty} \left( \text{const} ' \; c_0 \| \delta \hat{e} \|_{1, \infty} \right)^n
\]

\[
\leq \text{const} \frac{c_0^2 \| \delta \hat{e} \|_{1, \infty}}{1 - \| \delta \hat{e} \|_{1, \infty}}
\]

and

\[
\| C - C_0 \|_{\infty} \leq \| C_0 \|_{\infty} \sum_{n=1}^{\infty} \left( \| \delta \hat{e} \|_{1, \infty} \| \hat{C}_0 \|_{1, \infty} \right)^n
\]

\[
\leq \text{const} \sum_{n=1}^{\infty} \left( \text{const} ' \; \| \delta \hat{e} \|_{1, \infty} \right)^n
\]

\[
\leq \text{const} \frac{\| \delta \hat{e} \|_{1, \infty}}{1 - \text{const} ' \; \mu}
\]

\[
\leq \text{const} \| \delta \hat{e} \|_{1, \infty}
\]

ii) As

\[
\frac{d}{ds} C_s(k) \big|_{s=0} = -\frac{U(k) - \chi(k)}{|\kappa_0 - e(k) + \delta e(k)|^2} \delta e'(k)
\]

the first bound is a consequence of Proposition IV.8.i.

Let \( \hat{U}(k) \) and \( \hat{\chi}(k) \) be as in part (i) and set

\[
\hat{C}(k) = \frac{\hat{U}(k) - \hat{\chi}(k)}{|\kappa_0 - e(k) + \delta e(k)|}
\]

Then

\[
\frac{d}{ds} C_s(k) \big|_{s=0} = -C(k) \hat{C}(k) \delta e'(k)
\]
and
\[
\left\| \frac{d}{ds} C_s(k) \right\|_{s=0} \leq \| C \|_{1, \infty} \| \hat{C} \|_{1, \infty} \| \dot{\delta} \|_{1, \infty} \leq \text{const } \varepsilon_0 (\| \delta \|_{1, \infty})^2 \| \dot{\delta} \|_{1, \infty} \leq \text{const } \varepsilon_0 (\| \delta \|_{1, \infty}) \| \dot{\delta} \|_{1, \infty}
\]

by Corollary A.5.ii.
V. Insulators

An insulator is a many Fermion system as described in the introduction, for which the dispersion relation \( e(\mathbf{k}) \) does not have a zero on the support of the ultraviolet cutoff \( U(\mathbf{k}) \). We may assume that there is a constant \( \mu > 0 \) such that \( e(\mathbf{k}) \geq \mu \) for all \( \mathbf{k} \in \mathbb{R}^d \). We shall show in Theorem V.2 that for sufficiently small coupling constant the Green’s functions for the interacting system exist and differ by very little from the Green’s functions of the noninteracting system in the supremum norm.

Lemma V.1 Let \( \rho_{m,n} \) be a sequence of nonnegative real numbers such that \( \rho_{m,n'} \leq \rho_{m,n} \) for \( n' \leq n \). Define for \( f \in \mathcal{F}_m(n) \)

\[
\|f\| = \rho_{m,n} \|f\|_{1,\infty}
\]

where \( \|f\|_{1,\infty} \) is the \( L_1 - L_\infty \) norm introduced in Example II.6.

i) The seminorms \( \| \cdot \| \) are symmetric.

ii) For a covariance \( C \), let \( S(C) \) be the quantity introduced in Definition IV.1. Then \( 2S(C) \) is an integral bound for the covariance \( C \) with respect to the family of seminorms \( \| \cdot \| \).

iii) Let \( C \) be a covariance. Assume that for all \( m \geq 0 \) and \( n,n' \geq 1 \)

\[
\rho_{m+n+n'-2} \leq \rho_{m,n} \rho_{0,n'}
\]

Let \( c \in \mathbb{N}_{d+1} \) obey

\[
c \geq \|C\|_{1,\infty}
\]

\[
c_0 \geq \frac{\rho_{m'+m,n+n'-2}}{\rho_{m,n} \rho_{m',n'}} \|C\|_{1,\infty} \quad \text{for all } m,m',n,n' \geq 1
\]

where \( c_0 \) is the constant coefficient of the formal power series \( c \). Then \( c \) is a contraction bound for the covariance \( C \) with respect to the family of seminorms \( \| \cdot \| \).

Proof: Parts (i) and (ii) are trivial. To prove part (iii), let \( f \in \mathcal{F}_m(n) \), \( f' \in \mathcal{F}_{m'}(n') \) and \( 1 \leq i \leq n, \ 1 \leq j \leq n' \). Set

\[
g(\eta_1,\cdots,\eta_{m+m'};\xi_1,\cdots,\xi_{i-1},\xi_{i+1},\cdots,\xi_n,\xi_{n+1},\cdots,\xi_{n+j-1},\xi_{n+j+1},\cdots,\xi_{n+n'})
= \int d\zeta d\zeta' f(\eta_1,\cdots,\eta_m;\xi_1,\cdots,\xi_{i-1},\xi,\xi_{i+1},\cdots,\xi_n) C(\zeta,\zeta')
\]

\[
f'(\eta_{m+1},\cdots,\eta_{m+m'};\xi_{n+1},\cdots,\xi_{n+j-1},\zeta',\xi_{n+j+1},\cdots,\xi_{n+n'})
\]

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Then
\[ C_{i\rightarrow n+j} \text{Ant}_{\text{ext}} (f \otimes f') = \text{Ant}_{\text{ext}} g \]
and therefore
\[ \| C_{i\rightarrow n+j} \text{Ant}_{\text{ext}} (f \otimes f') \| \leq \| g \| \]
If \( m, m' \geq 1 \)
\[ \| g \|_{1,\infty} \leq \| f \|_{1,\infty} \| C \|_{\infty} \| f' \|_{1,\infty} \]
and consequently
\[ \| C_{i\rightarrow n+j} \text{Ant}_{\text{ext}} (f \otimes f') \| \leq \rho_{m+m';n+n'-2} \| C \|_{\infty} \| f \|_{1,\infty} \| f' \|_{1,\infty} \]
\[ \leq \epsilon_0 \rho_{m;n} \| f \|_{1,\infty} \rho_{m';n'} \| f' \|_{1,\infty} \]
\[ \leq \epsilon \| f \| \| f' \| \]
If \( m = 0 \) or \( m' = 0 \), by iterated application of Lemma II.7
\[ \| g \|_{1,\infty} \leq \left\| \int_{\mathcal{B}} d\zeta f(\zeta_1,\ldots,\zeta_m;\xi_1,\ldots,\xi_{n-1},\xi_{n+1},\ldots,\xi_n) C(\zeta,\zeta') \right\|_{1,\infty} \| f' \|_{1,\infty} \]
\[ \leq \| f \|_{1,\infty} \| C \|_{1,\infty} \| f' \|_{1,\infty} \]
and again
\[ \| C_{i\rightarrow n+j} \text{Ant}_{\text{ext}} (f \otimes f') \| \leq \epsilon \| f \| \| f' \| \]

To formulate the result about insulators, we define for a function \( f(x_1,\ldots,x_n) \), on \((\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\})^n\), the \( L_1-L_{\infty} \)-norm as in Example II.6 to be
\[ \| f \|_{1,\infty} = \max_{1 \leq j \leq n} \sup_{x_j} \int_{\prod_{i=1}^{n} / \neq j} dx_i |f(x_1,\ldots,x_n)| \]

**Theorem V.2 (Insulators)** Let \( r \) and \( r_0 \) be natural numbers. Let \( e(k) \) be a dispersion relation on \( \mathbb{R}^d \) that is at least \( r + d + 1 \) times differentiable, and let \( U(k) \) be a compactly supported, smooth ultraviolet cutoff on \( \mathbb{R}^d \). Assume that there is a constant \( 0 < \mu < \frac{1}{2} \) such that
\[ e(k) \geq \mu \quad \text{for all } k \text{ in the support of } U \]
Set
\[ g = \int_{\text{supp} U} d^d k \frac{1}{|e(k)|} \quad \gamma = \max \left\{ 1, \sqrt{\int d^d k U(k) \log \frac{E}{|e(k)|}} \right\} \]
where \( E = \max \{ 1, \sup_{k \in \text{supp } U} |e(k)| \} \). Let, for \( x = (x_0, \mathbf{x}, \sigma), x' = (x'_0, \mathbf{x}', \sigma') \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \)

\[
C(x, x') = \delta_{\sigma, \sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{ik \cdot (x - x')} \frac{U(k)}{i k_0 - e(k)}
\]

and set, for \( \xi = (x, a), \xi' = (x', a') \in \mathcal{B} \)

\[
C(\xi, \xi') = C(x, x') \delta_{a, 0} \delta_{a', 1} - C(x', x) \delta_{a, 1} \delta_{a', 0}
\]

Furthermore let

\[
\mathcal{V}(\psi, \bar{\psi}) = \int_{(\mathbb{R} \times \mathbb{R}^2 \times \{\uparrow, \downarrow\})^4} dx_1 dy_1 dx_2 dy_2 \ V_0(x_1, y_1, x_2, y_2) \ \bar{\psi}(x_1)\psi(y_1)\bar{\psi}(x_2)\psi(y_2)
\]

be a two particle interaction with a kernel \( V_0 \) that is antisymmetric in the variables \( x_1, x_2 \) and \( y_1, y_2 \) separately. Set

\[
u = \sup_{\mathcal{D} \text{ decay operator}} \mu^{\delta(\mathcal{D})} \|\| \mathcal{D} V_0\|_{1, \infty}
\]

Then there exists \( \varepsilon > 0 \) and a constant \( \text{const} \) such that

i) If \( \|\| V_0\|_{1, \infty} \leq \frac{\varepsilon \mu^d}{g^2} \), the connected amputated Green’s functions \( G_2^{\text{amp}}(x_1, y_1, \cdots, x_n, y_n) \) exist in the space of all functions on \((\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\})^2n\) with finite \( \|\| \cdot \|_{1, \infty} \) norms. They are analytic functions of \( V_0 \).

ii) Suppose that \( \nu \leq \frac{\varepsilon \mu^d}{g^2} \). For all decay operators \( \mathcal{D} \) with \( \delta_0(\mathcal{D}) \leq r_0 \) and \( |\delta(\mathcal{D})| \leq r \)

\[
\|\| \mathcal{D} G_2^{\text{amp}} \|_{1, \infty} \leq \frac{\text{const}^n \ g^{-2n}}{\mu^{d+\|\|\mathcal{D}\|\|}} \nu^2 \quad \text{if } n \geq 3
\]

\[
\|\| \mathcal{D}(G_2^{\text{amp}} - V_0)\|_{1, \infty} \leq \frac{\text{const}^2 \ g^{-2}}{\mu^{d+\|\|\mathcal{D}\|\|}} \nu^2
\]

\[
\|\| \mathcal{D}(G_2^{\text{amp}} - K)\|_{1, \infty} \leq \frac{\text{const} \ g^{-2}}{\mu^{d+\|\|\mathcal{D}\|\|}} \nu^2
\]

where

\[
K(x, y) = 4 \int dx' dy' \ V_0(x, y, x', y') C(x', y')
\]

The constants \( \varepsilon \) and \( \text{const} \) depend on \( r, r_0, U, \) and the suprema of the \( k \)–derivatives of the dispersion relation \( e(k) \) up to order \( r + d + 1 \), but not on \( \mu \) or \( V_0 \).

**Proof:** By (I.1), the generating functional for the connected amputated Green’s functions is

\[
G^{\text{amp}}_{\text{gen}}(\phi) = \Omega_C(\mathcal{V})(0, \phi)
\]
To estimate it, we use the norms \( \| \cdot \| \) of Lemma V.1 with \( \rho_{m:n} = 1 \). By part (ii) of Lemma V.1 and Proposition IV.5, there is a constant \( \text{const}_0 \) such that \( b = \text{const}_0 \gamma \) is an integral bound for the covariance \( C \) with respect to these norms. By part (iii) of Lemma V.1, Corollary IV.9 and part (ii) of Proposition IV.8, there is a constant \( \text{const}_1 \) such that

\[
\mathcal{C} = \frac{\text{const}_1 g}{\mu^2} \left( \sum_{|\delta| \leq r} \left( \frac{2}{\mu} \right)^{|\delta|} t^\delta + \sum_{|\delta| > r} \infty t^\delta \right)
\]

is a contraction bound for \( C \) with respect to these norms. Here we used that \( g \) is bounded below by a nonzero \( (E \text{ and } U(k)\text{–dependent}) \) constant. As in Definition III.9, we set for any Grassmann function \( \mathcal{W}(\phi, \psi) \) and any \( \alpha > 0 \)

\[
N(\mathcal{W}; \mathcal{C}, \mathcal{V}, \alpha) = \frac{1}{\mathcal{C}} \sum_{m,n \geq 0} \alpha^n b^n \| \mathcal{W}_{m,n} \|
\]

In particular

\[
N(\mathcal{V}; \mathcal{C}, \mathcal{V}, \alpha) = \alpha^4 b^2 c \| V_0 \|_{1, \infty}
\]

and

\[
N(\mathcal{V}; \mathcal{C}, \mathcal{V}, 8\alpha) \leq \text{const}_3 \frac{8^4 \alpha^4 b^2 g}{\mu^2} \| V_0 \|_{1, \infty}
\]

Observe that

\[
\mathcal{C} \| V_0 \|_{1, \infty} \leq \frac{\text{const}_1 g}{\mu^2} \left( \sum_{|\delta| \leq r} \left( \frac{2}{\mu} \right)^{|\delta|} t^\delta + \sum_{|\delta| > r} \infty t^\delta \right) \left( \sum_{|\delta| \leq r} \frac{1}{\mu^{v\delta}} t^\delta + \sum_{|\delta| > r} \infty t^\delta \right)
\]

\[
\leq \frac{\text{const}_2 g v}{\mu^2} \left( \sum_{|\delta| \leq r} \frac{1}{\mu^{v\delta}} t^\delta + \sum_{|\delta| > r} \infty t^\delta \right)
\]

Write \( \mathcal{V} = \mathcal{V}' : C \). By part (i) of Proposition A.2 of [FKTr1],

\[
\mathcal{V}' = \mathcal{V} + \int dx\, dy\, \psi(x) \psi(y) + \text{const}
\]

and by part (i) of Corollary II.29 of [FKTr1]

\[
N(\mathcal{V}'; \mathcal{C}, \mathcal{V}, \alpha) \leq N(\mathcal{V}; \mathcal{C}, \mathcal{V}, 2\alpha) = 16 \alpha^4 b^2 c \| V_0 \|_{1, \infty}
\]

\[
\leq \text{const}_3 \alpha^4 \gamma^2 \frac{g v}{\mu^2} \left( \sum_{|\delta| \leq r} \frac{1}{\mu^{v\delta}} t^\delta + \sum_{|\delta| > r} \infty t^\delta \right)
\]

We set \( \alpha = 2 \) and \( \varepsilon = \frac{1}{2 \text{const}_3} \). Then

\[
N(\mathcal{V}'; \mathcal{C}, \mathcal{V}, 16) \leq \frac{g v^2 \gamma^2}{2 \varepsilon \mu^2} \left( \sum_{|\delta| \leq r} \frac{1}{\mu^{v\delta}} t^\delta + \sum_{|\delta| > r} \infty t^\delta \right)
\]
and, by (V.1)
\[ N(V'; c, b, 16)_0 \leq \frac{\gamma^2}{2\pi \mu^2} \| V_0 \|_{1, \infty} \]

Therefore, whenever \( \| V_0 \|_{1, \infty} \leq \frac{\varepsilon \mu^d}{\gamma^r} \), \( V' \) fulfills the hypotheses of Theorem III.10 and \( \mathcal{G}_{\text{gen}}(\psi) = \Omega_C(V; \psi)(0, \psi) \) exists. Part (i) follows.

If, in addition, \( \nu \leq \frac{\varepsilon \mu^d}{\gamma^r} \), then
\[ N(G^{\text{amp}}_{\text{gen}} - V'; c, b, 2) \leq \frac{1}{2} N(V'; c, b, 16)^2 \leq \frac{1}{2} \left( \frac{\gamma^2 \nu^2}{2\pi \mu^2} \right)^2 f \left( \frac{1}{n} \right) \]

where
\[ f(t) = \left( \sum_{|\delta| \leq r} t^\delta + \sum_{|\delta| > r} \infty t^\delta \right)^2 \]

\[ = \sum_{|\delta| \leq r} F_\delta t^\delta + \sum_{|\delta| > r} \infty t^\delta \]

with \( F_\delta \) finite for all \( |\delta| \leq r, |\delta_0| \leq r_0 \). Hence
\[ N(G^{\text{amp}}_{\text{gen}} - V'; c, b, 2) \leq \text{const}_4 \left( \frac{\gamma^2 \nu^2}{2\pi \mu^2} \right)^2 \left( \sum_{|\delta| \leq r} \frac{1}{\mu^{|\delta|}} t^\delta + \sum_{|\delta| > r} \infty t^\delta \right) \]

with \( \text{const}_4 = \frac{1}{8} \max_{|\delta| \leq r} F_\delta \). As
\[ N(G^{\text{amp}}_{\text{gen}} - V'; c, b, 2) = \nu \left( 4 \left\| G^2_{\text{amp}} - K \right\| + 16b^2 \left\| G^4_{\text{amp}} - V_0 \right\| + \sum_{n=3}^\infty 4(2b)^{2n-2} \left\| G^{2n}_{\text{amp}} \right\| \right) \]

\[ \geq \frac{4\text{const}_1 \nu}{\mu^d} \left( \left\| G^2_{\text{amp}} - K \right\|_{1, \infty} + 4 \text{const}_0^2 \gamma^2 \left\| G^4_{\text{amp}} - V_0 \right\|_{1, \infty} + \sum_{n=3}^\infty (\text{const}_0 \gamma)^{2n-2} \left\| G^{2n}_{\text{amp}} \right\|_{1, \infty} \right) \]

we have
\[ \left\| G^2_{\text{amp}} - K \right\|_{1, \infty} + 4 \text{const}_0^2 \gamma^2 \left\| G^4_{\text{amp}} - V_0 \right\|_{1, \infty} + \sum_{n=3}^\infty (\text{const}_0 \gamma)^{2n-2} \left\| G^{2n}_{\text{amp}} \right\|_{1, \infty} \]

\[ \leq \text{const}_4 \frac{\gamma^2 \nu^2}{\text{const}_1 \varepsilon^2 \mu^2} \left( \sum_{|\delta| \leq r} \frac{1}{\mu^{|\delta|}} t^\delta + \sum_{|\delta| > r} \infty t^\delta \right) \]

The estimates on the amputated Green’s functions follow.

\[ \blacksquare \]

**Remark V.3**

i) In reasonable situations, for example if the gradient of \( e(k) \) is bounded below, the constants \( \gamma \) and \( g \) in Theorem V.2 are of order one and \( \log \mu \) respectively.
ii) Using Lemma A.3, one may prove an analog of Theorem V.2 with the constants $\varepsilon$ and $\text{const}$ independent of $r_0$ and

$$
\| \mathcal{D} G_{2n}^{\text{amp}} \|_{1, \infty} \leq \text{const}^n \delta(\mathcal{D})! g \gamma^{6-2n} \left( \frac{8(d+1)}{\mu} \right)^{d+|\delta(\mathcal{D})|} \nu^2 \quad \text{if } n \geq 3
$$

$$
\| \mathcal{D}(G_4^{\text{amp}} - V_0) \|_{1, \infty} \leq \text{const}^2 \delta(\mathcal{D})! g \gamma^2 \left( \frac{8(d+1)}{\mu} \right)^{d+|\delta(\mathcal{D})|} \nu^2
$$

$$
\| \mathcal{D}(G_2^{\text{amp}} - K) \|_{1, \infty} \leq \text{const} \delta(\mathcal{D})! g \gamma^4 \left( \frac{8(d+1)}{\mu} \right)^{d+|\delta(\mathcal{D})|} \nu^2
$$

iii) Roughly speaking, the connected Green’s function are constructed from the connected amputated Green’s functions by appending propagators $C$. The details are given in §VI. Using Proposition IV.8.ii, one sees that, under the hypotheses of Theorem V.2.i, the connected Green’s functions exist in the space of all functions on $(\mathbb{R} \times \mathbb{R}^d \times \{ \uparrow, \downarrow \})^{2n}$ with finite $\| \cdot \|_{1, \infty}$ and $\| \cdot \|_{\infty}$ norms.
Appendix A: Calculations in the Norm Domain

Recall from Definition II.4 that the \((d+1)\)-dimensional norm domain \(\mathfrak{N}_{d+1}\) is the set of all formal power series

\[
X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} X_\delta \, t_0^\delta_0 \, t_1^\delta_1 \cdots t_d^\delta_d = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} X_\delta \, t^\delta
\]

in the variables \(t_0, t_1, \ldots, t_d\) with coefficients \(X_\delta \in \mathbb{R}_+ \cup \{\infty\}\. \\

**Definition A.1** A nonempty subset \(\Delta\) of \(\mathbb{N}_0 \times \mathbb{N}_0^d\) is called saturated if, for every \(\delta \in \Delta\) and every multiindex \(\delta'\) with \(\delta' \leq \delta\), the multiindex \(\delta'\) also lies in \(\Delta\). If \(\Delta\) is a finite set, then

\[
N(\Delta) = \min \{ n \in \mathbb{N} \mid n\delta \notin \Delta \text{ for all } 0 \neq \delta \in \Delta \}
\]

is finite.

For example, if \(r, r_0 \in \mathbb{N}\) then the set \(\{ \delta \in \mathbb{N}_0 \times \mathbb{N}_0^d \mid \delta_0 \leq r_0, |\delta| \leq r \}\) is saturated and \(N(\Delta) = \max\{r_0 + 1, r + 1\}\).

**Lemma A.2** Let \(\Delta\) be a saturated set of multiindices and \(X, Y \in \mathfrak{N}_{d+1}\). Furthermore, let \(f(t_0, \ldots, t_d)\) and \(g(t_0, \ldots, t_d)\) be analytic functions in a neighbourhood of the origin in \(\mathbb{C}^{d+1}\) such that, for all \(\delta \in \Delta\), the \(\delta^{\text{th}}\) Taylor coefficients of \(f\) and \(g\) at the origin are real and nonnegative. Assume that \(g(0) < 1\) and that, for all \(\delta \in \Delta\),

\[
X_\delta \leq \frac{1}{\delta!} \left( \prod_{i=0}^d \frac{\partial^\delta_i f}{\partial t_i^{\delta_i}} \right) \left| f(t_0, \ldots, t_d) \right|_{t_0=\cdots=t_d=0}
\]

\[
Y_\delta \leq \frac{1}{\delta!} \left( \prod_{i=0}^d \frac{\partial^\delta_i g}{\partial t_i^{\delta_i}} \right) \left| g(t_0, \ldots, t_d) \right|_{t_0=\cdots=t_d=0}
\]

Set \(Z = \frac{X}{1-Y}\) and \(h(t) = \frac{f(t)}{1-g(t)}\). Then, for all \(\delta \in \Delta\),

\[
Z_\delta \leq \frac{1}{\delta!} \left( \prod_{i=0}^d \frac{\partial^\delta_i h}{\partial t_i^{\delta_i}} \right) \left| h(t_0, \ldots, t_d) \right|_{t_0=\cdots=t_d=0}
\]

**Proof:** Trivial.  

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Example A.3  Let $\Delta$ be a saturated set and $a \geq 0$, $0 \leq \lambda \leq \frac{1}{2}$. Then
\[
\frac{\left( \sum_{\delta \in \Delta} a^{l(\delta)} t^\delta + \sum_{\delta \not\in \Delta} \infty t^\delta \right)^2}{1 - \lambda \left( \sum_{\delta \in \Delta} a^{l(\delta)} t^\delta + \sum_{\delta \not\in \Delta} \infty t^\delta \right)} \leq \frac{16}{3} \sum_{\delta \in \Delta} \left( 4(d + 1) a \right)^{|\delta|} t^\delta + \sum_{\delta \not\in \Delta} \infty t^\delta
\]

Proof: Set
\[
X = \left( \sum_{\delta \in \Delta} a^{l(\delta)} t^\delta + \sum_{\delta \not\in \Delta} \infty t^\delta \right)^2 \quad f(t) = \left( \sum_{\delta} a^{l(\delta)} t^\delta \right)^2 = \prod_{i=0}^{d} \frac{1}{(1 - at_i)^2}
\]
\[
Y = \lambda \left( \sum_{\delta \in \Delta} a^{l(\delta)} t^\delta + \sum_{\delta \not\in \Delta} \infty t^\delta \right) \quad g(t) = \lambda \left( \sum_{\delta} a^{l(\delta)} t^\delta \right) = \lambda \prod_{i=0}^{d} \frac{1}{1 - at_i}
\]

Set
\[
h(t) = \frac{f(t)}{1 - g(t)} = \frac{1}{\Pi(1 - at_i) \Pi(1 - at_i) - \lambda}
\]

By the Cauchy integral formula, with $\rho = \frac{1}{\pi} \left( 1 - \sqrt{\frac{3}{4}} \right)$
\[
\frac{1}{2\pi i} \left( \prod_{j=0}^{d} \frac{\partial}{\partial t_j} \right) h(t_0, \ldots, t_d) \bigg|_{t_0=\ldots=t_d=0} = \int_{|z_0| = \rho} \cdots \int_{|z_d| = \rho} h(z) \prod_{j=0}^{d} \left( \frac{1}{z_j^{1/2}} \frac{dz_j}{2\pi i} \right)
\]
\[
\leq \frac{1}{\rho^{|\delta|}} \sup_{|z_0| = \ldots = |z_d| = \rho} |h(z)|
\]
\[
\leq \frac{1}{\rho^{|\delta|}} \frac{1}{(1 - a \rho)^{d+1}} \frac{1}{(1 - a \rho)^{d+1} - \lambda}
\]
\[
\leq \frac{a^{l(\delta)}}{(1 - (3/4)^{1/(d+1)})^{|\delta|}} \frac{4}{3} \frac{1}{3/4 - 1/2}
\]
\[
\leq \frac{16}{3} \left( 4(d + 1) a \right)^{|\delta|}
\]

Lemma A.4

i) Let $X, Y \in \mathcal{N}_{d+1}$ with $X_0 + Y_0 < 1$
\[
\frac{1}{1-X} \frac{1}{1-Y} \leq \frac{1}{1-(X+Y)}
\]

ii) Let $\Delta$ be a finite saturated set and $X, Y \in \mathcal{N}_{d+1}$ with $X_0 + Y_0 < \frac{1}{2}$. There is a constant, $\text{const}$ depending only on $\Delta$, such that
\[
\frac{1}{1-(X+Y)} \leq \text{const} \frac{1}{1-X} \frac{1}{1-Y} + \sum_{\delta \not\in \Delta} \infty t^\delta \in \mathcal{N}_{d+1}
\]

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Proof: i)

\[
\frac{1}{1-\mu} \frac{1}{1-Y} = \sum_{m,n=0}^{\infty} X^m Y^n = \sum_{p=0}^{p} \sum_{m=0}^{m} X^m Y^{p-m} \leq \sum_{p=0}^{p} \left( \frac{p}{m} \right) X^m Y^{p-m}
\]

\[
= \sum_{p=0}^{p} (X + Y)^p = \frac{1}{1-(X+Y)}
\]

ii) Set \( \hat{X} = X - X_0 \) and \( \hat{Y} = Y - Y_0 \). Then

\[
\frac{1}{1-(X+Y)} = \frac{1}{1-(X_0+Y_0)-(X+Y)} \leq \frac{1}{\frac{1}{2}-(X+Y)}
\]

\[
\leq 2 \sum_{n=0}^{N(\Delta)-1} (2\hat{X} + 2\hat{Y})^n + \sum_{\delta \in \Delta} \infty t^\delta
\]

\[
= 2 \sum_{n=0}^{N(\Delta)-1} \sum_{m=0}^{n} 2^n \left( \frac{n}{m} \right) \hat{X}^m \hat{Y}^{n-m} + \sum_{\delta \in \Delta} \infty t^\delta
\]

\[
\leq 2^{2N(\Delta)-1} \sum_{n=0}^{N(\Delta)-1} \sum_{m=0}^{n} \hat{X}^m \hat{Y}^{n-m} + \sum_{\delta \in \Delta} \infty t^\delta
\]

\[
\leq 2^{2N(\Delta)-1} \frac{1}{1-X} \frac{1}{1-Y} + \sum_{\delta \in \Delta} \infty t^\delta
\]

\[
\leq 2^{2N(\Delta)-1} \frac{1}{1-X} \frac{1}{1-Y} + \sum_{\delta \in \Delta} \infty t^\delta
\]

\[\blacksquare\]

Corollary A.5 Let \( \Delta \) be a finite saturated set, \( \mu, \Lambda > 0 \). Set \( c = \sum_{\delta \in \Delta} \Lambda^{\delta} t^\delta + \sum_{\delta \in \Delta} \infty t^\delta \). There is a constant, \( \text{const} \) depending only on \( \Delta \) and \( \mu \), such that the following hold.

i) For all \( X \in \mathcal{N}_{d+1} \) with \( X_0 < \min\{\frac{1}{2\mu}, 1\} \).

\[
\frac{1}{1-\mu X} \leq \text{const} \frac{c}{1-X}
\]

ii) Set, for \( X \in \mathcal{N}_{d+1} \), \( \epsilon(X) = \frac{c}{1-\mu X} \). If \( \mu + \Lambda X_0 < \frac{1}{2} \), then

\[
\epsilon(X)^2 \leq \text{const} \epsilon(X) \quad \frac{\epsilon(X)}{1-\mu \epsilon(X)} \leq \text{const} \epsilon(X)
\]

Proof: i) Decompose \( X = X_0 + \hat{X} \). Then, by Example A.3 and Lemma A.4,

\[
\frac{c}{1-\mu X} = \frac{c}{1-\mu X_0 \epsilon - \mu \epsilon X} \leq \text{const} \frac{c}{1-\mu X_0 \epsilon} \frac{1}{1-\mu \epsilon X} \leq \text{const} \frac{c}{1-\epsilon/2} \frac{1}{1-\mu \epsilon X} \leq \text{const} \frac{c}{1-\mu X}
\]

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Expanding in a geometric series

\[ \frac{c}{1 - \mu X} \leq \text{const} \ c \sum_{n=0}^{N(\Delta) - 1} (\mu c \hat{X})^n \leq \text{const} \ c^{N(\Delta)} (1 + \mu^{N(\Delta)}) \sum_{n=0}^{N(\Delta) - 1} \hat{X}^n \leq \text{const} \ \frac{c}{1 - X} \leq \text{const} \ \frac{c}{1 - X} \]

ii) The first claim follows from the second, by expanding the geometric series. By Lemma A.4.ii and part (i),

\[ \frac{\epsilon(X)}{1 - \mu \epsilon(X)} = \frac{c}{1 - \mu X} \leq \text{const} \ \frac{c}{1 - \lambda X - \mu} \leq \text{const} \ \frac{c}{1 - \lambda X} = \epsilon(X) \]

**Remark A.6** The following generalization of Corollary A.5 is proven in the same way. Let \( \Delta \) be a finite saturated set, \( \mu, \lambda, \Lambda > 0 \). Set \( c = \sum_{\delta \in \Delta} \lambda^{\delta_0} \Lambda^{\delta} t^\delta + \sum_{\delta \notin \Delta} \infty t^\delta \). There is a constant, \( \text{const} \) depending only on \( \Delta \) and \( \mu \), such that the following hold.

i) For all \( X \in \mathcal{R}_{d+1} \) with \( X_0 < \min\{\frac{1}{2\mu}, 1\} \),

\[ \frac{c}{1 - \mu X} \leq \text{const} \ \frac{c}{1 - X} \]

ii) Set, for \( X \in \mathcal{R}_{d+1} \), \( \epsilon(X) = \frac{c}{1 - \lambda X} \). If \( \mu + \Lambda X_0 < \frac{1}{2} \), then

\[ \epsilon(X)^2 \leq \text{const} \ \epsilon(X) \quad \quad \frac{\epsilon(X)}{1 - \mu \epsilon(X)} \leq \text{const} \ \epsilon(X) \]

**Lemma A.7** Let \( \Delta \) be a finite saturated set and

\[ X = \sum_{\delta \in \Delta} X_\delta t^\delta + \sum_{\delta \notin \Delta} \infty t^\delta \in \mathcal{R}_{d+1} \]

Let \( f(z) \) be analytic at \( X_0 \), with \( f^{(n)}(X_0) \geq 0 \) for all \( n \), whose radius of convergence at \( X_0 \) is at least \( r > 0 \). Let \( 0 < \beta < \frac{1}{X_0} \). Then there exists a constant \( C \), depending only on \( \Delta, \beta, r \) and \( \max_{|z - X_0| = r} |f(z)| \) such that

\[ f(X) \leq C \frac{1}{1 - \beta X} \]

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**Proof:** Set $\alpha = \frac{\beta}{1-\beta X_0}$ and $\tilde{X} = X - X_0$. Then

\[
f(X) = \sum_n \frac{1}{n!} f^{(n)}(X_0) \tilde{X}^n \\
\leq \sum_{n<\Delta} \frac{1}{n!} f^{(n)}(X_0) \tilde{X}^n + \sum_{\delta \notin \Delta} \infty t^\delta \\
\leq C \sum_{n<\Delta} \alpha^n \tilde{X}^n + \sum_{\delta \notin \Delta} \infty t^\delta
\]

where

\[
C = \max_{n<\Delta} \frac{f^{(n)}(X_0)}{n! \beta^n} > \max_{n<\Delta} \frac{f^{(n)}(X_0)}{n! \alpha^n}
\]

Hence

\[
f(X) \leq \frac{C}{1-\alpha X} + \sum_{\delta \notin \Delta} \infty t^\delta = \frac{C}{1-\alpha(X-X_0)} = \frac{C(1-\beta X_0)}{1-\beta X} \leq \frac{C}{1-\beta X}
\]

\[\blacksquare\]
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